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CONTENTS

PAGE

1. Some Philosophical Aspects of Modern Physics. (Inaugural Lecture as Tait Professor of Natural Philosophy, University of Edinburgh.) By Max Born, Hon. D.Sc., Dr.Phil. *Communicated by* Professor E. T. Whittaker, F.R.S. Issued separately January 20, 1937, ... 1

2. Some Formulæ for the Associated Legendre Functions of the Second Kind; with corresponding Formulæ for the Bessel Functions. By Professor T. M. MacRobert, D.Sc. Issued separately January 20, 1937, ... 19

3. Quantitative Evolution in Compositæ. By Professor James Small, D.Sc., and Miss I. K. Johnston, M.Sc., Department of Botany, Queen's University, Belfast. Issued separately January 21, 1937, ... 26

4. Studies in Clocks and Time-keeping: No. 6. The Arc Equation. By Professor R. A. Sampson, LL.D., F.R.S. Issued separately January 21, 1937, ... 55

* 5. Microphthalmia and Other Eye-defects throughout Fourteen Generations of Albino Rats. By A. M. Hain, M.A., Ph.D., D.Sc., Carnegie Research Fellow, Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D. (With Chart and One Figure.) Issued separately February 10, 1937, ... 64

* 6. Ovarian Rhythm in *Drosophila*. By H. P. Donald, Ph.D., and Rowena Lamy, Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D. (With Five Text-figures and Four Graphs.) Issued separately February 11, 1937, ... 78

* 7. On the Geometry of Dirac's Equations and their Expression in Tensor Form. By Professor H. S. Ruse, M.A., D.Sc., University College, Southampton. (With One Text-figure.) Issued separately March 11, 1937, ... 97

8. On the Ciliary Currents on the Gills of Some *Tellinacea* (Lamellibranchiata). By Alastair Graham, M.A., B.Sc. (From the Department of Zoology, Birkbeck College, University of London.) *Communicated by* Charles H. O'Donoghue, D.Sc. (With Two Figures.) Issued separately May 6, 1937, 128

9. The Gravitational Field of a Distribution of Particles Rotating about an Axis of Symmetry. By W. J. van Stockum, Mathematical Institute, University of Edinburgh. *Communicated by* Professor E. T. Whittaker, F.R.S. Issued separately May 5, 1937, ... 135

*10. The Revised Complete System of a Quadratic Complex. By Professor H. W. Turnbull, F.R.S. Issued separately May 6, 1937, ... 155

PAGE

11. The Time Lag of the Vacuum Photo-cell. By R. A. Houstoun, M.A., D.Sc., Natural Philosophy Department, University of Glasgow. (With Three Figures.) Issued separately May 6, 1937, ... 163

12. Studies in Practical Mathematics. I. The Evaluation, with Applications, of a Certain Triple Product Matrix. By A. C. Aitken, D.Sc., F.R.S., Mathematical Institute, University of Edinburgh. Issued separately May 7, 1937, 172

13. Ions and Isotopes. By Professor James Kendall, F.R.S., Department of Chemistry, University of Edinburgh. (With One Figure.) (Address delivered to the Royal Society of Edinburgh on February 1, 1937.) Issued separately May 10, 1937, ... 182

*14. The Genetical and Mechanical Properties of Sex Chromosomes. III. Man. By P. C. Koller, D.Sc., Ph.D., Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D., D.Sc. (With Two Plates and Twelve Figures.) Issued separately June 9, 1937, . 194

15. Quantitative Evolution. II. Compositæ Dp-ages in Relation to Time. By Professor James Small, D.Sc., Department of Botany, Queen's University, Belfast. (With One Figure.) Issued separately June 9, 1937, . 215

16. Quantitative Evolution. III. Dp-ages of Gramineæ. By Professor James Small, D.Sc., Department of Botany, Queen's University, Belfast. (With One Figure.) Issued separately June 9, 1937, ... 221

17. Tests for Randomness in a Series of Numerical Observations. By W. O. Kermack, M.A., D.Sc., and Lt.-Col. A. G. McKendrick, M.B., D.Sc. (From the Laboratory of the Royal College of Physicians, Edinburgh.) Issued separately June 10, 1937, ... 228

18. The Benthic Amphipoda of the North-Western North Sea and Adjacent Waters. By D. S. Raitt, D.Sc., Ph.D., F.L.S., The Marine Laboratory of the Fishery Board for Scotland, Aberdeen. (With One Figure.) Issued separately August 16, 1937, ... 241

19. "Spheroidal": A Mutant in *Drosophila funebris* affecting Egg Size and Shape, and Fecundity. By Professor F. A. E. Crew, M.D., D.Sc., and Charlotte Auerbach, Ph.D., Institute of Animal Genetics, University of Edinburgh. (With Eight Graphs and Two Figures.) Issued separately August 16, 1937, ... 255

20. Studies in Practical Mathematics. II. The Evaluation of the Latent Roots and Latent Vectors of a Matrix. By A. C. Aitken, D.Sc., F.R.S., Mathematical Institute, University of Edinburgh. Issued separately August 18, 1937, ... 269

*21. On the Immature Stages of some Scottish and other Psyllidæ. By K. B. Lal, M.Sc., Ph.D. (Edin.), F.R.E.S., Department of Agricultural and Forest Zoology, University of Edinburgh. *Communicated by* Dr A. E. Cameron. (With Nine Figures.) Issued separately October 4, 1937, . 305

PAGE

*22. Some Distributions associated with a Randomly Arranged Set of Numbers. By W. O. Kermack, M.A., D.Sc., LL.D., and Lt.-Col. A. G. McKendrick, M.B., D.Sc. (From the Laboratory of the Royal College of Physicians, Edinburgh.) Issued separately October 5, 1937, ... 332

23. On Rotating Mirrors at High Speed. By Sir Charles V. Boys, LL.D., F.R.S. Issued separately October 5, 1937, ... 377

24. *Geonemertes Dendyi* Dakin, a Land Nemertean, in Wales. By A. R. Waterston, B.Sc., and H. E. Quick, M.B., F.R.C.S., B.Sc. *Communicated by* Dr A. C. Stephen. Issued separately October 5, 1937, ... 379

25. An Histological Analysis of Eye Pigment Development in *Drosophila pseudo-obscura*. By Flora Cochrane, Ph.D., Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D., D.Sc. (With Three Coloured Plates and Two Text-figures.) Issued separately November 13, 1937, ... 385

OBITUARY NOTICES: Richard Anschütz; de Burgh Birch; Albert William Borthwick; Alfred Daniell; John Edwards; David Ellis; Mungo McCallum Fairgrieve; David Fraser Fraser-Harris; Alfred William Gibb; John Anderson Gilruth; Sir Patrick Hehir; William Hunter; Rev. G. A. Frank Knight; Canon Albert Ernest Laurie; Magnus Maclean; Henry Moir; William John Owen; Arthur George Perkin; Salvatore Pincherle; Rt. Hon. Lord Rutherford of Nelson; Grafton Elliot Smith; William Morton Wheeler; William Brodie Brodie; Robert Craig Cowan; James Knight; James A. Macdonald; John Smith Purdy; William Ramsay Smith, . . . 400–436

APPENDIX—

Proceedings of the Statutory General Meeting, October 1936, ... 439
Proceedings of the Ordinary Meetings, Session 1936–1937, ... 442
Proceedings of the Statutory General Meeting, October 1937, ... 447
The Keith, Makdougall-Brisbane, Neill, Gunning Victoria Jubilee, James Scott, Bruce, and David Anderson-Berry Prizes, and the Bruce-Preller Lecture Fund, ... 451
Awards, ... 455
Accounts of the Society, Session 1936–1937, ... 463
Voluntary Contributors under Law VI (end of para. 3), ... 471
The Council of the Society at October 25, 1937, ... 472
Fellows of the Society at October 25, 1937, ... 473
Honorary Fellows of the Society at October 25, 1937, ... 501
Changes in Fellowship during Session 1936–1937, ... 503
Fellows of the Society elected during Session 1936–1937, ... 503
Laws of the Society, ... 504
Additions to the Library—Presentations, etc.—1936–1937, ... 512
Index, ... 517
Index, under Authors' Names, of Papers published in *Transactions*, . . 520

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PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. LVII. 1936–37.

I.—Some Philosophical Aspects of Modern Physics. (Inaugural Lecture as Tait Professor of Natural Philosophy, University of Edinburgh.) By **Max Born,** Hon. D.Sc., Dr.Phil. *Communicated by* Professor E. T. WHITTAKER, F.R.S.

(MS. received October 19, 1936.)

THE Chair which I have been elected to occupy, in succession to Professor Darwin, is associated with the name of a great scholar of our fathers' generation, Peter Guthrie Tait. This name has been familiar to me from the time when I first began to study mathematical physics. At that time Felix Klein was the leading figure in a group of outstanding mathematicians at Göttingen, amongst them Hilbert and Minkowski. I remember how Klein, ever eager to link physics with mathematics, missed no opportunity of pointing out to us students the importance of studying carefully the celebrated *Treatise on Natural Philosophy* of Thomson and Tait, which became a sort of Bible of mathematical science for us.

To-day theoretical physics has advanced in very different directions, and "Thomson and Tait" is perhaps almost unknown to the younger generation. But such is the fate of all scientific achievement; for it cannot claim eternal validity like the products of great artists, but has served well if it has served its time. For myself this book has a special attraction by reason of its title. The subject known everywhere else in the world by the dull name "Physics" appears here under the noble title of "Natural Philosophy," the same title as is given to the two Chairs of Physics in this University. Our science acquires by virtue of this name a dignity of its own. Occupied by his tedious work of routine

measurement and calculation, the physicist remembers that all this is done for a higher task: the foundation of a philosophy of nature. I have always tried to think of my own work as a modest contribution to this task; and in entering on the tenure of the Tait Chair of Natural Philosophy at this University, though far from my fatherland, I feel intellectually at home.

The justification for considering this special branch of science as a philosophical doctrine is not so much its immense object, the universe from the atom to the cosmic spheres, as the fact that the study of this object in its totality is confronted at every step by logical and epistemological difficulties; and although the material of the physical sciences is only a restricted section of knowledge, neglecting the phenomena of life and consciousness, the solution of these logical and epistemological problems is an urgent need of reason.

For describing the historical development it is a convenient coincidence that the beginning of the new century marks the separation of two distinct periods, of the older physics which we usually call classical, and modern physics. Einstein's theory of relativity of 1905 can be considered as being at once the culmination of classical ideas and the starting-point of the new ones. But during the preceding decade research on radiation and atoms, associated with the names of Röntgen, J. J. Thomson, the Curies, Rutherford, and many others, had accumulated a great number of new facts which did not fit into the classical ideas at all. The new conception of the quantum of action which helped to elucidate them was first put forward by Planck in 1900. The most important consequences of this conception were deduced by Einstein, who laid the foundations of the quantum theory of light in 1905, the year in which he published his relativity theory, and by Niels Bohr in 1913, when he applied the idea of the quantum to the structure of atoms.

Every scientific period is in interaction with the philosophical systems of its time, providing them with facts of observation and receiving from them methods of thinking. The philosophy of the nineteenth century on which classical physics relied is deeply rooted in the ideas of David Hume. From his philosophy there developed the two systems which dominated science during the latter part of the classical period, critical philosophy and empiricism.

The difference between these systems concerns the problem of the *a priori*. The idea that a science can be logically reduced to a small number of postulates or axioms is due to the great Greek mathematicians, who first tried to formulate the axioms of geometry and to derive the complete system of theorems from them. Since then the question of

what are the reasons for accepting just these axioms has perpetually occupied the interest of mathematicians and philosophers. Kant's work can be considered as a kind of enormous generalisation of this question; he attempted to formulate the postulates, which he called categories *a priori*, necessary to build up experience in general, and he discussed the roots of their validity. The result was the classification of the *a priori* principles into two classes, which he called analytic and synthetic, the former being the rules of pure logical thinking, including arithmetic, the latter containing the laws of space and time, of substance, causality, and other general conceptions of this kind. Kant believed that the root of the validity of the first kind was "pure reason" itself, whereas the second kind came from a special ability of our brain, differing from reason, which he called "pure intuition" (*reine Anschauung*). So mathematics was classified as a science founded on *a priori* principles, properties of our brain and therefore unchangeable; and the same was assumed for some of the most general laws of physics, as formulated by Newton.

But I doubt whether Kant would have maintained this view if he had lived a little longer. The discovery of non-Euclidean geometry by Lobatchefsky and Bolyai shook the *a priori* standpoint. Gauss has frankly expressed his opinion that the axioms of geometry have no superior position as compared with the laws of physics, both being formulations of experience, the former stating the general rules of the mobility of rigid bodies and giving the conditions for measurements in space. Gradually most of the physicists have been converted to the empirical standpoint. This standpoint denies the existence of *a priori* principles in the shape of laws of pure reason and pure intuition; and it declares that the validity of every statement of science (including geometry as applied to nature) is based on experience. It is necessary to be very careful in this formulation. For it is of course not meant that every fundamental statement—as, for instance, the Euclidean axioms of geometry—is directly based on special observations. Only the totality of a logically coherent field of knowledge is the object of empirical examination, and if a sufficient set of statements is confirmed by experiment, we can consider this as a confirmation of the whole system, including the axioms which are the shortest logical expression of the system.

I do not think that there is any objection to this form of empiricism. It has the virtue of being free from the petrifying tendency which systems of *a priori* philosophy have. It gives the necessary freedom to research, and as a matter of fact modern physics has made ample use of this freedom. It has not only doubted the *a priori* validify of Euclidean

geometry as the great mathematicians did a hundred years ago, but has really replaced it by new forms of geometry; it has even made geometry depend on physical forces, gravitation, and it has revolutionised in the same way nearly all categories *a priori*, concerning time, substance, and causality.

This liberation from the idea of the *a priori* was certainly important for the development of science, but it already took place during the last century, and does not represent the deciding difference between classical and modern physics. This difference lies in the attitude to the objective world. Classical physics took it for granted that there is such an objective world, which not only exists independently of any observer, but can also be studied by this observer without disturbing it. Of course every measurement is a disturbance of the phenomenon observed; but it was assumed that by skilful arrangement this disturbance can be reduced to a negligible amount. It is this assumption which modern physics has shown to be wrong. The philosophical problem connected with it arises from the difficulty in speaking of the state of an objective world if this state depends on what the observer does. It leads to a critical examination of what we mean by the expression "objective world."

The fact that statements of observations depend on the standpoint of the observer is as old as science. The orbit of the earth round the sun is an ellipse only for an observer standing just at the centre of mass of the two bodies. Relativity gave the first example in which the intrusion of the observer into the description of facts is not so simple, and leads to a new conception to conserve the idea of an objective world Einstein has acknowledged that his studies on this problem were deeply influenced by the ideas of Ernst Mach, a Viennese physicist who developed more and more into a philosopher. From his writings sprang a new philosophical system, positivism, which is much in favour to-day. Traces of it can be seen in fundamental papers of Heisenberg on quantum theory; but it has also met with strenuous opposition, for instance, from Planck. In any case, positivism is a living force in science. It is also the only modern system of philosophy which by its own rules is bound to keep pace with the progress of science. We are obliged to define our attitude towards it.

The characteristic feature of this system is the sharp distinction it draws between real and apparent problems, and correspondingly between those conceptions which have a real meaning and those which have not. Now it is evident and trivial that not every grammatically correct question is reasonable; take, for instance, the well-known conundrum: Given the length, beam, and horse-power of a steamer, how old is the captain?—

or the remark of a listener to a popular astronomical lecture: "I think I grasp everything, how to measure the distances of the stars and so on, but how did they find out that the name of this star is Sirius?" Primitive people are convinced that knowing the "correct" name of a thing is real knowledge, giving mystical power over it, and there are many instances of the survival of such word-fetishism in our modern world. But let us now take an example from physics in which the thing is not so obvious. Everybody believes he knows what the expression "simultaneous events" means, and he supposes as a matter of course that it means the same for any other individual. This is quite in order for neighbours on this little planet. Even when science made the step of imagining an individual of similar brain-power on another star there seemed to be nothing problematical. The problem appeared only when the imagination was driven so far as to ask how an observer on the earth and another on, say, Mars could compare their observations about simultaneous events. It was then necessary to take into account the fact that we are compelled to use signals for this comparison. The fastest signal at our disposal is a flash of light. In using light, or even only thinking about it, we are no longer permitted to rely on our brain-power, our intuition. We have to consider facts revealed by experiments. We have not only the fact of the finite velocity of light, but another most important fact, disclosed by Michelson's celebrated experiment: that light on this earth travels with the same speed in all directions, independently of the motion of the earth round the sun. One usually expresses this by saying that these experiments disprove the existence of an ether-wind which we would expect from the analogy of the wind felt in a moving car.

An admirable logical analysis of these facts led Einstein to the result that the question of simultaneity of two distant events is almost as absurd as that regarding the age of the captain. Just as this question would become significant by adding some data, say about his life insurance, the problem of simultaneity becomes reasonable by adding data about the motion of the observer. In this way the conception of time loses its absolute character, and space becomes involved in this revolution. For it becomes meaningless to speak about "space at this moment"; if we assume two observers in relative motion just passing one another, then each has his own "space at this moment," but the events contained in this space are different for the two observers.

What has now become of the idea of a world independent of the observer? If one sticks to the meaning of a static assembly of things at one moment, this idea of an objective world is lost. But it can be

saved by considering as the world the assembly of events, each having not only a given position in space but also a given time of occurrence. Minkowski has shown that it is possible to get a description of the connection of all events which is independent of the observer, or invariant, as the mathematicians say, by considering them as points in a four-dimensional continuum with a quasi-Euclidean geometry. But the division of this four-dimensional world into space and time depends on the observer.

When I wrote a popular book on relativity in 1920 I was so impressed by this wonderful construction that I represented this method of objectivation as the central achievement of science. I did not then realise that we were soon to be confronted with a new empirical situation which would compel us to undertake a much deeper critical review of the conception of an objective world.

I have here used the phrase "new empirical situation," following Niels Bohr, the founder of modern atomic theory, and the deepest thinker in physical science. He has coined this expression to indicate that the birth of new and strange ideas in physics is not the result of free or even frivolous speculation, but of the critical analysis of an enormous and complicated body of collected experience. Physicists are not revolutionaries but rather conservative, and inclined to yield only to strong evidence before sacrificing an established idea. In the case of relativity this evidence was strong indeed, but consisted to a large extent of negative statements, such as that mentioned above regarding the absence of an ether-wind. The generalisation which was conceived by Einstein in 1915 combining the geometry of the space-time world with gravitation rested, and still rests, on a rather slender empirical basis.

The second revolution of physics, called quantum theory, is, however, built on an enormous accumulation of experience, which is still growing from day to day. It is much more difficult to talk about these matters, because they have a much more technical character. The problem is the constitution of matter and radiation, which can be adequately treated only in laboratories with refined instruments. The evidence provided there consists of photographic plates, and of tables and curves representing measurements. They are collected in enormous numbers all over the world, but known only to the experts. I cannot suppose that you are acquainted with these experiments. In spite of this difficulty, I shall try to outline the problem and its solution, called quantum mechanics.

Let us start with the old problem of the constitution of light. At the beginning of the scientific epoch two rival theories were proposed: the corpuscular theory by Newton, the wave theory by Huygens. About

a hundred years elapsed before experiments were found deciding in favour of one of them, the wave theory, by the discovery of interference. When two trains of waves are superposed, and a crest of one wave coincides with a valley of the other, they annihilate one another; this effect creates the well-known patterns which you can observe on any pond on which swimming ducks or gulls excite water-waves. Exactly the same kind of pattern can be observed when two beams of light cross one another, the only difference being that you need a magnifying-lens to see them; the inference is that a beam of light is a train of waves of short wave-length. This conclusion has been supported by innumerable experiments.

But about a hundred years later, during my student days, another set of observations began to indicate with equal cogency that light consists of corpuscles. This type of evidence can best be explained by analogy with two types of instruments of war, mines and guns. When a mine explodes you will be killed if you are near it, by the energy transferred to you as a wave of compressed air. But if you are some hundred yards away you are absolutely safe; the explosion-wave has lost its dangerous energy by continuously spreading out over a large area. Now imagine that the same amount of explosive is used as the propellant in a machine-gun which is rapidly fired, turning round in all directions. If you are near it you will almost certainly be shot, unless you hastily run away. When you have reached a distance of some hundred yards you will feel much safer, but certainly not quite safe. The probability of being hit has dropped enormously, but if you are hit the effect is just as fatal as before.

Here you have the difference between energy spread out from a centre in the form of a continuous wave-motion, and a discontinuous rain of particles. Planck discovered, in 1900, the first indication of this discontinuity of light in the laws governing the heat radiated from hot bodies. In his celebrated paper of 1905, mentioned already, Einstein pointed out that experiments on the energetic effect of light, the so-called photoelectric effect, can be interpreted in the way indicated as showing unambiguously the corpuscular constitution of light. These corpuscles are called quanta of light or photons.

This dual aspect of the luminous phenomenon has been confirmed by many observations of various types. The most important step was made by Bohr, who showed that the enormous amount of observations on spectra collected by the experimentalists could be interpreted and understood with the help of the conception of light-quanta. For this purpose he had also to apply the idea of discontinuous behaviour to the motion of material particles, the atoms, which are the source of light.

I cannot follow out here the historical development of the quantum idea which led step by step to the recognition that we have here to do with a much more general conception. Light is not the only "radiation" we know; I may remind you of the cathode rays which appear when electric currents pass through evacuated bulbs, or the rays emitted by radium and other radioactive substances. These rays are certainly not light. They are beams of fast-moving electrons, *i.e.* atoms of electricity, or ordinary atoms of matter like helium. In the latter case this has been proved directly by Rutherford, who caught the beam (a so-called α-ray of radium) in an evacuated glass vessel and showed that it was finally filled with helium gas. To-day one can actually photograph the tracks of these particles of radiating matter in their passage through other substances.

In this case the corpuscular evidence was primary. But in 1924 de Broglie, from theoretical reasoning, suggested the idea that these radiations should show interference and behave like waves under proper conditions. This idea was actually confirmed by experiments a short time later. Not only electrons, but real atoms of ordinary matter like hydrogen or helium have all the properties of waves if brought into the form of rays by giving them a rapid motion.

This is a most exciting result, revolutionising all our ideas of matter and motion. But when it became known, theoretical physics was already prepared to treat it by proper mathematical methods, the so-called quantum mechanics, initiated by Heisenberg, worked out in collaboration with Jordan and myself, and quite independently by Dirac; and another form of the same theory, the wave-mechanics, worked out by Schrödinger in close connection with de Broglie's suggestion. The mathematical formalism is a wonderful invention for describing complicated things. But it does not help much towards a real understanding. It took several years before this understanding was reached, even to a limited extent. But it leads right amidst philosophy, and this is the point about which I have to speak.

The difficulty arises if we consider the fundamental discrepancy in describing one and the same process sometimes as a rain of particles, and at other times as a wave. One is bound to ask, What is it really? You see here the question of reality appears. The reason why it appears is that we are talking about particles or waves, things considered as well known; but which expression is adequate depends on the method of observation. We thus meet a situation similar to that in relativity, but much more complicated. For here the two representations of the same phenomenon are not only different but contradictory. I think

everyone feels that a wave and a particle are two types of motion which cannot easily be reconciled. But if we take into account the simple quantitative law relating energy and frequency already discovered by Planck, the case becomes very serious. It is clear that the properties of a given ray when appearing as a rain of particles must be connected with its properties when appearing as a train of waves. This is indeed the case, and the connecting law is extremely simple when all the particles of the beam have exactly the same velocity. Experiment then shows that the corresponding train of waves has the simplest form possible, which is called harmonic, and is characterised by a definite sharp frequency and wave-length. The law of Planck states that the kinetic energy of the particles is exactly proportional to the frequency of vibration of the wave; the factor of proportionality, called Planck's constant, and denoted by the letter h, has a definite numerical value which is known from experiment with fair accuracy.

There you have the logical difficulty; a particle with a given velocity is, *qua* particle, a point, existing at any instant without extension in space. A train of waves is by definition harmonic only if it fills the whole of space and lasts from eternity to eternity! [The latter point may not appear so evident; but a mathematical analysis made by Fourier more than a hundred years ago has clearly shown that every train of waves finite in space and time has to be considered as a superposition of many infinite harmonic waves of different frequencies and wave-lengths which are arranged in such a way that the outer parts destroy one another by interference; and it can be shown that every finite wave can be decomposed into its harmonic components.] Bohr has emphasised this point by saying that Planck's principle introduces an irrational feature into the description of nature.

Indeed the difficulty cannot be solved unless we are prepared to sacrifice one or other of those principles which were assumed as fundamental for science. The principle to be abandoned now is that of causality as it has been understood ever since it could be formulated exactly. I can indicate this point only very shortly. The laws of mechanics as developed by Galileo and Newton allow us to predict the future motion of a particle if we know its position and velocity at a given instant. More generally, the future behaviour of a system can be predicted from a knowledge of proper initial conditions. The world from the standpoint of mechanics is an automaton, without any freedom, determined from the beginning. I never liked this extreme determinism, and I am glad that modern physics has abandoned it. But other people do not share this view.

To understand how the quantum idea and causality are connected, we must explain the second fundamental law relating particles and waves. This can be readily understood with the help of our example of the exploding mine and the machine-gun. If the latter fires not only horizontally but equally in all directions, the number of bullets, and therefore the probability of being hit, will decrease with distance in exactly the same ratio as the surface of the concentric spheres, over which the bullets are equally distributed, increases. But this corresponds exactly to the decrease of energy of the expanding wave of the exploding mine. If we now consider light spreading out from a small source, we see immediately that in the corpuscular aspect the number of photons will decrease with the distance in exactly the same way as does the energy of the wave in the undulatory aspect. I have generalised this idea for electrons and any other kind of particles by the statement that we have to do with "waves of probability" guiding the particles in such a way that the intensity of the wave at a point is always proportional to the probability of finding a particle at that point. This suggestion has been confirmed by a great number of direct and indirect experiments. It has to be modified if the particles do not move independently, but act on one another; for our purpose, however, the simple case is sufficient.

Now we can analyse the connection between the quantum laws and causality.

Determining the position of a particle means restricting it physically to a small part of space. The corresponding probability wave must also be restricted to this small part of space, according to our second quantum law. But we have seen that by Fourier's analysis such a wave is a superposition of a great number of simple harmonic waves with wave-lengths and frequencies spread over a wide region. Using now the first quantum law stating the proportionality of frequency and energy, we see that this geometrically well-defined state must contain a wide range of energies. The opposite holds just as well. We have derived qualitatively the celebrated uncertainty law of Heisenberg: exact determination of position and velocity exclude one another; if one is determined accurately the other becomes indefinite.

The quantitative law found by Heisenberg states that for each direction in space the product of the uncertainty interval of space and that of momentum (equal to mass times velocity) is always the same, being given by Planck's quantum constant h.

Here we have the real meaning of this constant as an absolute limit of simultaneous measurement of position and velocity. For more

complicated systems there are other pairs or groups of physical quantities which are not measureable at the same instant.

Now we remember that the knowledge of position and velocity at one given time was the supposition of classical mechanics for determining the future motion. The quantum laws contradict this supposition, and this means the break-down of causality and determinism. We may say that these propositions are not just wrong, but empty: the premise is never fulfilled.

The result that the discovery of the quantum laws puts an end to the strict determinism which was unavoidable in the classical period is of great philosophical importance by itself. After relativity has changed the ideas of space and time, another of Kant's categories, causality, has to be modified. The *a priori* character of these categories cannot be maintained. But of course there is not a vacuum now where these principles were previously; they are replaced by new formulations. In the case of space and time these are the laws of the four-dimensional geometry of Minkowski. In the case of causality there also exists a more general conception, that of probability. Necessity is a special case of probability; it is a probability of one hundred per cent. Physics is becoming a fundamentally statistical science. The mathematical theory called quantum mechanics which expresses these ideas in a precise form is a most wonderful structure, not only comparable with, but superior to, classical mechanics. The existence of this mathematical theory shows that the whole structure is logically coherent. But this proof is rather indirect, and convincing only for those who understand the mathematical formalism. It is therefore an urgent task to show directly for a number of important cases why, in spite of the use of two such different pictures as particles and waves, a contradiction can never arise. This can be done by discussing special experimental arrangements with the help of Heisenberg's uncertainty relation. In complicated cases this sometimes leads to rather puzzling and paradoxical results, which have been carefully worked out by Heisenberg, Bohr, and Darwin, my predecessor in this Chair.

I shall mention only one case. Looking through a microscope I can see a microbe and follow its motion. Why should it not be possible to do the same with atoms or electrons, simply by using more powerful microscopes? The answer is that "looking through" the microscope means sending a beam of light, of photons, through it. These collide with the particles to be observed. If these are heavy like a microbe or even an atom they will not be essentially influenced by the photons, and the deflected photons collected by the lenses give an image of the object.

But if this is an electron, which is very light, it will recoil on colliding with the photon, an effect first directly observed by Compton. The change of velocity of the electron is to some extent indeterminate, and depends on the physical conditions in such a way that Heisenberg's uncertainty relation is exactly fulfilled in this case also.

Bohr has introduced the expression "complementarity" for the two aspects of particles and waves. Just as all colours which we see can be arranged in pairs of complementary colours giving white when mixed, so all physical quantities can be arranged in two groups, one belonging to the particle aspect, the other to the wave aspect, which never lead to contradictions, but are both necessary to represent the full aspect of nature.

Such a short expression for a complicated and difficult situation is very useful, for instance, with respect to the naïve question: Now, what is a beam of light or a material substance "really," a set of particles or a wave? Anybody who has understood the meaning of complementarity will reject this question as too much simplified and missing the point. But this rejection does not solve the problem whether the new theory is consistent with the idea of an objective world, existing independently of the observer. The difficulty is not the two aspects, but the fact that no description of any natural phenomenon in the atomistic domain is possible without referring to the observer, not only to his velocity as in relativity, but to all his activities in performing the observation, setting up the instruments, and so on. The observation itself changes the order of events. How then can we speak of an objective world?

Some theoretical physicists, among them Dirac, give a short and simple answer to this question. They say: the existence of a mathematically consistent theory is all we want. It represents everything that can be said about the empirical world; we can predict with its help unobserved phenomena, and that is all we wish. What you mean by an objective world we don't know and don't care.

There is nothing to be objected against this standpoint—except one thing, that it is restricted to a small circle of experts. I cannot share this *l'art pour l'art* standpoint. I think that scientific results should be interpreted in terms intelligible to every thinking man. To do this is precisely the task of natural philosophy.

The philosophers to-day concentrate their interest on other questions, more important for human life than the troubles arising from a refined study of atomistic processes. Only the positivists, who claim to have a purely scientific philosophy, have answered our question. Their standpoint (Jordan, 1936) is even more radical than that of Dirac mentioned above.

Whereas he declares himself content with the formulæ and uninterested in the question of an objective world, positivism declares the question to be meaningless.

Positivism considers every question as meaningless which cannot be decided by experimental test. As I said before, this standpoint has proved itself productive by inducing physicists to adopt a critical attitude towards traditional assumptions, and has helped in the building of relativity and quantum theory. But I cannot agree with the application made by the positivists to the general problem of reality. If all the notions we use in a science had their origin in this science, the positivists would be right. But then science would not exist. Although it may be possible to exclude from the internal activity of science all reference to other domains of thinking, this certainly does not hold for its philosophical interpretation. The problem of the objective world belongs to this chapter.

Positivism assumes that the only primary statements which are immediately evident are those describing direct sensual impressions. All other statements are indirect, theoretical constructions to describe in short terms the connections and relations of the primary experiences. Only these have the character of reality. The secondary statements do not correspond to anything real, and have nothing to do with an existing external world; they are conventions invented artificially to arrange and simplify "economically" the flood of sensual impressions

This standpoint has no foundation in science itself; nobody can prove by scientific methods that it is correct. I would say that its origin is metaphysical were I not afraid of hurting the feelings of the positivists, who claim to have an entirely unmetaphysical philosophy. But I may safely say that this standpoint rests on psychology, only it is not a sound psychology. Let us consider it applied to examples of everyday life. If I look at this table or this chair I receive innumerable sense-impressions —patches of colour—and when I move my head these impressions change. I can touch the objects and get a great variety of new sense-impressions, of varying resistance, roughness, warmth, and so on. But if we are honest, it is not these unco-ordinated impressions that we observe, but the total object "table" or "chair." There is a process of unconscious combination, and what we really observe is a totality which is not the sum of the single impressions, not more or less than this sum, but something new. What I mean will perhaps become clearer if I mention an acoustical phenomenon. A melody is certainly something else than the sum of the tones of which it is composed; it is a new entity.

Modern psychology is fully aware of this fact. I allude to the

Gestalt-psychology of v. Ehrenfels, Köhler, and Wertheimer. The word *Gestalt*, which seems to have no adequate English translation, means not only shape, but the totality which is really perceived. I cannot explain it better than by referring again to the example of melody. These *Gestalten* are formed unconsciously; when they are considered by the conscious mind they become conceptions and are provided with words. The unsophisticated mind is convinced that they are not arbitrary products of his mind, but impressions of an external world on his mind. I cannot see any argument for abandoning this conviction in the scientific sphere. Science is nothing else than common sense applied under unaccustomed conditions. The positivists say that this assumption of an external world is a step into metaphysics, and meaningless, since we can never know anything about it except by the perceptions of our senses. This is evident. Kant has expressed the same point by distinguishing between the empirical thing and the "thing in itself" (*Ding an sich*) which lies behind it. If the positivists go on to say that all our assertions regarding the external world are only symbolical, that their meaning is conventional, then I protest. For then every single sentence would be symbolical, conventional; even if I merely say, "Here I am sitting on a chair." The "chair" is no primary sensual impression, but a notion connected with a *Gestalt*, an unconscious integration of the impressions to a new unit which is independent of changes in the impressions. For if I move my body, my hands, my eyes, the sensual impressions change in the most complicated way, but the "chair" remains. The chair is invariant with respect to changes of myself, and of other things or persons, perceived as *Gestalten*. This fact, a very obtrusive fact, of "invariance" is what we mean by saying that there is "really" a chair. It can be submitted to test, not by physical experiment, but by the wonderful methods of unconscious mind, which is able to distinguish between a "real" and a painted chair by merely moving the head a little. The question of reality is therefore not meaningless, and its use not merely symbolic or conventional.

The expression "invariant" which I have already used in speaking of relativity, and which appears here in a more general sense, is the link connecting these psychological considerations with exact science. It is a mathematical expression first used in analytical geometry to handle quantitatively spatial *Gestalten*, which are simple shapes of bodies or configurations of such. I can describe any geometrical form by giving a sufficient number of co-ordinates of its points; for instance, the perpendicular projections of its points on three orthogonal co-ordinate planes. But this is by far too much; it describes not only the form but

the position relative to the three arbitrary planes, which is entirely irrelevant. Therefore one has to eliminate all the superfluous, uninteresting parts of the co-ordinate description by well-known mathematical processes; the result is the so-called invariants describing the intrinsic form considered.

Exactly the same holds if we have to do not only with size and shape, but also with colour, heat, and other physical properties. The methods of mathematical physics are just the same as those of geometry, starting with generalised co-ordinates and eliminating the accidental things. These are now not only situation in space, but motion, state of temperature or electrification, and so on. What remains are invariants describing things.

This method is the exact equivalent of the formation of *Gestalten* by the unconscious mind of the man unspoiled by science. But science transcends the simple man's domain by using refined methods of research. Here unknown forms are found, for which the unconscious process does not work. We simply do not know what we see. We have to think about it, change conditions, speculate, measure, calculate. The result is a mathematical theory representing the new facts. The invariants of this theory have the right to be considered as representations of objects in the real world. The only difference between them and the objects of everyday life is that the latter are constructed by the unconscious mind, whereas the objects of science are constructed by conscious thinking. Living in a time in which Freud's ideas about the unconscious sphere are generally accepted, there seems to be no difficulty in considering this difference between common and scientific objects as of second order. This is also justified by the fact that the boundary between them is not at all sharp, and is continually changing. Conceptions which once were purely scientific have become real things. The stars were bright points on a spherical shell for the primitive man. Science discovered their geometrical relations and orbits. It met with furious opposition; Galileo himself became a martyr to truth. To-day these mathematical abstractions are common knowledge of school-children, and have become part of the unconscious mind of the European. Something similar has happened with the conceptions of the electromagnetic field.

This idea that the invariant is the link between common sense and science occurred to me as quite natural. I was pleased when I found the same idea in the presentation of the Philosophy of Mathematics by Hermann Weyl (1926), the celebrated Princeton mathematician. I think it is also in conformity with Bohr's (1933) ideas. He insists on the point that our difficulties in physics come from the fact that we are compelled to use

the words and conceptions of everyday life even if we are dealing with refined observations. We know no other way of describing a motion than either by particles or waves. We have to apply them also in those cases where observation shows that they do not fit completely, or that we really have to do with more general phenomena. We develop mathematically the invariants describing the new observations, and we learn step by step to handle them intuitively. This process is very slow, and it proceeds only in proportion as the phenomena become known in wider circles. Then the new conceptions sink down in the unconscious mind, they find adequate names, and are absorbed into the general knowledge of mankind.

In quantum theory we are only at the beginning of this process. Therefore I cannot tell you in a few words of ordinary language what the reality is which quantum mechanics deals with. I can only develop the invariant features of this theory and try to describe them in ordinary language, inventing new expressions whenever a conception begins to appeal to intuition. This is what teaching of physics means. Well-trained youth takes things for granted which seemed to us horribly difficult, and later generations will be able to talk about atoms and quanta as easily as we are able to talk about this table and this chair, and about the stars in heaven. I do not, however, wish to belittle the gap between modern and classical physics. The idea that it is possible to think about the same phenomena with the help of two entirely different and mutually exclusive pictures without any danger of logical contradiction is certainly new in science. Bohr has pointed out that it may help to solve fundamental difficulties in biology and psychology. A living creature, plant, or animal is certainly a physico-chemical system. But it is also something more than this. There are apparently two aspects again. The time of materialism is over; we are convinced that the physico-chemical aspect is not in the least sufficient to represent the facts of life, to say nothing of the facts of mind. But there is the most intimate connection between both spheres; they overlap and are interwoven in the most complicated way. The processes of life and mind need other conceptions for their description than the physico-chemical processes with which they are coupled. Why do these differing languages never contradict each other? Bohr has suggested the idea that this is another case of complementarity, just as between particles and waves in physics. If you want to study a specific biological or psychological process by the methods of physics and chemistry, you have to apply all kinds of physical apparatus, which disturbs the process. The more you learn about the atoms and molecules during the process, the less

you are sure that the process is that which you want to study. By the time you know everything about the atoms, the creature will be dead. This is briefly Bohr's suggestion of a new and deeper complementary relation between physics and life, life and mind.

The old desire to describe the whole world in one unique philosophical language cannot be fulfilled. Many have felt this, but to modern physics belongs the merit of having shown the exact logical relation of two apparently incompatible trends of thought, by uniting them into a higher unit.

But with this result physics has not come to rest. It is the achievement of a bygone period, and new difficulties have appeared since. Observations on nuclei, the innermost parts of the atoms, have revealed a new world of smallest dimensions, where strange laws hold. It has been shown that every kind of atom has a nucleus of definite structure, consisting of a very close packing of two kinds of particles, called protons and neutrons. The proton is the nucleus of the lightest atom, hydrogen, with a positive electric charge. The neutron is a particle of nearly the same weight, but uncharged. In the atom the nucleus is surrounded by a cloud of electrons, which we have mentioned several times. They are particles nearly 2000 times lighter than the proton or the neutron; they carry a negative charge equal and opposite to that of the proton. But recently positive electrons or "positrons" have also been discovered; in fact, their existence was predicted by Dirac on account of theoretical considerations. Hence we have four kinds of particles, two "heavy" ones, proton and neutron, and two "light" ones, the negative and positive electron, which can all move with any velocity less than that of light. But then there are the photons, which can move only with the velocity of light, and very likely another kind of particles called "neutrinos" the motion of which is restricted in the same way.

The question which modern physics raises is: Why just these particles? Of course a question put like this is rather vague, but it has a definite meaning. There is, for instance, the ratio of the masses of proton and electron, the exact value of which has been found to be 1845. Then there is another dimensionless number 137, connecting the elementary charge, Planck's quantum constant, and the velocity of light. To derive these numbers from theory is an urgent problem—only a theory of this kind does not exist. It would have to deal with the relations between the four ultimate particles. There has been made the fundamental discovery that a positive and a negative electron can unite to nothing, disappear, the energy liberated in this process being emitted in the form of photons; and *vice versa*, such a pair can be born out of light. Processes

of this type, transformations of ultimate particles including birth and death, seem to be the key to a deeper understanding of matter. We can produce these violent processes in the laboratory only on a very small scale, but nature provides us with plenty of material in the form of the so-called cosmic rays. In observing them we are witnesses of catastrophes in which by the impact of two particles large groups of new particles are generated, which have received the suggestive name of "showers." We seem here to be at the limit where the conception of matter as consisting of distinct particles loses its value, and we have the impression that we shall have to abandon some other accepted philosophical principle before we shall be able to develop a satisfying theory.

It would be attractive to analyse the indications which our present knowledge yields. But my time is over.

The purpose of my lecture has been to show you that physics, besides its importance in practical life, as the fundamental science of technical development, has something to say about abstract questions of philosophy. There is much scepticism to-day about technical progress. It has far outrun its proper use in life. The social world has lost its equilibrium through the application of scientific results. But Western man, unlike the contemplative Oriental, loves a dangerous life, and science is one of his adventures.

We cannot stop it, but we can try to fill it with a true philosophical spirit: the search of truth for its own sake.

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II.—Some Formulæ for the Associated Legendre Functions of the Second Kind; with corresponding Formulæ for the Bessel Functions. By Professor T. M. MacRobert, D.Sc.

(MS. received July 10, 1936. Read November 2, 1936.)

§ 1. *Introductory.*—In two former papers (*Proc. Roy. Soc. Edin.*, vol. liv, 1934, pp. 135–144; vol. lv, 1935, pp. 85–90) a number of integrals and series, involving Associated Legendre Functions of the First Kind regarded as functions of their degrees, were evaluated. In this paper similar integrals and series for the Associated Legendre Functions of the Second Kind, and also for Bessel Functions, are discussed. The latter are deduced from Bessel's Integral in its generalised form and from the corresponding integral for the Modified Bessel Function of the First Kind; the former from an analogous formula for the Associated Legendre Functions of the Second Kind.

§ 2. *Preliminary Formulæ for the Associated Legendre Functions of the Second Kind.*—If in the integral

$$\int_0^\infty e^{-\lambda x+\frac{1}{2}\lambda(t+1/t)}\lambda^{m-\frac{1}{2}}d\lambda=\frac{\Gamma(m+\frac{1}{2})}{\{x-\frac{1}{2}(t+1/t)\}^{m+\frac{1}{2}}},$$

where $R(m) > -\frac{1}{2}$, $R\{x-\frac{1}{2}(t+1/t)\} > 0$, the expansion

$$e^{\frac{1}{2}\lambda(t+1/t)}=I_0(\lambda)+\sum_{n=1}^{\infty}(t^n+t^{-n})I_n(\lambda) \qquad (1)$$

is substituted, and if the series is then integrated term by term by means of the formula

$$\int_0^\infty e^{-\lambda x}I_n(\lambda)\lambda^{m-\frac{1}{2}}d\lambda=\sqrt{\left(\frac{2}{\pi}\right)}(x^2-1)^{-\frac{1}{2}m}Q^m_{n-\frac{1}{2}}(x), \qquad (2)$$

where $R(x) > 1$, $R(m+n) > -\frac{1}{2}$, it is found that

$$\sqrt{\left(\frac{\pi}{2}\right)}\frac{\Gamma(m+\frac{1}{2})(x^2-1)^{\frac{1}{2}m}}{\{x-\frac{1}{2}(t+1/t)\}^{m+\frac{1}{2}}}=Q^m_{-\frac{1}{2}}(x)+\sum_{n=1}^{\infty}(t^n+t^{-n})Q^m_{n-\frac{1}{2}}(x), \qquad (3)$$

where $R(x) > 1$, $R\{x-\frac{1}{2}(t+1/t)\} > 0$, $R(m) > -\frac{1}{2}$.

This is an example of the method of obtaining Legendre Function formulæ from Bessel Function formulæ elaborated in a recent paper (*Phil. Mag.*, vol. xxi, 1936, pp. 697–703).

Now in (3) put $t=e^{i\theta}$, and it becomes

$$\sqrt{\left(\frac{\pi}{2}\right)}\frac{\Gamma(m+\frac{1}{2})(x^2-1)^{\frac{1}{2}m}}{(x-\cos\theta)^{m+\frac{1}{2}}}=Q^m_{-\frac{1}{2}}(x)+2\sum_{n=1}^{\infty}\cos n\theta\, Q^m_{n-\frac{1}{2}}(x), \qquad (4)$$

where $R(x) > 1$, $R(m) > -\frac{1}{2}$. It follows that

$$Q^m_{n-\frac{1}{2}}(x) = \frac{\Gamma(m+\frac{1}{2})}{\sqrt{(2\pi)}}(x^2-1)^{\frac{1}{2}m}\int_0^\pi \frac{\cos n\theta d\theta}{(x-\cos\theta)^{m+\frac{1}{2}}}, \qquad (5)$$

where n is zero or a positive integer, $R(x) > 1$, $R(m) > -\frac{1}{2}$. This formula may be verified by expanding the integrand in descending powers of x and integrating term by term.

If x is equal to $\cosh\phi$,

$$\frac{1}{(x-\cos\theta)^{m+\frac{1}{2}}} = \frac{2^{m+\frac{1}{2}}e^{-(m+\frac{1}{2})\phi}}{(1-2\cos\theta\, e^{-\phi}+e^{-2\phi})^{m+\frac{1}{2}}}.$$

Hence, from (4) and the expansion for $(1-2z\cos\theta+z^2)^{-m-\frac{1}{2}}$, we deduce the identity

$$e^{(m+\frac{1}{2})\phi}(\sinh\phi)^{-m}\left\{Q^m_{-\frac{1}{2}}(\cosh\phi)+2\sum_{n=1}^{\infty}\cos n\theta\, Q^m_{n-\frac{1}{2}}(\cosh\phi)\right\}$$
$$= \pi(\sin\theta)^{-m}\sum_{n=0}^{\infty}\frac{\Gamma(n+2m+1)}{n!}e^{-n\phi}T^{-m}_{n+m}(\cos\theta), \qquad (6)$$

where $R(m) > -\frac{1}{2}$, $\phi > 0$.

The formula (5) can be generalised as follows: If $\lambda > 0$,

$$I_n(\lambda) = \frac{1}{2\pi i}\int_C e^{\frac{1}{2}\lambda(\zeta+1/\zeta)}\zeta^{-n-1}d\zeta, \qquad (7)$$

where the contour starts from $-\infty$ on the real axis, passes round the origin in the positive direction, and returns to $-\infty$ on the real axis, and amp $\zeta = -\pi$ initially. On substituting from (7) into (2), and changing the order of integration, we find that

$$\sqrt{\left(\frac{2}{\pi}\right)}(x^2-1)^{-\frac{1}{2}m}Q^m_{n-\frac{1}{2}}(x) = \frac{\Gamma(m+\frac{1}{2})}{2\pi i}\int_C \frac{\zeta^{-n-1}d\zeta}{\{x-\frac{1}{2}(\zeta+1/\zeta)\}^{m+\frac{1}{2}}}, \qquad (8)$$

where $R(x) > 1$, $R(m+n) > -\frac{1}{2}$, and the path C is chosen so that $R\{x-\frac{1}{2}(\zeta+1/\zeta)\} > 0$.

Now deform the contour into the real axis from $-\infty$ to -1, the unit circle described positively, and the real axis from -1 to $-\infty$, and multiply by $\pi/\Gamma(m+\frac{1}{2})$. Then *

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(x^2-1)^{-\frac{1}{2}m}Q^m_{n-\frac{1}{2}}(x) = \int_0^\pi \frac{\cos n\theta d\theta}{(x-\cos\theta)^{m+\frac{1}{2}}} - \sin n\pi\int_0^\infty \frac{e^{-nu}du}{(x+\cosh u)^{m+\frac{1}{2}}}, \qquad (9)$$

where $R(x) > 1$, $R(m+n) > -\frac{1}{2}$.

§ 3. *Integrals involving Associated Legendre Functions of the Second Kind.*—From (9), if $\lambda > 0$, $R(m) > -\frac{1}{2}$, $\phi > 0$,

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(\sinh\phi)^{-m}\int_0^\lambda \cos\lambda\theta\, Q^m_{\lambda-\frac{1}{2}}(\cosh\phi)d\lambda$$
$$= \tfrac{1}{2}\int_0^\pi\left\{\frac{\sin\lambda(\psi+\theta)}{\psi+\theta}+\frac{\sin\lambda(\psi-\theta)}{\psi-\theta}\right\}\frac{d\psi}{(\cosh\phi-\cos\psi)^{m+\frac{1}{2}}}$$

* *Cf.* F. J. W. Whipple, *Proc. Lond. Math. Soc.*, vol. xvi, 1917, p. 308.

$$+\frac{1}{2}\int_0^\infty \frac{e^{-\lambda u}\{u\sin\lambda(\pi+\theta)+(\pi+\theta)\cos\lambda(\pi+\theta)\}-(\pi+\theta)}{u^2+(\pi+\theta)^2}\,\frac{du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}$$

$$+\frac{1}{2}\int_0^\infty \frac{e^{-\lambda u}\{u\sin\lambda(\pi-\theta)+(\pi-\theta)\cos\lambda(\pi-\theta)\}-(\pi-\theta)}{u^2+(\pi-\theta)^2}\,\frac{du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}.$$

Hence, when $\lambda\to\infty$, if $R(m)>-\frac{1}{2}$, $\phi>0$,

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(\sinh\phi)^{-m}\int_0^\infty \cos\lambda\theta\, Q^m_{\lambda-\frac{1}{2}}(\cosh\phi)\,d\lambda$$

$$=\begin{cases}\dfrac{\pi}{2}\,\dfrac{1}{(\cosh\phi-\cos\theta)^{m+\frac{1}{2}}}-K, & -\pi<\theta<\pi,\\[2mm] \dfrac{\pi}{4}\,\dfrac{1}{(\cosh\phi+1)^{m+\frac{1}{2}}}-K, & \theta=\pm\pi,\\[2mm] -K, \quad \theta<-\pi \text{ or } \theta>\pi, & \end{cases} \quad . \quad . \quad (10)$$

where

$$K=\frac{1}{2}\int_0^\infty\left\{\frac{\pi+\theta}{u^2+(\pi+\theta)^2}+\frac{\pi-\theta}{u^2+(\pi-\theta)^2}\right\}\frac{du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}.$$

Similarly, if $R(m)\geqq-\frac{1}{2}$, $\phi>0$,

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(\sinh\phi)^{-m}\int_0^\infty \sin\lambda\theta\, Q^m_{\lambda-\frac{1}{2}}(\cosh\phi)\,d\lambda$$

$$=\frac{1}{2}P\int_{-\pi}^{\pi}\frac{1}{\psi+\theta}\cdot\frac{d\psi}{(\cosh\phi-\cos\psi)^{m+\frac{1}{2}}}$$

$$-\frac{1}{2}\int_0^\infty\left\{\frac{u}{u^2+(\pi-\theta)^2}-\frac{u}{u^2+(\pi+\theta)^2}\right\}\frac{du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}, \quad . \quad (11)$$

if $-\pi<\theta<\pi$, $\theta<-\pi$ or $\theta>\pi$.

If $\theta=\pm\pi$, the two integrals on the right are divergent. In these cases the value of the integral can be obtained by putting $m=-\frac{1}{2}$ in the equation before proceeding to the limit: the *L.H.S.* then vanishes. This equation is then divided by $(\cosh\phi+1)^{m+\frac{1}{2}}$ and subtracted from the given equation; it is then possible to make $\lambda\to\infty$. Thus, when $\theta=\pi$, the *L.H.S.* of (11) is equal to

$$\frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{\psi+\pi}\left\{\frac{1}{(\cosh\phi-\cos\psi)^{m+\frac{1}{2}}}-\frac{1}{(\cosh\phi+1)^{m+\frac{1}{2}}}\right\}d\psi$$

$$-\frac{1}{2}\int_0^\infty\left(\frac{1}{u}-\frac{u}{u^2+4\pi^2}\right)\left\{\frac{1}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}-\frac{1}{(\cosh\phi+1)^{m+\frac{1}{2}}}\right\}du. \quad (11')$$

Formulæ (10) and (11) may be checked by applying Fourier's Integral Theorem to them: in each case, after simplification, formula (9) is obtained.

§ 4. *Series for the Associated Legendre Functions of the Second Kind.*—From (9), if $R(m+\alpha) > -1$, $\phi > 0$, it can be deduced that

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(\sinh\phi)^{-m}\sum_{n=0}^{p}\cos(n+\alpha+\tfrac{1}{2})\theta\, Q_{n+\alpha}^{m}(\cosh\phi)$$

$$=\tfrac{1}{2}\int_0^{\pi}\left[\begin{aligned}&\cos\left\{\left(\alpha+\frac{p+1}{2}\right)(\psi-\theta)\right\}\frac{\sin\frac{p+1}{2}(\psi-\theta)}{\sin\frac{1}{2}(\psi-\theta)}\\&+\cos\left\{\left(\alpha+\frac{p+1}{2}\right)(\psi+\theta)\right\}\frac{\sin\frac{p+1}{2}(\psi+\theta)}{\sin\frac{1}{2}(\psi+\theta)}\end{aligned}\right]\frac{d\psi}{(\cosh\phi-\cos\psi)^{m+\frac{1}{2}}}$$

$$-\cos\alpha\pi\int_0^{\infty}e^{-(\alpha+\frac{1}{2})u}\frac{\Theta\, du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}},$$

where

$$\Theta=\frac{\cos(\alpha+\frac{1}{2})\theta+e^{-u}\cos(\alpha-\frac{1}{2})\theta+e^{-(p+1)u}\Lambda}{1+2e^{-u}\cos\theta+e^{-2u}},$$

and $|\Lambda|$ is bounded.

Hence, if $-\pi<\theta<\pi$, $R(m+\alpha)>-1$, $\phi>0$,

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(\sinh\phi)^{-m}\sum_{n=0}^{\infty}\cos(n+\alpha+\tfrac{1}{2})\theta\, Q_{n+\alpha}^{m}(\cosh\phi)=\frac{\pi}{2}\,\frac{1}{(\cosh\phi-\cos\theta)^{m+\frac{1}{2}}}$$

$$-\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{\sin\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}\cdot\frac{d\psi}{(\cosh\phi-\cos\psi)^{m+\frac{1}{2}}}$$

$$-\cos\alpha\pi\int_0^{\infty}e^{-(\alpha+\frac{1}{2})u}\frac{\cos(\alpha+\frac{1}{2})\theta+e^{-u}\cos(\alpha-\frac{1}{2})\theta}{1+2e^{-u}\cos\theta+e^{-2u}}\cdot\frac{du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}.\quad (12)$$

If $\alpha=\frac{1}{2}$, this reduces to (4).

When $\theta=\pm\pi$, the device used to obtain (11′) may again be employed. In these cases also, when $\alpha=\frac{1}{2}$ the formula reduces to (4).

Similarly it can be shown that, if $-\pi<\theta<\pi$, $R(m+\alpha)>-1$, $\phi>0$,

$$\frac{\sqrt{(2\pi)}}{\Gamma(m+\frac{1}{2})}(\sinh\phi)^{-m}\sum_{n=0}^{\infty}\sin(n+\alpha+\tfrac{1}{2})\theta\, Q_{n+\alpha}^{m}(\cosh\phi)$$

$$=\tfrac{1}{2}P\int_{-\pi}^{\pi}\frac{\cos\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}\cdot\frac{d\psi}{(\cosh\phi-\cos\psi)^{m+\frac{1}{2}}}$$

$$-\cos\alpha\pi\int_0^{\infty}e^{-(\alpha+\frac{1}{2})u}\frac{\sin(\alpha+\frac{1}{2})\theta+e^{-u}\sin(\alpha-\frac{1}{2})\theta}{1+2e^{-u}\cos\theta+e^{-2u}}\cdot\frac{du}{(\cosh\phi+\cosh u)^{m+\frac{1}{2}}}.\quad (13)$$

§ 5. *Integrals and Series for Bessel Functions.*—For the Bessel Function of the First Kind the generalised Bessel's Integral

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-x\sin\theta)d\theta-\frac{\sin n\pi}{\pi}\int_0^{\infty}e^{-nu-x\sinh u}du,\quad (14)$$

where $R(x) > 0$, is used. From it the following results are derived:—

$$\int_0^\infty \cos(\lambda\theta)\,J_\lambda(x)\,d\lambda = \begin{cases} \tfrac{1}{2}\cos(x\sin\theta) + L, & -\pi < \theta < \pi, \\ \tfrac{1}{4} + L, & \theta = \pm\pi, \\ L, & \theta < -\pi \text{ or } \theta > \pi, \end{cases} \tag{15}$$

where $R(x) \geqq 0$ and

$$L = \frac{1}{2\pi}P\int_{-\pi}^{\pi} \frac{1}{\psi+\theta}\sin(x\sin\psi)\,d\psi - \frac{1}{2\pi}\int_0^\infty e^{-x\sinh u}\left\{\frac{\pi+\theta}{u^2+(\pi+\theta)^2} + \frac{\pi-\theta}{u^2+(\pi-\theta)^2}\right\}du;$$

$$\int_0^\infty \sin(\lambda\theta)\,J_\lambda(x)\,d\lambda = \begin{cases} \tfrac{1}{2}\sin(x\sin\theta) + M, & -\pi \leqq \theta \leqq \pi, \\ M, & \theta \leqq -\pi \text{ or } \theta \geqq \pi; \end{cases} \tag{16}$$

where $R(x) \geqq 0$ and

$$M = \frac{1}{2\pi}P\int_{-\pi}^{\pi} \frac{1}{\psi+\theta}\cdot\cos(x\sin\psi)\,d\psi - \frac{1}{2\pi}\int_0^\infty e^{-x\sinh u}\left\{\frac{u}{u^2+(\pi-\theta)^2} - \frac{u}{u^2+(\pi+\theta)^2}\right\}du;$$

unless $\theta = \pm\pi$, when the integrals diverge: in these cases put $x=0$ and subtract before proceeding to the limit;

$$\begin{aligned}\sum_{n=0}^{\infty}\cos(n+\alpha+\tfrac{1}{2})\theta\,J_{n+\alpha+\frac{1}{2}}(x) &= \tfrac{1}{2}\cos(x\sin\theta) \\ &- \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\psi)\frac{\sin\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}d\psi \\ &+ \frac{1}{2\pi}P\int_{-\pi}^{\pi}\sin(x\sin\psi)\frac{\cos\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}d\psi \\ &- \frac{\cos\alpha\pi}{\pi}\int_0^\infty e^{-x\sinh u-(\alpha+\frac{1}{2})u}\,\frac{\cos(\alpha+\frac{1}{2})\theta + e^{-u}\cos(\alpha-\frac{1}{2})\theta}{1+2e^{-u}\cos\theta+e^{-2u}}du,\end{aligned} \tag{17}$$

where $R(x) \geqq 0$, $\alpha > -\frac{1}{2}$, $-\pi < \theta < \pi$;

$$\begin{aligned}\sum_{n=0}^{\infty}\sin(n+\alpha+\tfrac{1}{2})\theta\,J_{n+\alpha+\frac{1}{2}}(x) &= \tfrac{1}{2}\sin(x\sin\theta) \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(x\sin\psi)\,\frac{\sin\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}d\psi \\ &+ \frac{1}{2\pi}P\int_{-\pi}^{\pi}\cos(x\sin\psi)\frac{\cos\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}d\psi \\ &- \frac{\cos\alpha\pi}{\pi}\int_0^\infty e^{-x\sinh u-(\alpha+\frac{1}{2})u}\,\frac{\sin(\alpha+\frac{1}{2})\theta + e^{-u}\sin(\alpha-\frac{1}{2})\theta}{1+2e^{-u}\cos\theta+e^{-2u}}du,\end{aligned} \tag{18}$$

where $R(x) \geqq 0$, $\alpha > -\frac{1}{2}$, $-\pi < \theta < \pi$: if $\theta = \pm\pi$, put $x=0$ and subtract;

$$\int_0^\infty J_\lambda(\lambda x)\,d\lambda = \frac{1}{2(1-x)} - \int_0^\infty \frac{du}{(u+x\sinh u)^2+\pi^2}, \tag{19}$$

where $0 \leqq x < 1$;

$$\begin{aligned}\sum_{n=0}^{\infty}J_{n+\alpha+\frac{1}{2}}\{(n+\alpha+\tfrac{1}{2})x\} &= \frac{1}{2(1-x)} - \frac{1}{\pi}\int_0^\pi \frac{\sin\alpha(\theta-x\sin\theta)}{2\sin\frac{1}{2}(\theta-x\sin\theta)}d\theta \\ &- \frac{\cos\alpha\pi}{\pi}\int_0^\infty e^{-\alpha(u+x\sinh u)}\frac{du}{2\cosh\frac{1}{2}(u+x\sinh u)},\end{aligned} \tag{20}$$

where $0 \leqq x < 1$, $\alpha > -\frac{1}{2}$.

Similarly, from the formula

$$I_n(x)=\frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta d\theta-\frac{\sin n\pi}{\pi}\int_0^\infty e^{-x\cosh u-nu}du, \quad . \quad . \quad (21)$$

where $R(x)>0$, it can be deduced that:

$$\int_0^\infty \cos(\lambda\theta)I_\lambda(x)d\lambda=\begin{cases}\frac{1}{2}e^{x\cos\theta}-N, & -\pi<\theta<\pi,\\ \frac{1}{2}e^{-x}-N, & \theta=\pm\pi,\\ -N, & \theta<-\pi \text{ or } \theta>\pi,\end{cases} \quad . \quad (22)$$

where $R(x)\geqq 0$ and

$$N=\frac{1}{2\pi}\int_0^\infty e^{-x\cosh u}\left\{\frac{\pi+\theta}{u^2+(\pi+\theta)^2}+\frac{\pi-\theta}{u^2+(\pi-\theta)^2}\right\}du;$$

$$\int_0^\infty \sin(\lambda\theta)I_\lambda(x)d\lambda=\frac{1}{2\pi}P\int_{-\pi}^{\pi} e^{x\cos\psi}\frac{d\psi}{\psi+\theta}$$
$$-\frac{1}{2\pi}\int_0^\infty e^{-x\cosh u}\left\{\frac{u}{u^2+(\pi-\theta)^2}-\frac{u}{u^2+(\pi+\theta)^2}\right\}du, \quad (23)$$

where $R(x)\geqq 0$ and $-\pi<\theta<\pi$ or $\theta<-\pi$ or $\theta>\pi$: if $\theta=\pm\pi$, put $x=0$, multiply by e^{-x} and subtract;

$$\sum_{n=0}^\infty \cos(n+\alpha+\tfrac{1}{2})\theta\, I_{n+\alpha+\frac{1}{2}}(x)=\tfrac{1}{2}e^{x\cos\theta}-T, \quad . \quad . \quad (24)$$

where $R(x)\geqq 0$, $\alpha>-\frac{1}{2}$, $-\pi<\theta<\pi$, and

$$T=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\psi}\frac{\sin\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}d\psi$$
$$+\frac{\cos\alpha\pi}{\pi}\int_0^\infty e^{-x\cosh u-(\alpha+\frac{1}{2})u}\,\frac{\cos(\alpha+\frac{1}{2})\theta+e^{-u}\cos(\alpha-\frac{1}{2})\theta}{1+2e^{-u}\cos\theta+e^{-2u}}du;$$

if $\theta=\pm\pi$, put $x=0$, multiply by e^{-x} and subtract: when $\alpha=\frac{1}{2}$, (24) reduces to

$$\tfrac{1}{2}e^{x\cos\theta}=\tfrac{1}{2}I_0(x)+\sum_{n=1}^\infty \cos n\theta\, I_n(x),$$

which is a form of (1) and is valid for all values of x and θ;

$$\sum_{n=0}^\infty \sin(n+\alpha+\tfrac{1}{2})\theta\, I_{n+\alpha+\frac{1}{2}}(x)=\frac{1}{2\pi}P\int_{-\pi}^{\pi} e^{x\cos\psi}\frac{\cos\alpha(\psi+\theta)}{2\sin\frac{1}{2}(\psi+\theta)}d\psi$$
$$-\frac{\cos\alpha\pi}{\pi}\int_0^\infty e^{-x\cosh u-(\alpha+\frac{1}{2})u}\frac{\sin(\alpha+\frac{1}{2})\theta+e^{-u}\sin(\alpha-\frac{1}{2})\theta}{1+2e^{-u}\cos\theta+e^{-2u}}du, \quad (25)$$

where $R(x)\geqq 0$, $\alpha>-\frac{1}{2}$, $-\pi<\theta<\pi$: if $\theta=\pm\pi$ put $x=0$, multiply by e^{-x} and subtract.

NOTE.—From the recurrence formula $2J_n'(x)=J_{n-1}(x)-J_{n+1}(x)$ it is easily deduced that, if $\alpha>-\frac{1}{2}$,

$$\sum_0^\infty J_{2n+\alpha+\frac{1}{2}}(x)=\tfrac{1}{2}\int_0^x J_{\alpha-\frac{1}{2}}(x)dx,$$

and therefore

$$\sum_0^\infty J_{n+\alpha+\frac{1}{2}}(x) = \tfrac{1}{2}\int_0^x \{J_{\alpha-\frac{1}{2}}(x) + J_{\alpha+\frac{1}{2}}(x)\}dx.$$

This may be deduced from (17) when $\theta=0$ by putting $x=0$ in that formula and subtracting.

Similarly, from the formula $2I_n'(x) = I_{n-1}(x) + I_{n+1}(x)$, it follows that, if $\alpha > -\frac{1}{2}$,

$$\sum_{n=0}^\infty (-1)^n I_{2n+\alpha+\frac{1}{2}}(x) = \tfrac{1}{2}\int_0^x I_{\alpha-\frac{1}{2}}(x)dx.$$

This formula may also be proved—as an exercise—by the methods employed in this paper.

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III.—Quantitative Evolution in Compositæ. By Professor James Small, D.Sc., and Miss I. K. Johnston, M.Sc.,

Department of Botany, Queen's University, Belfast.

(MS. received October 23, 1936. Read December 7, 1936.)

CONTENTS.

	PAGE
I. Introduction	26
II. Udny Yule's Formulæ and the Goodness of Fit . . .	30
III. Dp-ages of Sub-tribes and Tribes	37
IV. Exceptions to the Seriation of Dp-ages	43
V. Dp-ages in Relation to Time .	46
Conclusion and Summary .	50
References to Literature . .	50
Appendix A. Basic Table I .	51
Appendix B. Basic Data for Compositæ . . .	52

I. Introduction.

Any reasonable generalisation on possibly fundamental rules of specific and generic differentiation would appear to be worth consideration. Defining our main terms briefly, we may here state that the genera and species of quantitative evolution are the genera and species recognised as such by responsible systematists who have considered entire groups in detail: the relatively few "critical" genera do not yield comparable data.

In relation to Compositæ, a compact group and the largest family of flowering plants, the general lines of evolution were suggested (Small, 1919) after a discussion of all the then known data, including the facts of structure, physiology, cytology, geographical distribution as shown by maps, and fossil distribution in time. This phylogenetic scheme (fig 1) gave a basis upon which the working of the Age and Area Law (Willis, 1915; *ibid.*, 1922) could be examined. In order to apply the Age and Area Law it was necessary to extend Willis's restricted hypothesis to absolute age and total area. This extension was first suggested by Sinnott (1917) and used by Small (1919, pp. 207, 214). It was adopted later by Willis (1922, p. 107). The examination of Compositæ was then made, and the results gave very clear mutual support to the previous phylogenetic conclusions and to the Age and Area hypothesis (Willis, 1922, table ii, pp. 127, 107–108).

A further hypothesis was put forward by Willis that "on the whole the area occupied by a genus (taking a great many; say ten allies at

least) varies in the same sense as the number of species it contains"; and this was developed to a certain extent as "Size and Space" (Willis, 1922, chap. xii). This hypothesis was also examined in relation to Compositæ (Small in Willis, 1922, p. 132) and found to support the previous conclusions.

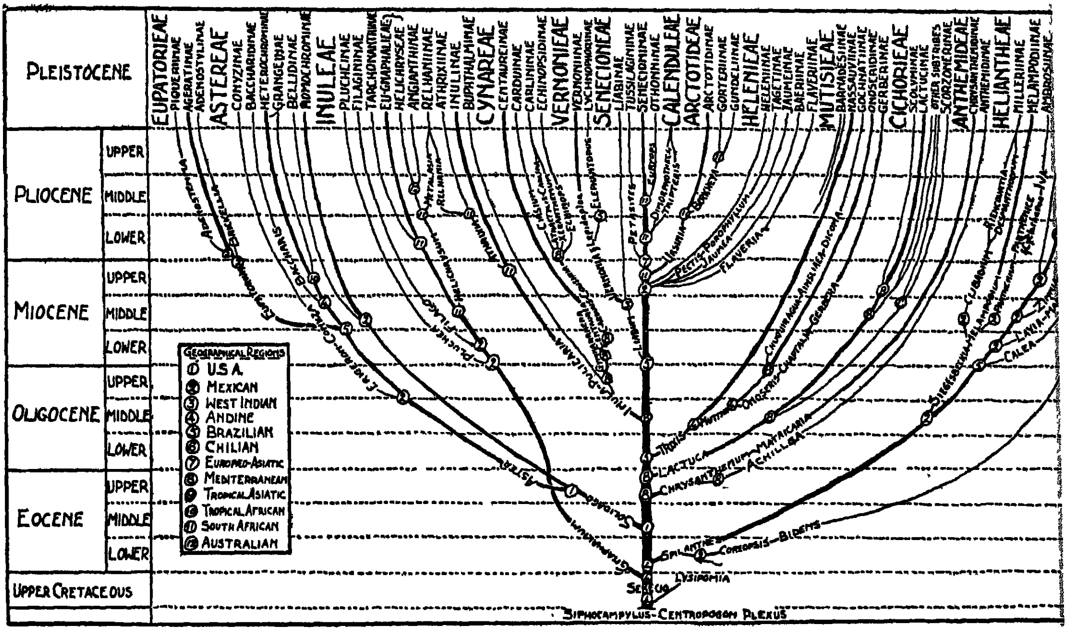


FIG. 1.—The "family tree" of the Compositæ, showing the evolution of chief sub-divisions and genera in time and space.

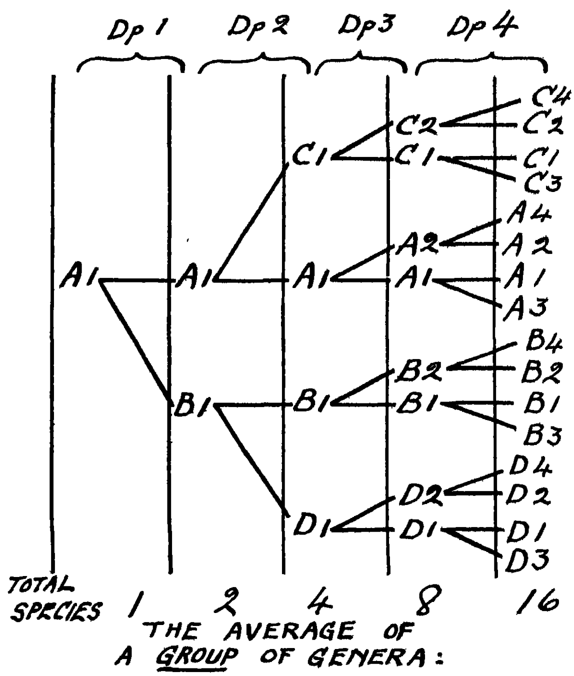
Since then another fact has been noted. If one multiplies all the figures for average generic area for the five geological periods concerned (fig. 1) by 2·5 the result is a remarkable similarity between these and the average number of species per genus for the same periods, except in the case of the Pliocene Calenduleæ which is a small group of eight genera with 114 species. The actual figures for the four older periods are 2·46, 2·39, 2·17, 2·69, or $2{\cdot}43 \pm 0{\cdot}26$. This means an average of two and a half species for every million square miles of average distribution (Table I). It would, therefore, appear that there is some justification for the statement by Small (Willis, 1922, chap. xiii) that "in the Compositæ on the whole both the average generic area and the average number of species per genus are closely related to absolute age."

Willis connected Age with Area, and Area with Size; Small in the above-mentioned passage clearly connected absolute age with size or average number of species per genus. Udny Yule (1924) then developed

TABLE I.

Time.	Tribes.	Average Generic Area.	Average Spp. per Genus.	Average Area × 2·5.	Average Dp-ages from Table VI.
Pliocene	Calenduleæ	3·6	14·2	9·0	2·432 (2·66)
	Arctotideæ Helenieæ				
Miocene	Vernonieæ Cynareæ Eupatorieæ	4·3	10·6	10·7	3·58 (4·192)
Oligocene	Inuleæ (ltd.) Mutisieæ	4·6	11·0	11·5	5·06
Eocene	Cichorieæ Anthemideæ Astereæ Heliantheæ	5·9	12·8	14·7	5·46
Earlier	Gnaphalieæ Senecioneæ	7·2	19·2	18·0	5·96

TABLE II.



this Age and Size into a mathematical theory of evolution, and elaborated formulæ by means of which the frequencies of the various sizes of genera after a given number of doubling periods could be calculated. This mathematical theory depends essentially upon the throwing of new species by mutation, which makes it possible to apply the laws of chance to the increase in number of species.

Since the whole matter is viewed statistically the behaviour of species and genera must be considered as an average behaviour within a group and not as the actual behaviour of any one species or genus. Keeping this "average" point of view in mind, Yule's view of what happens in general may be briefly summarised firstly as a simple diagram (Table II) and then in the form of more accurate equations. Each species is assumed to continue while derivative species arise by mutation; thus in successive doubling periods the species number rises to 2, 4, 8, 16, etc., provided all the species mutate. From this it will be seen that a group of genera, where the average number of species per genus is sixteen, is likely to be at least four doubling-periods in age. The initial history of the older larger genera would, of course, extend backwards as a less certain quantity for a few more doubling periods (Table II). In order to allow for species which do not throw new species and for species which throw new genera instead of new species, Yule develops the scheme in detail. For the details readers are referred to Yule's original memoir. For present purposes it is sufficient to note that Yule derived formulæ, from the observed frequency distributions, which were used to draw up certain tables. From these tables, by interpolation, the Dp-age or age in doubling periods can be determined provided that M, the average number of species per genus, and $f1$, the proportion of monotypic genera, are both known.

The basic equations for our work are

$$(a) \quad f1 = \frac{1}{1+\rho} + \frac{\rho}{\rho+1} \cdot e^{-\alpha\left(\frac{1+\rho}{\rho}\right)\tau},$$

$$(b) \quad M_{s\tau} = \frac{1}{1-\rho} \cdot \left(\rho \cdot e^{\alpha\left(1-\frac{1}{\rho}\right)\tau} - 1\right),$$

where $\rho = s/g$, s being a constant proportional to the chance of a specific mutation and g the same for a generic mutation; τ being the age in doubling periods; and $e^{\alpha} = 2$, $e^{-\alpha} = \frac{1}{2}$.

The tables give the values of M and $f1$ which correspond to all values of ρ and τ within the normal range. Thus with $f1$ and M known, ρ and τ can be obtained by interpolation. "Dp-age" is a convenient term used in this work for τ values or age in doubling periods.

The essential mathematical calculations were made by K. M. Bourke, K. I. Small, and I. K. Johnston.

II. Yule's Formulæ and the Goodness of Fit.

In order to calculate the distribution of genera of various sizes Yule begins with the present proportion of monotypes ($f1$), the present mean number of species per genus (M), and the present total number of genera in the group. From these data he calculates the time τ and the ratio ρ, and determines the theoretical "frequency distribution" which can then be tested against the actual "frequency distribution." This is one point upon which direct comparison between theory and fact can be used with any group of genera.

In order to find ρ and τ for any given data, Udny Yule had a table drawn up giving all the values of $f1$ (proportion of monotypes) and M (mean numbers of species per genus) which correspond to values of ρ from 1·0 to 3·0, and of τ from 1 to 10. The tables are essential for this work and they are given in Yule's memoir (1924). When the application to Compositæ was begun it was found that Yule's range of values for ρ (1·0 to 3·0) did not cover all the possibilities in Compositæ. The tables were therefore extended by my sister, Miss K. I. Small, for values of ρ from 3·1 to 4·0. This table (Appendix A) was essential for this work, and the writer is very much indebted to K. I. Small for her help. It is interesting to note that all values are reduced to a point at *one* species, s/g 0·1, but this does not concern the present contribution.

Although the main scheme was to determine the ages of the subdivisions of Compositæ in terms of Dp-ages and to revise the details of evolution within the family on this basis, it was considered necessary to check the agreement between theory and fact at the only possible point, namely, the "frequency distribution" of genera of various sizes The calculations of ρ and τ from the tables were originally made by Miss K. M. Bourke, to whom the writer is indebted for the original lists of Dp-ages for Compositæ. The testing of the goodness of fit of the frequency distributions, and the extensive reviewing of the basic data in connection with this, was done by Miss I. K. Johnston, to whom the completion of the investigation of the τ values is mainly due.

Goodness of Fit (I. K. Johnston).

The goodness of fit test involves the calculation of the numbers of genera with 1, 2, 3, 4, 5, etc., species and the comparison of those calculated numbers with the observed numbers. The values of ρ and τ

for the whole family is obtained by interpolation from the tables of $f1$ and M values. From these values of ρ and τ the frequency distribution of genera with 1, 2, 3, 4, 5, etc., species, *i.e.* $f1$, $f2$, $f3$, $f4$, $f5$, etc., is calculated when time is infinite and then the correction is applied to reduce it to finite time, since infinite time involves the neglecting of the primordial number of genera.

For the Compositæ the f ($\tau = \infty$) values were calculated up to $f65$ and the C^1 (correction) values up to C^1 12. In the case of grouped distributions the values of the series were added for the group and the correction applied collectively by adding the binomial coefficients of the corresponding corrections. We thus obtain a list showing the numbers of genera of each size, from one species to eight species, and upwards in groups such as 9–11, 12–14, 15–20, 21–30, 31–40, 41–50, 51–65, 66 or more; and this list can be checked against a corresponding list of the values as they actually occur. By the method of calculation the first (monotypic) group should agree very closely, and in actual practice the divergence in the first four groups is very slight, but in the higher groups divergences appear which are sometimes considerable. In order to test whether these divergences are large enough to prove that the basal assumptions are insufficient the χ^2 test of the statisticians was applied. It must be remembered that Yule's original assumptions neglect the phenomena involved in Harshberger's generic coefficient which is lower with more varied topography, and also Guppy's somewhat similar view of the differentiation of the organism in response to differentiation of climate. On the other hand, taking the whole of this world-wide family such climatic or topographical differentiation should show an average result.

The χ^2 test is applied by adding the square of divergences divided by the calculated number in each class or $\frac{x^2}{m}$, *i.e.* $\chi^2 = S\left(\frac{x^2}{m}\right)$. The number of the arbitrarily fixed classes (n) is in this case two less than the total number of classes because the number of monotypes is fixed by the observed number, and the last group "66 or more" is the residue which is fixed by subtracting the genera already classified from the total number of genera; therefore with these 16 groups $n = 14$. The form of distribution of χ^2 is known, and therefore it is possible by using tables to calculate the proportion (P) of cases in which any particular value of χ^2 will be exceeded. This P value corresponds to the probability that χ^2 shall exceed any specified value, and gives a general idea of whether the agreement is good enough to support the original hypothesis or so bad that it suggests that the hypothesis is not sufficient to account for

all the facts. Fisher (1930) gives the necessary tables and assesses the relative values of P thus: "If P is between 0·1 and 0·9 there is certainly no reason to suspect the hypothesis tested. If it is below 0·02 it is strongly indicated that the hypothesis fails to account for the whole of the facts. We shall not often be astray if we draw a conventional line at ·05 and consider that higher values of χ^2, *i.e.* lower values of P, indicate a real discrepancy."

Basic Data of Genera and Species (I. K. Johnston).

Miss K. M. Bourke in her work on the problem used as her basal numbers 780 genera and 9846 species, with 255 monotypic genera; these figures give $\rho=2{\cdot}089$ and $\tau=5{\cdot}37$. The corresponding frequency distribution was determined and the χ^2 test applied by Miss I. K. Johnston. With $n=14$, then $\chi^2=32{\cdot}85$ and $P<{\cdot}01$. The goodness of fit was not satisfied. The basal data in this case were modified from Bentham (*Genera Plantarum*) with uncertain numbers taken as average values, as by Small (Willis, 1922, pp. 132–133).

Bentham's figures (*Genera Plantarum*) were used again with a rule for genera with uncertain numbers of species, *e.g.* genera with 2–3 species were divided, half becoming ditypes and the other half tritypes. The figures were 768 genera, 10,450 species, with 250 monotypic genera, giving $\rho=2{\cdot}09$ and $\tau=5{\cdot}559$. The result of the test in this case was $n=14$, $\chi^2=37{\cdot}79$, and P again less than ·01, giving a worse fit than Bourke's figures.

In these cases curves were also plotted on a large scale and the observed curve varied fairly closely around the smoother calculated curve, but it was observed that a constant divergence occurred which brought the line of observed numbers below the line of calculated numbers in the region of genera with 21–30 or more species; as an example the figures for Bentham with uncertain numbers divided as described may be taken (Table III). Here it will be seen that while the significant divergences (with $\frac{x^2}{m}$ more than 0·5) are all negative for the small genera, they are uniformly positive for the genera with 21–30 or more species. In this case there are fewer large genera and more small genera than would be expected on Yule's assumptions.

Thus the χ^2 test apparently showed that Udny Yule's formulæ for frequency distribution could not be applied to the Compositæ as arranged by Bentham, but the regularity of the divergence stimulated further inquiry.

There are in Compositæ three genera with unusually large numbers

of species, *Senecio*, *Eupatorium*, and *Centaurea*. It was therefore decided that for the goodness of fit test these genera might be broken up, and their sub-genera or other smaller sections given temporary generic rank. *Eupatorium* was split into 8 sections, following Hoffman's sub-genera; *Centaurea* into 41 sections after the same authority, with

TABLE III.

Bentham. *Genera Plantarum*. Uncertain numbers distributed. I. K. J.

Spp./Gen.	Calc.	Obs.	Div.	x^2/m.
1	250·2	250	+ 0·2	·000
2	102·0	102	0	·000
3	59·3	59	+ 0·3	·002
4	40·2	41	− 0·8	·016
5	29·8	41	− 11·2	4·209
6	23·3	31	− 7·7	2·544
7	19·0	21	− 2·0	·210
8	15·9	25	− 9·1	5·208
9–11	35·9	34	+ 1·9	·100
12–14	25·4	38	− 12·6	6·250
15–20	35·1	41	− 5·9	·992
21–30	36·5	26	+ 10·5	3·021
31–40	23·5	16	+ 7·5	2·394
41–50	16·6	6	+ 10·6	6·767
51–65	17·2	11	+ 6·2	2·235
66 up	38·1	26	+ 12·1	3·843

$\chi^2 = 37·791$

$n = 14$, $P = < ·01$

TABLE IV.

Bentham. *Genera Plantarum*. Three large genera divided. I. K. J.

Spp./Gen.	Calc.	Obs.	Div.	x^2/m.
1	263·8	264	− 0·2	·000
2	110·4	110	+ 0·4	·001
3	65·7	64	+ 1·7	·044
4	45·5	47	− 1·5	·049
5	34·2	43	− 8·8	2·264
6	27·2	35	− 7·8	2·237
7	22·5	23	− 0·5	·011
8	19·1	29	− 9·9	5·131
9–11	44·0	38	+ 6·0	·818
12–14	32·0	40	− 8·0	2·000
15–20	45·1	53	− 7·9	1·383
21–30	47·4	40	+ 7·4	1·155
31–40	29·9	20	+ 9·9	3·278
41–50	20·2	10	+ 10·2	5·150
51–65	19·6	12	+ 7·6	2·946
66 up	31·4	30	+ 1·4	·062

$\chi^2 = 26·531$

$n = 14$, $P > ·02 < ·05$

Bentham's *Leuzea*, already calculated as a genus, reducing the number to 40; *Senecio* gave some difficulty, but *Kleinia* gave one sub-genus; *Emilia* and *Notonia* had already been calculated as genera after Bentham; and *Eu-senecio* was then divided into geographical and general habit sections, giving a total of 45 provisional genera. With this revision the basal data became 858 genera, 10,565 species, with 264 monotypes, then $\rho=2{\cdot}30$, $\tau=5{\cdot}09$, giving $n=14$, $\chi^2=26{\cdot}531$. From the tables when $n=14$ and $\chi^2=26{\cdot}873$, $P={\cdot}02$, so that with $\chi^2=26{\cdot}531$ P is slightly greater than ·02; and thus by breaking up these three abnormally large genera into more normal sections we can reach a point where it is no longer "strongly indicated that the hypothesis fails to account for the whole of the facts," although the P value is still below Fisher's conventional limit of ·05, and further real discrepancies may be suspected. This splitting is of course only an experiment, undertaken in order to discover if possible how the divergence from the expected distribution arises in the case of Compositæ.

Table IV gives these revised values. It will be noted that the larger proportional divergences are all negative for the smaller genera and still all positive for genera with 21–30 or more species. Although the three large genera have been split up, the total for "66" is increased by four and the calculated number reduced by about seven, giving the insignificant proportional divergence of ·062. Then if we add the integrals the symmetry of the divergences becomes significant; 11 positive for larger genera being nearly balanced by 12 negative for smaller genera.

Udny Yule, in dealing with the Chrysomelidæ (1924, p. 56), at first used a grouping similar to our Table IV, and then regrouped "according to the runs of sign of the differences from expectation." Taking this, therefore, as a recognised statistical method of smoothing the curve, we can regroup the data of Table IV as in Table V, where P is very nearly ·20 and therefore the χ^2 test indicates that the hypothesis is in reasonable accordance with the facts, although a supplementary factor may have been neglected, as will be seen later.

Thus for Compositæ we may say that Udny Yule's interpretation is in reasonable accordance with all the facts except the three abnormally large genera, *Senecio*, *Eupatorium*, and *Centaurea*, which fit into the mathematical scheme only when divided into sub-genera. As these genera all have more than 400 species, they are clearly marked for exceptional treatment, and division into sub-genera for statistical purposes would appear to be quite a reasonable method of treatment, especially when one is really considering the average behaviour within the family. The evolutionary history of these three genera has clearly been something

widely removed from the family average. In any case a regrouping of the original data with no division of any of the genera gives a reasonable χ^2 test with $P < \cdot 05 > \cdot 02$, so that the calculated frequency distribution of generic sizes in Compositæ cannot be said to be "not in accordance with the facts."

TABLE V.

Bentham. As in Table IV. Three genera divided. Regrouped. J. S.

Spp./Gen.	Calc.	Obs.	Div.	x^2/m.
1	263·8	264	− 0·2	·000
2	110·4	110	+ 0·4	·001
3	65·7	64	+ 1·7	·044
4	45·5	47	− 1·5	·049
5	34·2	43	− 8·8	2·264
6	27·2	35	− 7·8	2·237
7	22·5	23	− 0·5	·011
8–11	63·1	67	− 3·9	·241
12–40	154·4	153	+ 1·4	·013
41–50	20·2	10	+10·2	5·150
51–65	19·6	12	+ 7·6	2·946
66 up	31·4	30	+ 1·4	·062

$\chi^2 = 13{\cdot}018$

$n = 10$, $P > \cdot 20 < \cdot 30$ (very nearly ·20)

With $n = 10$, $P = \cdot 10$ for χ^2 15·987,
$P = \cdot 20$,, 13·442,
and $P = \cdot 30$,, 11·781.

Having thus reached a stage where the calculated distribution of generic sizes in Compositæ is shown to be in reasonable accordance with practically all the facts, we feel a certain confidence in proceeding to the consideration of Dp-ages which can be checked only by applying them to the previous phylogenetic scheme as given in fig. 1.

Original Monotypes.—Udny Yule (1924, p. 58) recognises, from the small variation in calculated Dp-ages, that time in relation to most groups must be considered as relatively short. "It may be as well to emphasise that our unit of time being a relative unit, its equivalent in years or in geological time will vary from group to group and can only be determined if the results can effectively and without fallacy be collated with the geological record." He then proceeds to consider the fact that "in every case the number of genera at zero time required to fit the data is very substantial." These "original genera" or N_0 vary from "5 per cent. of the existing number of genera for the Cerambycinæ" to "nearly 15 per cent. for the Lizards." Yule uses N_0 as the basis of a check upon the interpolation, because (given N_0) M and $f1$ can be recalculated from ρ and τ. This has been done for the Compositæ data given in Tables III and IV, with results as follows:—

Consequential Data

	from Table III.	from Table IV.
No. of genera	768	858
No. of species	10450	10565
M	13·61	12·31
$f1$	·3254	·3077
ρ	2·09	2·30
τ	5·559	5·09
N_0	121·12 or 15·8 per cent.	185·7 or 21·6 per cent.
M from ρ, τ	13·39	12·23
$f1$ from ρ, τ	·3258	·3075
χ^2	37·79	26·53
n	14	14
P	$<$ ·01	$>$ ·02

It will be seen that both values of N_0 for Compositæ are high. The percentage values of N_0 are controlled by the ratio of τ/ρ, and for Compositæ the value of ρ is higher than that for any of the four groups of animals given by Yule, while the τ value is not higher in proportion. There may be a general difference between plants and animals in this respect.

Yule's values for $\tau : \rho$ are as follows:—

6·28 : 1·925	giving	N_0 10·43	per cent.	for	Chrysomelidæ
4·980 : 1·188	,,	N_0 5·27	,,	,,	Cerambycinæ
4·260 : 1·253	,,	N_0 9·5	,,	,,	Snakes
4·281 : 1·496	,,	N_0 13·9	,,	,,	Lizards

Compare—

5·559 : 2·09	giving	N_0 15·8	per cent.	for	Compositæ
5·09 : 2·30	,,	N_0 21·6	,,	,,	,,

The large proportion of monotypes at zero time led Yule to consider various hypothetical cases (*op. cit.*, p. 59) such as the simple one of a cataclysm, *e.g.* a glacial epoch, killing off all except one species of each genus within an existing group, leaving evolution to begin again from a group of monotypic genera instead of from only one or a few genera; but he writes: "In fact, of course, we must expect matters to be far more complex even than this." He proceeds to consider "the action of a cataclysm of less than limiting severity." "It is evident that the distribution will remain of the same general form to the eye, with a maximum frequency for the monotypics, and I am inclined to suspect that it may be fairly closely of the same mathematical form." From these and other points it is clear that Yule recognises his analysis as being of simple or ideal conditions and as awaiting amendment or "reconstruction" when the facts are well enough known to admit of some simple modification (cp. *op. cit.*, p. 25, lines 9–12). The fact that

the proportion of original genera, as calculated from ρ and τ for Compositæ, is high does not, therefore, invalidate the original assumptions made by Yule, it indicates that some other factor or assumption has to be added to those already made. That this is correct will be seen in future developments of this work, since, as Yule says (*loc. cit.*): "It is only by a full development of the consequences that an effective test of the assumptions can be made."

III. The Dp-ages of Sub-tribes and Tribes.

The general phylogenetic investigation (Small, 1919) gives suggested points in time for the origin of tribes and sub-tribes. The only really fundamental criticism which has been published on this scheme was made by Berry (1924). He objected to the place and time of origin for the family thus: Small finds that "the great alliance of the Compositæ originated in the mountains of north-western South America at a time when there was neither mountains nor even land in that region, but seas."

Berry (1925) was still of the opinion that the Andes were below sea-level in late Cretaceous time, but Schuchert and Dunbar (1933, vol. ii, p. 354) refer to a great Andean geosyncline thus: "During the last half of Cretaceous time this geosyncline was folding and rising, now here and now there, into a great mountain chain that was completed at the end of the Mesozoic." Berry himself is not consistent, since in the following year (Berry and Singewald, 1926, p. 438) he writes: "Throughout most of its extent the Andean region has been above sea-level since before the close of the Upper Cretaceous." He also gives a diagram (*loc. cit.*, fig. 2) indicating a considerable rise of the Andean region above sea-level in late Cretaceous time. We are, therefore, quite justified in adhering to the original scheme as a basis for further examination.

The comparison, of the seriation of Dp-ages with the relative times of origin already suggested, involves the assumption that the actual unit-doubling period has nearly the same value throughout the family. The close correlation obtained for the majority of sub-tribes makes this assumption appear to be reasonable, while the divergences are grouped in an interesting geographical fashion which is considered in the next section.

Determination of Dp-ages.—Working with the basic tables for interpolation, the Dp-ages can be calculated by means of the methods given by Yule (1924). The data necessary are (1) the average number of species per genus in each group (M), and (2) the proportion of monotypic genera as a decimal of the total genera ($f1$). The determinations are

made by interpolation and are worked with seven-figure logarithms so that the second decimal figure may be reliable. Since the details of such work tends only to confuse and repel the normal biologist, readers are referred to Yule's original memoir if they desire such information. As a matter of fact, although the lengthy process of arithmetical interpolation was used for each of the figures in the present basic data (Appendix B), it has been found possible to graph the whole of the basic tables (Yule, 1924, and Appendix A) in sections on a large scale, and to obtain from the graph Dp-age values correct to the second decimal place in a fraction of the time necessary for the arithmetical interpolation.

There are two ways of regarding these Dp-ages. The values obtained can be treated as simple arithmetical characters of the sub-tribes and tribes which can be averaged arithmetically; or they can be treated further statistically and new values determined for any groupings which are suggested. Both methods have been used.

(*a*) *Relative Dp-ages of Sub-tribes within Tribes.*—Taking the Dp-age values obtained for the various sub-tribes of the tribes in the previously suggested chronological order (Small, 1919), we can analyse them as follows. The divergent values are annotated, and the original series are given as "order."

Senecioneæ.—This is the basal tribe and has four sub-tribes.

	Dp-age.	Order.	Time.
Senecioninæ . .	8·41	1	Upper Cretaceous.
Liabinæ . . .	3·89	2	Lower Miocene.
Tussilagininæ . .	2·65	3	Upper Miocene.
Othonninæ . . .	(5·20)	4	Middle Pliocene.

The South African Othonninæ is divergent, while the others are in series.

Gnaphalieæ (ltd.).—This more ancient part of the Inuleæ shows two series, one of which (B) is almost entirely South African and diverges from the normal values for the tribe while showing seriation within itself. The Gnaphaliinæ is best considered as two groups, Eu-gnaphalieæ and Helichryseæ (Small, 1919, pp. 210–211).

	Dp-age.	Series.	Order.	Time.
Eu-gnaphalieæ . .	4·27	A1	1	Paleocene.*
Plucheinæ . .	4·02	A2	2	Lower Miocene.
Filagininæ . .	2·84	A3	3	,, ,,
Tarchonanthinæ . .	2·17	A4	5	,, ,,
Helichryseæ . .	(6·18)	B1	4	Middle Miocene.
Relhaniinæ . .	(4·02)	B2	6	Middle Pliocene.
Angianthinæ . .	(3·55)	B3	7	,, ,,

* Paleocene comes between Cretaceous and Eocene.

Again the divergent values in brackets are those of South African groups.

Helianthеæ.—This large tribe shows three series—A1–6, B, and C1–3. The data for the last two sub-tribes are not sufficient for the determination of Dp-ages; the Lagasceinæ is one genus of 7 species; the Petrobiinæ three genera with 4 species (Appendix B).

	Dp-age.	Series.	Order.	Time.
Verbesininæ	4·301	A1	1	Lower Eocene.
Coreopsidinæ	4·134	B	2	,, ,,
Galinsoginæ	3·915	A2	4	Lower Miocene.
Madiinæ	3·692	A3	5	,, ,,
Melampodiinæ *	2·862	C1	3 *	Lower Miocene.*
Ambrosiinæ	2·822	C2	6 *	Middle Miocene.
Milleriinæ	2·33	C3	6 *	,, ,,
Zinniinæ	2·27	A4	6 *	,, ,,
Lagasceinæ	..	A5	7	Upper Miocene.
Petrobiinæ	< 1·0	A6	8	Lower Pliocene.

* As revised below in Section IV, *Siegesbeckia* in fig. 1 belongs to Verbesininæ, not to Melampodiinæ.

The seriation here is complete with the one exception in Melampodiinæ. This case is considered in Section IV, and the relative position of the tribe revised as shown. The Petrobiinæ data, M = 1·3 and $f1$ = ·66, do not occur even on the extended table; on the graphs the Dp-age is certainly less than 1·0; with such scanty data the actual Dp-age as calculated is not really reliable, but the indications are in the right direction.

Astereæ.—In this tribe there are six sub-tribes which, although in origin they form two series (fig. 1), fall into a simple sequence with one very decided exception.

	Dp-age.	Series.	Order.	Time.
Homochrominæ	5·363	A1	1	Middle Eocene.
Heterochrominæ	5·198	B1	2	Upper Eocene.
Conyzinæ	4·057	B2	3	Upper Oligocene.
Baccharidinæ	(11·111)	B3	4	Middle Miocene.
Bellidinæ	3·185	A2	5	,, ,,
Grangeinæ	1·652	B4	6	Upper Miocene.

The exceptional Baccharidinæ is a small group of three genera with about 280 species, of which all but six belong to *Baccharis*. This case is discussed below in Section IV.

Anthemideæ.—Here there are only two sub-tribes.

	Dp-age.	Order.	Time.
Chrysantheminæ	6·950	1	Upper Eocene.
Anthemidinæ	(8·218)	2	,, ,,

The seriation of Dp-age is not in accordance with the conclusions derived from the formal morphology; and the average generic areas of the two groups (6·2 for Chrysantheminæ and 3·7 for Anthemidinæ) support the previous conclusions rather than the Dp-age sequence. Since both sub-tribes are more or less equally old, and the Dp-age values are both rather higher than normal, the Yule method of age calculation is supported to some extent. It is, however, interesting to inquire into the source of these values, especially on account of the relative development of these two groups in South Africa (see Section IV).

Cichorieæ.—In the original scheme of phylogeny (fig. 1) the details of this tribe were not given, because of the admittedly artificial grouping of the genera both by Bentham (1873, p. 475) and by Hoffmann. The Scolyminæ as a single genus does not yield the necessary data; the Hieraciinæ with only three genera, none of which are monotypic, gives a zero *f*1 which again makes calculation impossible; the Dendroseridinæ with only two genera and eight species yields a Dp-age value which is not at all reliable. Hoffmann has three main groups: (1) Cichoriinæ with coroniform or scaly pappus; (2) Leontodontinæ with plumose pappus; and (3) Crepidinæ with setose pappus. The setose pappus is obviously primitive, and the derivative genera in (1) and (2) have probably arisen from various genera within (3); so that Hoffmann's first two groups are to that extent more artificial than Bentham's sub-tribes. Hoffmann's separation of *Taraxacum* from *Hypochæris* and *Leontodon* is a striking case of this artificiality; another is the separation of *Crepis* and *Picris*. In order to get related groups for our statistical purposes we may combine Bentham and Hoffmann so that related genera are at least in the same larger group, thus:

Sub-tribes.	Series.	New Groups.	Dp-age.	Time.
Lactucinæ	A1	Crepidinæ S.	6·04	Upper Eocene.
Crepidinæ	A2			
Hieraciinæ	A3			
Scorzonerinæ	B1	Leontodontinæ S.	5·87	Middle Oligocene.
Hypochæridinæ	B2			
Hyoseridinæ	C1	Cichoriinæ S.	3·03	Middle Miocene.
Lapsaninæ	C2			
Rhagadiolinæ	C3			
Scolyminæ	D	..	..	Upper Miocene.
Dendroseridinæ	E	..	(4·294)	,, ,,

With this natural grouping there is again a seriation of Dp-ages with the suggested times of origin.

Mutisieæ.—Here the Barnadesiinæ is a group of only two genera

with 11 species, not sufficient to give a reliable basis for calculation. The other four sub-tribes have Dp-ages as follows:—

	Dp-age. Bentham.	Dp-age. Hoffmann.	Series.	Order.	Time.
Nassauviinæ .	(5·700)	7·01	A1	1	Lower Oligocene.
Onoseridinæ .	7·648	5·90	A2	2	Middle Oligocene.
"Gerberinæ" .	6·019	6·92	B1	3	" "
Gochnatiinæ .	5·523	3·49	B2	4	Upper Oligocene.
Barnadesiinæ .	(6·488)	(7·94)	C	5	?

With Bentham's numbers A2, B1, B2 form a sequence, leaving the primitive Nassauviinæ with a Dp-age which is low for its position. With Hoffmann's numbers A1 has the highest Dp-age, but A2 and B1 both in the Middle Oligocene are reversed. There is therefore only one exception to the seriation in each case, with a possible reversal of sequence for A2 and B1 within the Middle Oligocene in the case of Hoffmann's data.

Inuleæ (ltd.).—This more recent part of the Inuleæ consists of three sub-tribes and part of the Relhaniinæ.

	Dp-age.	Order.	Time.
Inulinæ . .	6·599	1	Middle Oligocene.
Buphthalminæ .	(2·074)	2*a*	Upper Oligocene.
"Cynareæ" . .	"4·93"	2*b*	Lower Miocene.
Athrixiinæ . .	3·094	3	Upper Miocene.
Relhaniinæ . .	(4·02)	4	Middle Pliocene.

In this case the Relhaniinæ has to be considered (Section IV) in connection with other South African sub-tribes. The Buphthalminæ is a relatively small tribe which appears to have developed into the tribe Cynareæ, which is so closely connected that it may be taken with the Buphthalminæ in order to get a better approximation to the Dp-age of the two groups combined. Thus, although the seriation is not as striking as in previous tribes, the sequence can still be followed.

Cynareæ.—Only three of the four sub-tribes yield Dp-age values, the Echinopsidinæ having no monotypic genus.

	Dp-age.	Series.	Order.	Time.
Centaureinæ . .	6·17	A	1	Lower Miocene.
Carduinæ . .	(10·62)	B1	2	" "
Carlininæ . .	3·304	B2	3	Lower Pliocene.
Echinopsidinæ .	..	B3	4	" "

The exceptionally large value for the Carduinæ is discussed in Section IV; the other two Dp-age values show the usual parallel seriation.

Eupatorieæ.—The three sub-tribes here show a straight seriation of Dp-ages with time thus:

	Dp-age.	Order.	Time.
Ageratinæ . .	14·538	1	Lower Miocene.
Adenostylinæ .	3·498	2	Upper Miocene.
Piqueriinæ . .	3·226	3	Lower Pliocene.

The exceptionally high Dp-age for the Ageratinæ is discussed in Section IV.

Vernonieæ.—This includes only two sub-tribes, the Dp-ages of which show the usual seriation, but with high values (see Section IV).

	Dp-age.	Order.	Time.
Vernoniinæ . .	10·829	1	Middle Miocene.
Lychnophorinæ .	8·613	2	Lower Pliocene.

Helenieæ.—The five sub-tribes are given in fig. 1 as arising in the Upper Miocene in the order indicated by the "order" numbers below:

	Dp-age.	Series.	Order.	Time.
Tagetinæ . .	4·802	A1	1	Upper Miocene.
Flaveriinæ . .	3·999	A2	2′	,, ,,
Heleniinæ . .	3·04	B	2″	,, ,,
Bæriinæ . .	2·425	A3	3′	,, ,,
Jaumeinæ . .	1·617	A4	3″	,, ,,

Again there is a complete seriation of Dp-ages with the suggested order of origin. The value of s/g for the Flaveriinæ is below 1·0, and the Dp-age value is therefore unreliable, but in series.

Arctotideæ.—There are three sub-tribes but the Gundeliinæ has only two genera and three species, indicating a Dp-age below 1·0.

	Dp-age.	Order.	Time.
Arctotidinæ . .	4·908	1	Upper Miocene.
Gorteriinæ . .	4·503	2	Middle Pliocene.
Gundeliinæ . .	< 1·0	3	Upper Pliocene.

Again there is a parallel seriation of Dp-ages with the suggested order of origin.

Calenduleæ.—There are no sub-tribes here, but the Dp-age value for the tribe is 4·864, which is high for a Lower Pliocene origin. This tribe is predominantly South African, and is discussed with other groups of that region in Section IV.

Summarising the main points of this section (III*a*)—

1. The Dp-ages form series within the tribes, and these series in nearly all cases are graded in the same direction as the relative ages of the sub-tribes as indicated by other criteria. The Dp-age values

are therefore normally quite comparable for *sub-tribes within tribes*.

2. Some of the values are not in the normal seriation. Many of these are the Dp-ages of sub-tribes which are entirely or predominantly South African. A few others are exceptional although they are not South African. These are all discussed in Section IV.

(*b*) *Dp-ages of the Tribes*.—The values for the Dp-ages of the tribes, as calculated direct from the data, are given in Appendix B. They form no clear sequence and no obvious parallel seriation when compared with the suggested order of origin. This apparent disagreement has been one of the factors which have delayed these investigations over a period of ten years; but the very clear and detailed seriation of sub-tribes within tribes indicates that the tribal values do not fall into series because (1) the tribes are less closely related to one another than are the sub-tribes within the tribes, and (2) the Dp-ages of the tribes are average values for a series of sub-tribes of very different ages. For example, when all the sub-tribes are relatively old the tribe will have a high Dp-age value; but when most of the sub-tribes are relatively young the apparent Dp-age of the tribe will be less even if the basal sub-tribe of the tribe is very ancient. When, however, the exceptions to the sub-tribal seriation are considered two general corrections appear which, when applied to certain tribes, lead to a reasonable seriation in the tribal values also.

IV. Exceptions to Seriation of Dp-ages.

When the Dp-age values of the sub-tribes are arranged on the scheme given in fig. 1, it becomes clear that a number of recent sub-tribes, which are predominantly South African, show Dp-age values which are relatively high. As this is closely correlated with a South African habitat, it is possible to suppose that there is something peculiar to that region which either gives rise to a more rapid doubling of species or results in the species living much longer than they do elsewhere. Considering the diversified topography and the long-continued continental history of South Africa both suggestions are quite probable; combined they can scarcely be duplicated anywhere else in the world.

In order to get comparable Dp-ages for general consideration of the family, a correction may be introduced to compensate for this South African effect. The numerical value of this correction can be only approximate, and it is indicated by the Othonninæ which is the only exception to the seriation within the Senecioneæ. The minimum correction which will bring the Othonninæ into this series is $\times \cdot 5$, giving a Dp-age value of 2·60 as compared with 2·65 for the Tussilagininæ.

When this factor is adopted for South Africa it applies consistently to all the sub-tribes and tribes which are mainly South African. These include (1) the Helichryseæ series with the Relhaniinæ and Angianthinæ which are in series but acquire lower and more normal values for their times of origin, and give a complete seriation with time for the Gnaphalieæ and also for the Inuleæ (ltd.); (2) the Arctotideæ with three sub-tribes in series but now with Dp-age values more consistent with their suggested Pliocene origin; (3) the Calenduleæ which has no sub-tribes but which now shows a normal Pliocene Dp-age. See Appendix B, Basic Data, for the corrected seriation.

The results of the application of the above correction, consistently to all South African sub-tribes, leads us to consider another obvious exception to the seriation. The Astereæ show a complete series with the exception of the Baccharidinæ. This Dp-age value is rather unreliable in that it is derived for a small group of three genera with about 280 species, of which all but six belong to *Baccharis*, a peculiar genus with diœcious capitula and polymorphic in habit. Since the floral characters are so constant that Hoffmann's sections of this genus are based on well-marked vegetative characters, it would appear that the exceptional Dp-age for this sub-tribe may be due to an unusual instability in the main genus. If we consider *Baccharis* as showing autonomous instability and excessive mutation and take again the minimum factor which yields a normal Dp-age we arrive at $\times \cdot 33$ as a correction. This brings the Baccharidinæ into series according to the suggested times of origin in the Astereæ. The close similarity in the Brazilian habitat, multiplicity of species and exceptionally high Dp-age values in the Baccharidinæ, Ageratinæ, and Vernonieæ, all suggest the application of this Brazilian instability factor to the latter groups, and this is done in the table of Basic Data, Appendix B. The initial values for the Ageratinæ and Vernonieæ are in series, but are peculiar and obviously require special consideration for a comparative account of the whole family.

The application of these two factors leaves only two outstanding exceptions to the general seriation of sub-tribes within tribes. The Anthemideæ shows a reversed order in its two sub-tribes. Since both sub-tribes show a relatively high Dp-age and both are supposed to be ancient groups (Eocene) the method of calculation is upheld in a general sense, and the actual values appear to be affected by a special development of oligotypic genera in South Africa. The geographical distribution is such that neither of the correcting factors can be applied, and the initial values are left as exceptional in seriation but approximately correct in magnitude.

The other outstanding exception to the general seriation is the Carduinæ. This sub-tribal Dp-age is in series for the B1–3 series of sub-tribes in the Cynareæ, but it is higher than that of the apparently earlier Centaureinæ (A), although both sub-tribes are given (fig. 1) as arising in the Lower Miocene. The reversal in position of these two sub-tribes may be explained by the diphyletic origin suggested for the Cynareæ (Small, 1919, pp. 92, 115, 211, 303); the Centaureinæ coming from the Buphthalminæ and the Carduinæ with its two derivatives arising from the Plucheinæ. In that case the Carduinæ might antedate the Centaureinæ. The Dp-age value for the Carduinæ still remains high, and the origin of this abnormal value for a Miocene sub-tribe can be traced in the sizes of the seventeen genera which have species as follows: 150, 126, 60, 60, 40; 12, 7, 6, 6, 3; 1, 1, 1, 1, 1, 1, 1. There is thus a large proportion of large genera and a large proportion of monotypes with only five medium-sized genera. The throwing of monotypic genera and the abnormal specific differentiation within the large genera may be taken as exceptional instability, and the Dp-age corrected towards normal by a factor $\times \cdot 5$, as for South Africa.

The Melampodiinæ have already been given a revised position within the Heliantheæ. All the other sub-tribes of this large tribe show the usual seriation, and a review of the evidence discounting the fossil Oligocene Silphium-like leaves of Massalongo (Small, 1919, p. 246) would move the time of origin from Middle Oligocene to Lower Miocene. This also brings the average generic area and average generic size into series with those of the other sub-tribes; and indicates an origin as in fig. 1, with *Siegesbeckia* still in Verbesininæ.

The general study of Dp-age values for the sub-tribes within tribes yields so many examples of seriation of Dp-ages with order of origin that we are justified in the assumption of an approximately normal value for the Dp-age unit throughout the Compositæ, although it may vary to some extent from tribe to tribe. With series-sequences in such a large proportion of the sub-tribes the correction of abnormal values by the application of logically supported factors is the obvious preliminary to a comparative study of the whole.

The question may be raised of the treatment of *Senecio*, but this genus differs from all the other abnormal genera in being completely cosmopolitan, not mainly developed in one restricted region like those here considered. There is, further, no suggestion from any point of view that *Baccharis* or *Othonna* or Arctotideæ or Vernonieæ are really either old or primitive. The groups with Dp-age factors are all relatively advanced, young, and more or less unstable.

The result is an almost complete set of series-sequences of Dp-ages with order of origin. When it is considered that the basic data are merely the total number of genera and species, and the total number of monotypic genera in each group, and that the calculations throughout are confined within relatively narrow limits, as can be seen by inspecting the range of associated M and $f1$ values in the basic tables (Appendix A), this result certainly corroborates the agreement found in the goodness of fit test. We can say with increased confidence that the hypothesis is in general accordance with the facts and that there is still no reason to suspect the hypothesis which is being tested. The Yule formulæ and the Yule theory are supported in rather remarkable detail, and so are the main lines of evolution as previously suggested.

V. Dp-ages in Relation to Time.

Udny Yule (1924, p. 59) considers that, on account of the high values of N_0, "great caution will have to be used in interpreting doubling periods in terms of geological time." N_0 = the number of original genera, which should not be a large proportion of the whole. The need for caution becomes clear if we consider the short range of values for the Dp-ages as compared with the long range in time for the history of the family; but with Compositæ there is available a quite independent placing of the tribes throughout the Tertiary (fig. 1).

In all estimates of the lengths of the periods there is an overlap between the Lower Pliocene and Upper Miocene and, therefore, alternative positions for the origins of the younger tribes are given in Table VI. While the initial Dp-age values for the tribes vary erratically from youngest to oldest, there is a vague general increase in the values from top to bottom of column 2, Table VI. With the corrections applied as in column 3, Table VI, this gradual increase becomes more apparent. When, however, averages are taken for the periods or sub-divisions of periods the seriation becomes complete, whether we take the old positions of the younger tribes (column 4) or their revised positions (column 5).

The first case gives 2·43 Pliocene; 2·77 Upper Miocene; 3·59 Middle Miocene; and 4·39 Lower Miocene. With the revised positions, where the Helenieæ and Arctotideæ are given places in a *longer* Pliocene, the Pliocene average is 2·66, while Upper Miocene Vernonieæ is 3·59; Middle Miocene Eupatorieæ is 3·85; and Lower Miocene Cynareæ is 4·93; with a Miocene average of 4·192, as compared with the previous Miocene average of 3·58.

The average values of the Dp-ages of the tribes arising in each geological period show a complete seriation with time of origin.

TABLE VI.—TRIBAL DP-AGES.

Tribes.	Dp-ages.	*x* Factors.	Arith. Aver.	Revised.*	Time.
Calenduleæ	4·864	×·5 = 2·432	2·43 2·43	2·66	Lower Pliocene.
Arctotideæ	4·710	×·5 = 2·355	2·77		Upper Miocene. (Lower Pliocene).*
Helenieæ	3·201	3·201			" "
Vernonieæ	10·784	×·3 = 3·59	3·59 3·58	4·192	Upper Miocene.
Eupatorieæ	6·46	3·85†	4·39		Middle Miocene.
Cynareæ	6·89	4·93‡			Lower Miocene.
Inuleæ (ltd.)	3·987	3·987	5·06		Middle Oligocene.
Mutisieæ	6·144	6·144			Lower Oligocene.
Cichorieæ	5·899	5·899	5·46		Upper Eocene.
Anthemidieæ	7·036	7·036			" "
Astereæ	5·06	5·06			Middle Eocene.
Heliantheæ	3·86	3·86			Lower Eocene.
Gnaphalieæ (ltd.)	4·65	4·65	5·96		Paleocene.
Senecioneæ	7·263	7·263			(Basal) Upper Cretaceous.

* Revised tribal positions, see text, p. 46.
† Ageratinæ ×·33, and an arithmetical average taken of the sub-tribes.
‡ Carduinæ ×·5, and an arithmetical average taken of the sub-tribes.

Since the various sub-tribes arise within the tribes at periods which differ considerably (*a*) amongst themselves within each tribe and (*b*) from the times of origin of the containing tribes, it is necessary to examine the sub-tribal Dp-ages in relation to time. These values are tabulated in Table VII. The Dp-ages given are all arithmetical averages, except those within brackets in column 2, which are the values obtained by calculating the Dp-age of all the sub-tribes arising in each geological period. The differences between the arithmetical averages and the Dp-ages for grouped sub-tribes are so small that the arithmetical averages are used as being more readily calculated and discussed.

A brief inspection of Table VII, column 2, shows that the average sub-tribal Dp-ages are also in seriation with suggested age, although the actual values differ in every case from those for the average tribal Dp-ages in Table VI, column 4. Even the average sub-tribal Dp-ages for the sub-divisions of the Pliocene and Miocene show a complete seriation with time. The irregularities in the sub-divisions of the older periods are caused by a few abnormal values in a few sub-tribes, but they are smoothed out when period averages are taken. In these two older periods the values calculated for the grouped sub-tribes differ more reasonably, as 5·44–5·88, than do the simple arithmetical averages, as 5·37–5·39. The tribal averages for Oligocene, Eocene, Paleocene, as 5·06–5·46–5·96 vary more smoothly than the sub-tribal averages for these periods, which run 5·44–5·88–6·34 or 5·37–5·39–6·34.

TABLE VII.—SUB-TRIBAL DP-AGES.

	Averages.		
Gundeliinæ < 1·0.	Pliocene	< 1·0	Upper.
Angianthinæ (1·77), Relhaniinæ (2·01), Othonninæ (2·60), Gorteriinæ (2·75).	(2·57) 2·54	2·28	Middle.
Piqueriinæ 3·23, Tarchonanthinæ 2·17, (Echinopsidinæ?), Carlininæ 3·30, Lychnophorinæ [2·87], Calenduleæ (2·43), (Petrobiinæ < 1·0).		2·80	Lower.
Adenostylinæ 3·50, Grangeinæ 1·65, Athrixiinæ 3·09, Tussilagininæ 2·66, Arctotidinæ (2·45), Tagetinæ 4·80, Flaveriinæ 4·00, Heleniinæ 3·04, Bæriinæ 2·42, Jaumeinæ 1·62, (Dendroseridinæ 4·29?), (Scolyminæ?), (Lagasceinæ?).	Miocene	3·05	Upper.
Ageratinæ [4·85], Baccharidinæ [3·70], Bellidinæ 3·18, Helichryseæ (3·09), Vernoniinæ [3·61], Cichoriinæ 3·03, Ambrosiinæ 2·82, Milleriinæ 2·33, Zinniinæ 2·27.	(3·40) 3·45	3·21	Middle.
Plucheinæ 4·02, Filagininæ 2·84, Centaureinæ 6·17, Carduinæ (5·31), Liabinæ 3·89, Melampodiinæ 2·86, Madiinæ 3·69, Galinsoginæ 3·91.		4·08	Lower.
Conyzinæ 4·06, Buphthalminæ 2·07, Gochnatiinæ 5·52.	Oligocene	3·88	Upper.
Inulinæ 6·60, Onoseridinæ 7·65, Gerberinæ 6·02, Leontodontinæ 5·87.	(5·44) 5·37	6·53	Middle.
Nassauviinæ 5·70.		5·70	Lower.
Heterochrominæ 5·20, Chrysanthemidinæ 6·95, Anthemidinæ 8·22, Crepidinæ 6·04.	Eocene	6·60	Upper.
Homochrominæ 5·36.	(5·88) 5·39	5·36	Middle.
Verbesininæ 4·30, Coreopsidinæ 4·13.		4·21	Lower.
Eugnaphalieæ 4·27, Senecioninæ 8·41.	Basal	(6·34) 6·34	

( ) = × ·5 for South Africa or instability.
[ ] = × ·33 for Brazil and instability.
(Sub-tribe?) = omitted as uncalculated or uncertain.

The range of Dp-age values for the sub-tribes in each division of the various periods may be considered. The Upper Pliocene has only one sub-tribe at < 1·0; the significant Middle Pliocene values vary from 1·77–2·75; those of the Lower Pliocene from 2·17–3·30. The Miocene values vary thus: Upper 1·62–4·80, Middle 2·27–4·85, Lower 2·84–6·17. Thus for these two periods both the lower and the upper limits of the range become higher with time, with only one exception in Upper Miocene. The two exceptionally low values here are the Jaumeinæ which would be moved to the Pliocene with the rest of the Helenieæ in the revised position of that tribe, and the Grangeinæ which, as the youngest sub-tribe of the Astereæ, might well be moved up also; but these are retained in their originally suggested positions in the meantime because we are comparing the seriation of Dp-ages with the scheme given originally (fig. 1).

In the older periods the range is for Oligocene—Upper 2·07–5·52, Middle 5·87–7·65, Lower 5·70 only; for Eocene—Upper 5·20–8·22, Middle 5·36 only, Lower 4·13–4·30. Hoffmann's data give 7·01 instead of 5·70 for Lower Oligocene.

The Upper Oligocene is low because of the abnormally low Buphthalminæ, which should be taken together with the derivative Centaureinæ which has an exceptionally high value for the Lower Miocene. The Middle Oligocene is high on account of the exceptional Onoseridinæ which with Hoffmann's data is 5·90 instead of 7·65; similarly Hoffmann's data for the Nassauviinæ raise that Dp-age from 5·70–7·01 and give a complete seriation for the divisions of the Oligocene.

The Eocene values show a reversed seriation, but they depend on very few sub-tribes, including two both from the Heliantheæ in the Lower Eocene. There seems to be little doubt that the Heliantheæ as a tribe is much more advanced and younger than would appear in the original scheme. This will be discussed in a later contribution, but the floral morphology, the geographical distribution, and all the other data, except Berry's fossil fruit (Small, 1919, p. 246), all point in the same direction. The Upper Eocene has all high values, including both the abnormally high sub-tribes of the Anthemideæ.

A detailed examination of Table VII demonstrates clearly that, with a few exceptions in the sub-divisions of the two older periods, the average sub-tribal Dp-ages vary directly with suggested age taking each of the periods in three sub-divisions, and that they show complete seriation with time when averages are taken for the five periods concerned. The Senecioninæ, which is taken as the oldest sub-tribe, shows the highest Dp-age, 8·41.

Conclusion and Summary.

1. The frequency distribution of generic sizes in Compositæ agrees reasonably well with the calculated frequency distribution, using Udny Yule's formulæ. This agreement is very good for genera up to about 40 species in size, but not so good beyond that size. The reasons for this discrepancy will be made clear in a later contribution.

2. The ages in doubling-periods (Dp-ages) for the sub-tribes within tribes show a clear seriation with suggested times of origin.

3. The Dp-ages of both tribes and sub-tribes show a complete seriation with suggested times of origin when averages are taken for each of the geological periods concerned; and this seriation extends to the sub-divisions of both Pliocene and Miocene.

4. Although caution is advised in the interpretation of Dp-ages in terms of geological time, this seriation, in the only case which is available for testing the hypothesis in relation to plants, gives some hope of measuring evolutionary sequence in terms of time. This step has already been taken, but the present contribution deals only with the basic data which show the necessary seriation with suggested time of origin and a reasonable degree of agreement with the previous presentation of evolution within the Compositæ.

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APPENDIX A.—BASIC TABLE.

Values of ρ from 3·1 to 4·0, and Values of τ from 1 to 10.

s/g or ρ.		Time τ (or Dp-age).									
		1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
3·1	f1	·54620	·36477	·29224	·26322	·25163	·24699	·24514	·24440	·24410	·24397
	M	1·885	3·299	5·562	9·179	14·966	24·219	39·018	62·684	100·532	161·061
3·2	f1	·54486	·36160	·28782	·25812	·24616	·24134	·23940	·23862	·23831	·23818
	M	1·888	3·318	5·621	9·330	15·304	24·925	40·417	65·371	105·556	170·271
3·3	f1	·54358	·35861	·28364	·25326	·24095	·23556	·23394	·23312	·23278	·23265
	M	1·891	3·336	5·678	9·474	15·628	25·605	41·778	67·998	110·499	179·400
3·4	f1	·54238	·35577	·27967	·24864	·23699	·23083	·22872	·22786	·22751	·22737
	M	1·894	3·352	5·729	9·606	15·930	26·243	43·064	70·497	115·237	188·205
3·5	f1	·54124	·35307	·27589	·24424	·23125	·22587	·22372	·22284	·22247	·22233
	M	1·897	3·369	5·783	9·744	16·243	26·906	44·400	73·102	120·193	197·453
3·6	f1	·54016	·35051	·27229	·24003	·22673	·22124	·21898	·21805	·21766	·21750
	M	1·900	3·384	5·832	9·871	16·535	27·527	45·663	75·580	124·936	206·360
3·7	f1	·53914	·34807	·26886	·23602	·22241	·21676	·21442	·21345	·21305	·21288
	M	1·902	3·398	5·879	9·993	16·816	28·131	46·833	78·009	129·608	215·176
3·8	f1	·53817	·34575	·26559	·23219	·21827	·21247	·21006	·20905	·20863	·20846
	M	1·905	3·412	5·924	10·111	17·188	28·716	48·093	80·388	134·207	223·897
3·9	f1	·53724	·34354	·26246	·22852	·21455	·20836	·20587	·20483	·20440	·20421
	M	1·907	3·425	5·968	10·224	17·351	29·285	49·255	82·708	138·731	232·515
4·0	f1	·53616	·34126	·25936	·22494	·21048	·20440	·20185	·20078	·20033	·20014
	M	1·909	3·438	6·009	10·333	17·606	29·836	50·406	85·000	143·180	241·026

Calculated for the senior author by Miss K. I. Small (Mrs D. A. McCrea).

APPENDIX B.—TABLE OF BASIC DATA. I. K. J.

Tribes and Sub-tribes.	Spp. Gen.	Mon. Gen.	M.	f1.	ρ.*	τ.*	$\tau\times$ Factors.	Series.
Senecioneæ	$\frac{1407}{44}$	$\frac{13}{44}$	31·977	·29545	2·39	7·263		
1. Senecioninæ	$\frac{1194}{26}$	$\frac{8}{26}$	45·923	·30769	2·25	8·410		
2. Liabinæ	$\frac{50}{5}$	$\frac{1}{5}$	10·000	·20000	4·13	3·892		
3. Tussilagininæ	$\frac{24}{7}$	$\frac{3}{7}$	3·429	·42857	1·38	2·658		
4. Othonninæ	$\frac{139}{6}$	$\frac{1}{6}$	23·167	·16667	5·19	5·201	$\times$·5 = 2·60	
Gnaphalieæ (ltd.)	$\frac{970}{100}$	$\frac{31}{100}$	9·700	·31000	2·132	4·650		
1. Eu-gnaphalieæ	$\frac{215}{20}$	$\frac{5}{20}$	10·750	·25000	3·10	4·271		A1
2. Plucheinæ	$\frac{158}{16}$	$\frac{4}{16}$	9·875	·25000	3·46	4·016		A2
3. Filagininæ	$\frac{46}{11}$	$\frac{4}{11}$	4·182	·36364	1·80	2·839		A3
5. Tarchonanthinæ	$\frac{10}{3}$	$\frac{1}{3}$	3·333	·33333	2·10	2·171		A4
4. Helichryseæ	$\frac{392}{26}$	$\frac{9}{26}$	15·077	·34615	1·90	6·184	$\times$·5 = 3·09	B1
6. Relhaniinæ	$\frac{92}{14}$	$\frac{5}{14}$	6·571	·35714	1·82	4·016	$\times$·5 = 2·01	B2
7. Angianthinæ	$\frac{67}{10}$	$\frac{3}{10}$	6·700	·30000	2·44	3·547	$\times$·5 = 1·77	B3
Heliantheæ	$\frac{1140}{138}$	$\frac{37}{138}$	8·261	·26812	2·81	3·864		
1. Verbesininæ	$\frac{595}{57}$	$\frac{15}{57}$	10·439	·26316	2·81	4·301		A1
2. Coreopsidinæ	$\frac{207}{17}$	$\frac{3}{17}$	12·176	·17647	4·77	4·135		B
3. Galinsoginæ	$\frac{81}{7}$	$\frac{1}{7}$	11·571	·14286	6·23	3·915		A2
4. Madiinæ	$\frac{52}{7}$	$\frac{2}{7}$	7·429	·28571	2·63	3·692		A3
5. Melampodiinæ	$\frac{87}{20}$	$\frac{7}{20}$	4·350	·35000	1·92	2·862		C1
6. Ambrosiinæ	$\frac{40}{9}$	$\frac{3}{9}$	4·444	·33333	2·09	2·822		C2
7. Milleriinæ	$\frac{41}{11}$	$\frac{3}{11}$	3·727	·27273	2·77	2·33		C3
8. Zinniinæ	$\frac{26}{6}$	$\frac{1}{6}$	4·333	·16667	5·34	2·27		A4
9. Lagasceinæ	$\frac{7}{1}$	$\frac{0}{1}$	7·000	·00000	...	...		A5
10. Petrobiinæ	$\frac{4}{3}$	$\frac{2}{3}$	1·333	·66667	...	...		A6
Astereæ	$\frac{1420}{92}$	$\frac{23}{92}$	15·435	·25000	3·08	5·06		
1. Homochrominæ	$\frac{329}{27}$	$\frac{9}{27}$	12·185	·33333	2·03	5·363		A1
2. Heterochrominæ	$\frac{601}{36}$	$\frac{9}{36}$	16·694	·25000	3·07	5·198		B1
3. Conyzinæ	$\frac{109}{10}$	$\frac{2}{10}$	10·900	·20000	4·13	4·057		B2
4. Baccharidinæ	$\frac{281}{3}$	$\frac{1}{3}$	93·667	·33333	2·00	11·111	$\times$·3 = 3·70	B3
5. Bellidinæ	$\frac{82}{10}$	$\frac{1}{10}$	8·200	·10000	9·40	3·185		A2
6. Grangeinæ	$\frac{18}{6}$	$\frac{1}{6}$	3·000	·16667	5·38	1·652		B4
Anthemideæ	$\frac{775}{46}$	$\frac{17}{46}$	16·848	·36957	1·71	7·036		
1. Chrysantheminæ	$\frac{503}{30}$	$\frac{11}{30}$	16·767	·36667	1·73	6·950		
2. Anthemidinæ	$\frac{272}{15}$	$\frac{6}{15}$	18·133	·40000	1·50	8·218		

* $\rho = s/g$. τ = Dp-age.

APPENDIX B (*continued*).

Tribes and Sub-tribes.	Spp. Gen.	Mon. Gen.	M.	f1.	p.*	τ.*	$\tau\times$Factors.	Series.
Cichorieæ	878/56	18/56	15·679	·32143	2·13	5·89		
1. Lactucinæ . .	154/11	3/11	14·000	·27273	2·72	5·055		A1
3. Scorzonerinæ . .	186/10	3/10	18·600	·30000	2·35	6·034		B1
4. Scolyminæ . .	3/1	0/1	3·000	·00000	...	...		D
2. All other sub-tribes .	535/34	12/34	15·735	·35294	1·84	6·429		...
Crepidinæ . .	163/6	2/6	27·167	·33333	2·00	7·585		A2
Hypochœridinæ .	127/5	1/5	25·400	·20000	4·09	5·607		B2
Dendroseridinæ	8/2	1/2	4·000	·50000	1·01	4·294		E
Hyoseridinæ .	50/10	4/10	5·000	·40000	1·55	3·562		C1
Lapsaninæ . .	5/3	2/3	1·667	·66667	·52	1·602		C2
Rhagadiolinæ	11/5	2/5	2·200	·40000	1·61	1·393		C3
Hieraciinæ . .	171/3	0/3	57·000	·00000	...	...		A3
A1–A3 Crepidinæ S. .	488/20	5/20	24·400	·25000	3·05	6·04		
B1–B2 Leontodontinæ S.	313/15	4/15	20·867	·26667	2·79	5·87		
C1–C3 Cichoriinæ S. .	66/18	8/18	3·667	·44444	1·36	3·03		
D Scolyminæ .	3/1	0/1	...	...	...	...		
E Dendroseridinæ .	8/2	1/2	...	...	...	(4·294)		
Mutisieæ (Bentham) .	441/53	23/53	8·321	·43396	1·30	6·144		
1. Nassauviinæ . .	168/14	5/14	12·000	·35714	1·81	5·700		
2. Onoseridinæ . .	63/10	5/10	6·300	·50000	1·00	7·648		
3. Gerberinæ . .	101/10	4/10	10·100	·40000	1·50	6·019		
4. Gochnatiinæ . .	98/17	8/17	5·765	·47059	1·13	5·523		
? Barnadesiinæ . .	11/2	1/2	5·500	·50000	1·00	(6·488)		
Hoffmann numbers.								
1. Nassauviinæ . .	257/14	5/14	18·357	·35714	1·80	7·01		
2. Onoseridinæ . .	98/10	4/10	9·800	·40000	1·50	5·90		
3. Gerberinæ . .	129/10	4/10	12·900	·40000	1·50	6·92		
4. Gochnatiinæ . .	135/19	5/19	7·105	·26136	2·90	3·49		
? Barnadesiinæ. .	13/2	1/2	6·500	·50000	1·00	(7·94)		
Inuleæ (ltd.) . . .	305/56	23/56	5·446	·41071	1·47	3·987		
1. Inulinæ . .	129/19	9/19	6·789	·47368	1·11	6·599		
2. Buphthalminæ .	45/16	7/16	2·813	·43750	1·43	2·074		
3. Athrixiinæ . .	39/7	2/7	5·571	·28571	2·66	3·094		
4. Relhaniinæ . .	92/14	5/14	6·571	·35714	1·82	4·016	×·5 = 2·01	
4*a*. Heterogamous Relhaniinæ .	35/6	2/6	5·833	·33333	2·07	3·45	×·5 = 1·72	

$p = s/g$. τ = Dp-age.

APPENDIX B (*continued*).

Tribes and Sub-tribes.	Spp. Gen.	Mon. Gen.	M.	$f1$.	ρ.*	τ.*	$\tau\times$Factors.	Series.
Cynareæ	$\frac{992}{38}$	$\frac{11}{38}$	26·840	·29730	2·37	6·89	×† 4·93†	
1. Centaureinæ . .	$\frac{405}{11}$	$\frac{2}{11}$	36·818	·18182	4·60	6·17		A
2. Carduinæ . .	$\frac{474}{17}$	$\frac{7}{17}$	28·882	·41176	1·43	10·62	×·5=5·31	B1
3′. Carlininæ . .	$\frac{39}{8}$	$\frac{3}{8}$	4·875	·37500	1·70	3·304		B2
3″. Echinopsidinæ .	$\frac{75}{2}$	$\frac{0}{2}$	37·500	·00	...	...		B3
Eupatorieæ . . .	$\frac{720}{35}$	$\frac{11}{35}$	20·571	·31428	2·19	6·46	×‡ 3·857‡	
1. Ageratinæ . .	$\frac{588}{20}$	$\frac{9}{20}$	29·400	·45000	1·22	14·538	×·$\dot{3}$=4·846	
2. Adenostylinæ. .	$\frac{77}{8}$	$\frac{1}{8}$	9·625	·12500	7·18	3·498		
3. Piqueriinæ . .	$\frac{55}{7}$	$\frac{1}{7}$	7·857	·14286	6·24	3·226		
Vernonieæ . . .	$\frac{534}{40}$	$\frac{22}{40}$	13·350	·55000	·816	10·784	×·$\dot{3}$=3·595	K.M.B.
1. Vernoniinæ . .	$\frac{482}{29}$	$\frac{15}{29}$	16·622	·51726	·929	10·829	×·$\dot{3}$=3·609	,,
2. Lychnophorinæ .	$\frac{52}{11}$	$\frac{7}{11}$	4·727	·63636	·572	8·613	×·$\dot{3}$=2·871	,,
Helenieæ . . .	$\frac{314}{60}$	$\frac{20}{60}$	5·233	·33333	2·08	3·201		
1. Tagetinæ . .	$\frac{124}{14}$	$\frac{5}{14}$	8·857	·35714	1·83	4·802		A1
2′. Flaveriinæ . .	$\frac{9}{3}$	$\frac{2}{3}$	3·000	·66667	·50	3·999		A2
2″. Heleniinæ . .	$\frac{48}{7}$	$\frac{1}{7}$	6·857	·14286	6·27	3·04		B
3′. Bæriinæ . .	$\frac{120}{30}$	$\frac{9}{30}$	4·000	·30000	2·49	2·425		A3
3″. Jaumeinæ . .	$\frac{13}{6}$	$\frac{3}{6}$	2·167	·50000	1·07	1·617		A4
Arctotideæ . . .	$\frac{236}{17}$	$\frac{4}{17}$	13·882	·23529	3·31	4·710	×·5=2·355	
1. Arctotidinæ . .	$\frac{115}{8}$	$\frac{2}{8}$	14·375	·25000	3·08	4·908	×·5=2·454	
2. Gorteriinæ . .	$\frac{118}{7}$	$\frac{1}{7}$	16·857	·14286	6·21	4·503	×·5=2·251	
3. Gundeliinæ . .	$\frac{3}{2}$	$\frac{1}{2}$	1·500	·50000	...	...		
Calenduleæ . . .	$\frac{113}{8}$	$\frac{2}{8}$	14·125	·25000	3·08	4·864	×·5=2·432	
Compositæ as a whole .	$\frac{9846}{780}$	$\frac{255}{780}$	12·623	·32692	2·089	5·37		K.M.B.

* $\rho=s/g$. τ=Dp-age.
† Arithmetical average for sub-tribes with Carduinæ ×·5.
‡ Arithmetical average for sub-tribes with Ageratinæ ×·$\dot{3}$.

(*Issued separately January* 21, 1937.)

IV.—Studies in Clocks and Time-keeping: No. 6. The Arc Equation. By Professor R. A. Sampson, LL.D., F.R.S.

(MS. received October 17, 1936. Read December 7, 1936.)

THE isochronism of a pendulum is only true to a first approximation. If we make the arc large, the period increases, reaching infinity just before the pendulum makes complete circuits. Assuming the full value of the Circular Error, the actual calculation is given in No. 2 of this series (*Proc. Roy. Soc. Edin.*, vol. xxxviii, 1918, p. 169) for the first few degrees, where it is seen that even with variations that may occur in practice, the change of daily rate may easily amount to several seconds. The defect was recognized before regulation of pressure and temperature were considered feasible, as early, for instance, as in 1673, when Huygens published his *Horologium Oscillatorium*. In fact Huygens invented the cycloid to prevent it, and gave an ingenious device to make the pendulum describe true cycloids—a device that has been given up for various reasons, among others, that it involves disturbance of the pendulum which it is very desirable to avoid.

In spite of the amount being so large, the arc of precision clocks is now so constant that the effect is very difficult to ascertain. Several people have sought it. Bloxam about 1856, by an experiment with a pendulum, concluded that it was present at about half its full value. Recently E. C. Atkinson (*Proc. Phys. Soc.*, vol. xlviii, 1936, p. 606) shows remarkable results that make it double as much, namely, its full value without any diminution. These will be referred to later. The effect was often suspected at this observatory, but the first unmistakable case that occurred was in the going of *Shortt* No. 4. In 1930 this clock was troubled by a leak, and accordingly the arc fell as the pressure rose. At low pressures the arc depends very much more upon the barometer than at high pressures. As a mathematical problem, we have then three data—the barometer, the arc, and the rate, from which we can find theoretically the fall of arc as the barometer rises, and also the dependence of the rate upon barometer and arc. In their effect upon the rate the barometer and the arc are in conflict, a rise in the barometer producing a losing rate and a consequent fall in arc producing a gaining rate. It happens, also, that a falling barometer produces more and more effect in the arc, so that finally we have a balance of the two, and there exists a pressure where the

rate is a minimum (cf. *Monthly Notices R.A.S.*, Jackson and Bower, vol. xc, 1930, p. 268; Loomis, vol. xci, 1931, p. 569).

During the period mentioned, the barometer was read directly, and also the arc, once a day about 9.30 in the morning. These are marked in fig. 1. The rate was taken, at seven days' interval, from wireless comparisons, *i.e.* from the mean rate as determined at several observatories. It differs little from the rate as taken from the microchronograph comparisons of the four main clocks and from transit-circle determinations. It was adopted as being more impersonal.

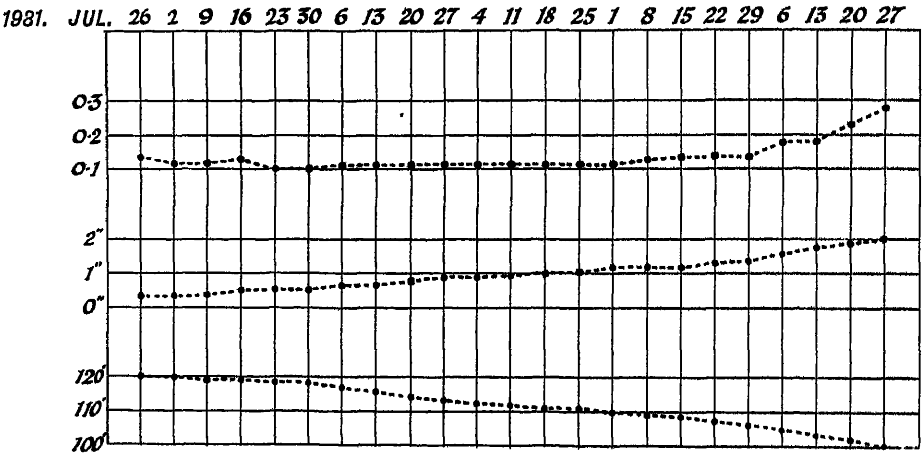

FIG. 1.
Abscissa: Date.
Ordinates: Rate, Barometer, Arc.

Using ten days' intervals, we get the following sixteen determinations, which should be completely reliable:—

Barometer = 1·00 in. + α; arc = 112′ + β.

	α. in.	β. ′	Rate. s.	Obsd. − Calcd. ′
1	−0·50	+ 8·4	+0·13	[+1·9]
2	− ·48	+ 7·4	+ ·12	[+1·1]
3	− ·45	+ 7·2	+ ·13	[+1·3]
4	− ·40	+ 6·1	+ ·10	[+0·8]
5	− ·38	+ 5·0	+ ·11	[+0·0]
6	− ·30	+ 3·9	+ ·11	[+0·0]
7	− ·17	+ 2·4	+ ·11	[+0·2]
8	− ·11	+ 1·1	+ ·11	[−0·4]
9	− ·02	0·0	+ ·12	[0·0]
10	+ ·04	− 1·0	+ ·12	[−0·5]
11	+ ·17	− 2·7	+ ·12	[−0·4]
12	+ ·28	− 3·9	+ ·14	[−0·2]
13	+ ·40	− 5·0	+ ·15	[+0·2]
14	+ ·53	− 6·2	+ ·17	[+0·7]
15	+ ·78	− 8·5	+ ·18	[+1·5]
16	+0·94	−10·5	+ ·25	[+1·8]

Solving these, first, to find the effect in arc of a change in the barometer, we get, from the whole group, by Least Squares (which is a very simple matter when there are only two variables and the coefficient of one of them is unity),

$$+1 \text{ inch} = -6'{\cdot}6 \text{ of semi-arc.}$$

The calculated comparison, shown in [ ], shows strong marks of a curvature; and, on inspecting the chart, there is sign that the barometer produces a greater effect on the arc at lower pressures, which is confirmed by taking the first six equations together and the last six together, when we find respectively

$$+1 \text{ inch} = -15' \quad \text{and} \quad +1 \text{ inch} = -5';$$

but these results are less determinative than what is given above, and are only made for information.

Several years ago I made many experiments—to be referred to later—on the clocks I then possessed or could borrow, to ascertain the constants of them. Naturally these differ according to the clock. I did not then possess *Shortt* No. 4; but the Synchronome Clock, which is described in No. 1 of this series (*Proc. Roy. Soc. Edin.*, vol. xxxviii, 1918, p. 75), seems sufficiently like it to justify its inclusion. It is an enclosed clock, made by the same makers, with practically the same pendulum and outside container. It is true it differs in the important matter of escapement, but it differs not radically, but only in being an earlier form; I then found

$$+1 \text{ inch} = -38'',$$

between 24 inches and 29·5 inches, from numerous experiments, which were mutually confirmatory, remarking that there appeared to be no sign of a progression (or curvature) as one proceeded. This is very different from the earlier statement, but is not in conflict with it. We have only to suppose that a fall of pressure produces a continually greater effect in the arc. The semi-arc cannot reduce by as much as 6′ or 5′ throughout the 30 inches of pressure, or otherwise, with a total arc of 112′, we should have negative values. Nor, we may add, can it become infinite at zero pressure. There is a certain limit to the energy fed in by the escapement, as indicated in No. 1, p. 97 (16*a*).

Assuming that the two data refer to the same clock, there is, naturally, considerable inconclusiveness, but they may be represented on one figure and may be united with a certain amount of probability. I now turn to a consideration of the rate. In the experience of this observatory, for fine work, a small change of the barometer cannot be counted to produce a proportionate change in rate. But one must assume that it does, and

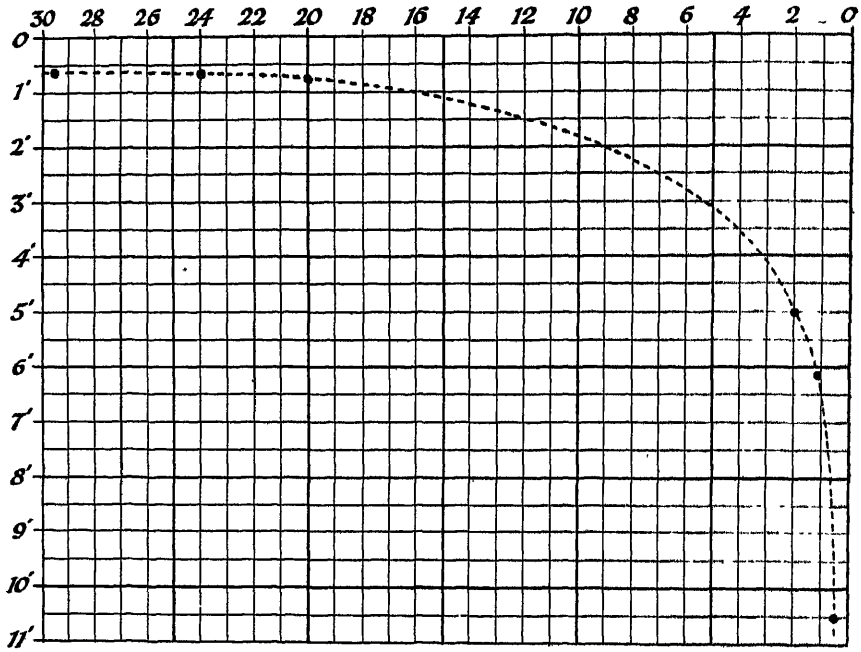

FIG. 2.—Barometer-Arc Equation.
Ordinates: Decrease of Semi-Arc in Minutes.
Abscissæ: Barometer in Inches.

equally one must assume a calculated effect in the Circular Error. Possibly any variations may be ascribed to accidental changes in the impulse or other circumstances. From this point of view they form a limit to the present-day performances of clocks, and have been considered in No. 4 of this series (*Proc. Roy. Soc. Edin.*, vol. xlviii, 1928, p. 161). But ascribing the rates recorded on p. 56 as regularly produced by the barometer and arc there recorded, and putting

$x = const.$,
y = value in seconds added to the daily rate for 1 inch added to the barometer,

and

z = proportion of the Circular Error that is actually found,

we have the following sixteen equations of condition:—

1	$x - \cdot 50y$	$+ \cdot 66z$	$= + \cdot 13$		9	$x - \cdot 02y$	$+ \cdot 43z$	$= + \cdot 12$
2	$- \cdot 48$	$+ \cdot 63$	$= + \cdot 12$		10	$+ \cdot 04$	$+ \cdot 41$	$= + \cdot 12$
3	$- \cdot 45$	$+ \cdot 62$	$= + \cdot 13$		11	$+ \cdot 17$	$+ \cdot 37$	$= + \cdot 12$
4	$- \cdot 40$	$+ \cdot 59$	$= + \cdot 10$		12	$+ \cdot 28$	$+ \cdot 34$	$= + \cdot 14$
5	$- \cdot 38$	$+ \cdot 56$	$= + \cdot 11$		13	$+ \cdot 40$	$+ \cdot 31$	$= + \cdot 15$
6	$- \cdot 30$	$+ \cdot 53$	$= + \cdot 11$		14	$+ \cdot 53$	$+ \cdot 28$	$= + \cdot 17$
7	$- \cdot 17$	$+ \cdot 50$	$= + \cdot 11$		15	$+ \cdot 78$	$+ \cdot 22$	$= + \cdot 18$
8	$- \cdot 11$	$+ \cdot 46$	$= + \cdot 11$		16	$+ \cdot 94$	$+ \cdot 17$	$= + \cdot 25$

These lead to the normal equations:

$$+16{\cdot}00x+0{\cdot}33y+7{\cdot}08z=+2{\cdot}17 \quad . \quad . \quad . \quad \text{(i)}$$
$$+\ 0{\cdot}33x+3{\cdot}15y-0{\cdot}89z=+0{\cdot}26 \quad . \quad . \quad . \quad \text{(ii)}$$
$$+\ 7{\cdot}08x-0{\cdot}89y+3{\cdot}47z=+0{\cdot}90 \quad . \quad . \quad . \quad \text{(iii)}$$

Eliminating x from (i), (ii), we have

$$y-{\cdot}3296z=+{\cdot}0685;$$

doing the same from (i), (iii), we have

$$y-{\cdot}3254z=+{\cdot}0579,$$

which are practically the same. It would be very uncertain to determine y and z separately from these equations, so we must confine ourselves to concluding, as the result of this investigation,

$$y-{\cdot}33z=+{\cdot}06.$$

This was an unexpected result, though perhaps it should not have been. From a preliminary inspection of the chart, I had concluded z to be about $\frac{3}{4}$. Indeed, on inspection of the chart, the effect seems very obvious. I felt that I must examine the matter further, so I divided the sixteen equations on p. 58 into two parts, consisting of the first eight equations, and the last eight, and derived the normal equations:

$$\text{I.}\quad \begin{aligned} 8{\cdot}00x-2{\cdot}79y+4{\cdot}55z&=+0{\cdot}92\\ -2{\cdot}74x+1{\cdot}11y-1{\cdot}66z&=-0{\cdot}33\\ +4{\cdot}55x-1{\cdot}66y+2{\cdot}62z&=+0{\cdot}53 \end{aligned}$$

and

$$\text{II.}\quad \begin{aligned} 8{\cdot}00x+3{\cdot}12y+2{\cdot}53z&=+1{\cdot}25\\ +3{\cdot}12x+2{\cdot}04y+0{\cdot}77z&=+0{\cdot}59\\ +2{\cdot}53x+0{\cdot}77y+0{\cdot}85z&=+0{\cdot}37 \end{aligned}$$

These, of course, are independent, and very different from one another; they also rest upon sufficient mean values to make it difficult to neglect them. But as to being determinate, it is obvious that the third of each set is nearly a repetition of the first. Putting it aside, then, and eliminating x from the first two equations of each set, we have

$$\text{from I:}\quad y-{\cdot}53z=-{\cdot}07,$$
$$\text{from II:}\quad y-{\cdot}26z=+{\cdot}12.$$

Comparing these with the (not independent) equation obtained above,

$$y-{\cdot}33z=+{\cdot}06,$$

we see that the value of the coefficient of z is not permanent, but is a mere expression of the increasing effect of the Circular Error in countering the barometric error as the pressure diminishes.

These are the only material for finding the effect of the barometer upon rate. If we substitute $z=1$, so that the full value of the circular error enters, we have, respectively,

$$\begin{aligned} &\text{from I:} \quad y = +^{s}{\cdot}46, \\ &\text{from II:} \quad\quad +^{s}{\cdot}38, \end{aligned}$$

and from the combined equation

$$+^{s}{\cdot}39.$$

These appear to me to be divergent; one must conclude that the barometric effect increases as the air becomes exhausted; on the other hand, if we take $z=0{\cdot}8$, so that four-fifths of the theoretical error is seen, we have respectively

$$+^{s}{\cdot}35, \qquad +^{s}{\cdot}33 \quad \text{and} \quad +^{s}{\cdot}33,$$

which are practically identical.

Calculation of the barometric effect is rather uncertain, and experiment with different pendulums shows that it differs largely, depending I suppose upon the shape of the pendulum, especially any sharp edges, and the quantity of "dead air" that it carries with it. One can only say that the last results are certainly credible.

When there is no air, I see no reason to suppose that the Circular Error should not produce its full theoretical effect; but it is a different matter when air is present. Possibly also it is dependent on the suspension spring. As to whether the Circular Error shows at its full amount, or by a fraction only, some considerations are given in No. 1, pp. 90, 94. I quote from the latter, which are fuller; take the equations

$$0 = x'' + \kappa x' + n^2 x + \lambda_1 x^3 + \lambda_2 x^2 x' + \lambda_3 x x'^2 + \lambda_4 x'^3 \ \ldots \qquad (8a)$$

and

$$0 = x'' + [\kappa + \tfrac{1}{4}\lambda_2 a^2 + \tfrac{3}{4}\lambda_4 n'^2 a^2]x' + [n^2 + \tfrac{3}{4}\lambda_1 a^2 + \tfrac{1}{4}\lambda_3 n'^2 a^2]x, \qquad (8b)$$

of which the second represents all the sensible effects of the motion, where $2a$ is the whole arc. Evidently the coefficients vary with the arc. Writing

$$\begin{aligned} \kappa + \Delta\kappa &\equiv \kappa + \tfrac{1}{4}\lambda_2 a^2 + \tfrac{3}{4}\lambda_4 n'^2 a^2, \\ (n+\Delta n)^2 &\equiv n^2 + \tfrac{3}{4}\lambda_1 a^2 + \tfrac{1}{4}\lambda_3 n'^2 a^2, \end{aligned}$$

and solving the equation as if a were constant, *i.e.* solving it approximately, we have

$$x = \mathrm{A} \exp\left(-\tfrac{1}{2}t\overline{\kappa+\Delta\kappa}\right) \sin\left(\overline{n+\Delta n}t + \gamma\right).$$

It is seen that observations of decrement of motion suffice to determine the coefficient of x', but the period must be observed as well, in order to get the coefficient of x. Also, it is not possible to separate λ_2 and λ_4, or λ_1 and λ_3 It is verified in No. 1, p. 94, that this equation leads to the usual value of the Circular Error, but this is only theoretical. To find out what actually happens, I made many experiments upon different pendulums,

with a view to ascertaining the quantities λ, removing any maintenance of the pendulum, and determining the time when a certain arc was realised. Though many of these are technically complete and serve to determine the value of the coefficient of x', I have not found it possible to determine the third coefficient, seeing that this involves the inference of many thousand swings of the pendulum, the period of which is changing as the arc diminishes. Hence my experiments throw little light upon the value of λ_1, λ_3, and the effect of the Circular Error upon the rate.

I now make some remarks on a paper by E. C. Atkinson above referred to (p. 55), in which he concludes from experiment that the Circular Error is present at its full theoretical value. The experiments are recorded in that paper on p. 609, Tables I, II. He determines the rate for different arcs, and then subtracts the "Impulse Deviation" and the "Circular Deviation," both being calculated theoretically, and finds that the residue is a constant. The pendulum, of course, is fitted with a maintenance, and the constants of the "Impulse Deviation"—which corresponds, I suppose, to what is usually called the Escapement Error—was ascertained experimentally with care, but he gives a reason why it might be systematically in error. All the observations took place nearly at pressure 21·3 inches. It seems to me it would be fairer to ascertain whether the impulse deviation and the circular deviation do appear in the observations at their full theoretical values, and this may be done by equating the observed rates, as recorded in Table I, to

$$x + by + cz,$$

where

$x =$ a constant,
$y =$ the fraction of the "impulse deviation" that actually is sensible,
$z =$ the fraction of the "circular deviation",

where we notice that z has the same meaning as before; and, from Table II,

$b =$ recorded impulse deviation,
$c =$,, circular deviation.

Combining the given observations in this manner, we have the normal equations

$$+\ 8{\cdot}00x + 19{\cdot}61y - 8{\cdot}44z = +\ 2{\cdot}37, \qquad (1)$$
$$+\ 19{\cdot}61x + 59{\cdot}42y - 26{\cdot}30z = +\ 11{\cdot}51, \qquad (2)$$
$$-\ 8{\cdot}44x - 26{\cdot}30y + 11{\cdot}75z = -\ 5{\cdot}33. \qquad (3)$$

Eliminating x, we have

from (1), (2) $\quad y - {\cdot}494z = +{\cdot}511,$
from (1), (3) $\quad y - {\cdot}508z = +{\cdot}504,$

which are almost the same.

Treated in this way, the observations are quite unequal to separating the two desiderata. Indeed what choice we may take makes little difference to the constant arrived at.

Thus, take

$$y = 1{\cdot}00;$$

then

$$z = 1{\cdot}00; \quad x = -1{\cdot}10,$$

which is Atkinson's solution; but if we take

$$y = 0{\cdot}90,$$

then

$$z = 0{\cdot}80; \quad x = -1{\cdot}07,$$

which is the value of z indicated on p. 60; but if

$$y = 0{\cdot}76,$$

then

$$z = 0{\cdot}50; \quad x = -1{\cdot}06,$$

which is the value Bloxam reached. I conclude therefore that Atkinson's data are insufficient to settle the question, but that his agreement is evidence that he has chosen one of a number of possible solutions.

We may approach the same question differently, by considering an ordinary run of the clock Riefler No. 258. The instance I took proved unsuccessful, as previous attempts had been. But I shall give it briefly, because readers may wish to know the evidence.

The clock Riefler is kept on mean time, as near as may be, and consequently its barometer is changed deliberately and repeatedly, always by small amounts. The arc also alters slightly, presumably owing to accidental variation of the impulse, but it must change too from the changes in the barometer; and if we take a long enough period, we may expect the effect of arc upon rate to declare itself. But we must not expect the comparison to be clean, for Riefler's "base rate" also alters, for no cause that I can assign.

The period chosen was from November 29, 1931, to July 2, 1932—217 days. During this period the temperature kept so constantly to 19°·0 C. that I have not thought that any correction would be an improvement. Time was adopted from the wireless comparisons. Sometimes not all the observatories sent, and occasionally the record of none of them was taken, but as a rule there were 6 determinations per week, giving 173 equations of condition. The equations were treated separately, because changes of barometer or arc might take place on any day of the week. They were combined by Least Squares, giving the Normal Equations :

$$\begin{aligned}
+173{\cdot}00x + 12{\cdot}26y - 1{\cdot}21z &= -5{\cdot}20,\\
+\ \ 12{\cdot}26x + \ \ 1{\cdot}20y - 0{\cdot}50z &= -0{\cdot}70,\\
-\ \ \ \ 1{\cdot}21x - \ \ 0{\cdot}05y + 0{\cdot}25z &= -1{\cdot}52,
\end{aligned}$$

where x, y, z have the same meanings as on p. 58, except that y now denotes the change of rate corresponding to 100 mm. in the barometer. These give on solution

$$y = -\cdot 188, \qquad z = -7\cdot 502,$$

which are certainly not right. It will be seen that the third equation gives a comparatively weak determination of z, but there is no gain in adopting z; if z were even put at zero, the sign given to y would be wrong, corresponding to a gaining rate for a rising barometer. Nor does adopting y produce any better effect upon z.

If we take an even longer run, we risk a change of what I call "base rate," and that, I think, enters to confuse this determination.

I leave the question in this unsatisfactory position. What is required is an adaptable clock, permitting a wide range of pressure variation, and at least two very different modes of suspension.

(*Issued separately January* 21, 1937.)

V.—Microphthalmia and Other Eye-defects throughout Fourteen Generations of Albino Rats.

By **A. M. Hain**, M.A., Ph.D., D.Sc., Carnegie Research Fellow, Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D. (With Chart and One Figure.)

(MS. received October 1, 1936. Read December 7, 1936.)

The albino rats used in this study were obtained originally from the Wistar Institute of Philadelphia and belong to a strain in which microphthalmia was first observed by Dr Helen Dean King in the F_4 generation extracted from a cross between an albino female of this stock and a wild grey Norway male. The defect occurred only in the "S" strain which formed a small proportion of the main stock. Its inheritance was observed by King (1931) through eleven generations, and 538 rats having either both or a single eye microphthalmic were obtained in a strain comprising 1884 individuals. The greater frequency of microphthalmia in the American stock is a marked feature, since only 154 rats with eye defects have been observed in the Edinburgh branch of Wistar rats in fourteen generations comprising over 5000 rats, and this total includes animals with eyes of normal size but possessing defects such as aniridia, cataract, etc., and also rare cases of anophthalmia and buthalmos. It is probable that 80 per cent. of the total affected were microphthalmics, and apparently the defect was generally more pronounced than in the parent stock, that is if the rat illustrated in King's monograph is to be taken as typical. According to King, the eye-defect could easily be detected at birth, "since a small eye did not protrude from the socket." In the total given here no still-borns are included, as it was found that this method of diagnosis, as also the opacity of the lens in such rats immediately or soon after birth, were unreliable criteria of abnormality. We are in agreement that prenatal mortality did not play an important rôle in determining the ratio of normal to abnormal individuals, because litters containing affected were of the average litter size and generally contained some normals (Table I). However, where a large number of still-born rats occurred in litters containing defectives, it is possible that the abnormality involved proved lethal, since still-births are not of common occurrence in this stock (Table II). The same may be true in those instances in which a litter consisted of only 2 or 3 rats all of which were

TABLE I.—SHOWING THE PROPORTION OF AFFECTED TO NORMALS IN LITTERS OF VARIOUS SIZES.

Total in Litter.	No. of Litters.	No. of Affected Rats.	No. Dead.
13	2	1, 5	0, 3
12	1	3	3
11	7	4, 1, 1, 1, 3, 1, 1	
10	6	1, 1, 1, 1, 3, 3	1
9	4	1, 4, 8, 2	3
8	5	1, 1, 2, 4, 4	
7	4	2, 4, 5, 1	
6	11	{ 1, 2, 1, 2, 2, 1, 4 1, 3, 1, 1	2
5	4	1, 1, 2, 1	
4	5	1, 1, 1, 4, 1	
3	8	1, 1, 3, 3, 3, 3, 2, 2	
2	1	1	

TABLE II.—ANALYSIS OF LITTERS IN WHICH DEFECTIVES OCCURRED AND IN WHICH THE INCIDENCE OF STILL-BIRTHS WAS HIGH.

No. of Rats Affected.	No. of Rats Dead.	No. of Rats Normal.
1	13	
3	10	2
1	10	2
3	7	
1	6	1
1	6	1
1	5	6
1	5	5
1	4	4
1	4	
2	4	3
2	4	
1	4	
1	3	
1	2	2
* 1	..	1
* 1		
* 1		

* It was presumed from the size of the mother late in pregnancy that more young were born but were eaten at birth.

defective, since such small litters are comparatively rare (Hain, 1934); six litters of this type were encountered: one of which consisted of 4 rats, four of 3, and one of 2. Moreover, 7 rats which had given birth to defectives subsequently had still-born litters, and absorption occurred 14 times.

Growth and Fertility.—It was found by King that "although affected rats grew rapidly and often attained a large size, in vitality and fertility they were definitely below the average of the stock from which they came, and showed a marked susceptibility to pneumonia." This was not observed in the Edinburgh stock. Occasionally the defective rat was smaller than others in the same litter as a youngster, but later the average of the litter was generally attained and sometimes surpassed. Table III, which gives the weights and ages of a number of affected rats at the time they were killed, illustrates this point. The largest rat ever obtained in the Wistar stock—a male weighing 520 g. when killed—was a microphthalmic, and another was almost three years old when killed. The number living to a great age would probably have been greater had it not been necessary to kill many when only 3 to 6 months old for the histological study of the eyes, which forms a separate investigation.

TABLE III.—SHOWING WEIGHTS AND AGES REACHED BY ♂♂ AND ♀♀ AT THE TIME THEY WERE KILLED.

Weight at Death.			Age at Death.		
Weight in Grams.	No. of ♂♂.	No. of ♀♀.	Age in Days.	No. of ♂♂.	No. of ♀♀.
>200	..	28	200–365 (*i.e.* 1 year)	23	11
>250	13	7	366–549 (*i.e.* 1½ years)	17	8
>300	26	2	550–730 (*i.e.* 2 years)	3	10
>400	13	..	731–912 (*i.e.* 2½ years)	5	8
>450	3				
>500	1	..	913–1095 (*i.e.* 3 years)	1	

N.B.—1 ♂ was aged 1076 days, 1 ♀ 807 days;
1 ♂ weighed 520 g., 1 ♀ 362 g.

There is, moreover, nothing to indicate that females with eye-defects were appreciably below the average in fertility. Of 50 ♀♀ which littered,

one gave birth to 65 and another to 48 living young (each had six litters); four had between 30 and 40 young, seven between 20 and 30, and twenty had between 10 and 20 young. The average was 7·6 per litter. A random selection of 18 ♂♂ indicates that their fertility was in no way impaired. One of these fathered 137 offspring (of which only 2 had eye-defects), three others were each responsible for 105 to 115 offspring, two for between 65 and 85 each, and the remainder for from 15 to 40 each.

Other Anomalies.—No other structural abnormalities were ever found in association with microphthalmia in the American stock of Wistar rats. In the Edinburgh stock renal and genital anomalies, including unilateral renal agenesia, have been observed in animals related to microphthalmics, but are of much rarer occurrence than microphthalmia (Hain and Robertson, 1936). That there is probably a connection between the two defects—microphthalmia and urogenital developmental anomalies—is suggested not only by the fact that the latter are transmitted, but also by their association in man (Collins, 1932). In addition, and for the same reasons, hydrocephaly may be considered as an associated anomaly. It was found in 15 rats, of which 12 were males; five of these were themselves anophthalmic, microphthalmic, or buthalmic; two were brothers of an anophthalmic female, six had microphthalmic sisters, and of six one or both parents were microphthalmic.

DATA.

Results of Selected Matings.—The defect first manifested itself in the British branch of the Wistar stock in 1932, and during the four years that have elapsed since its appearance, 154 rats suffering with eye-defects have been encountered in fourteen generations, *i.e.* out of some 5000 animals. In the pedigree chart (following p. 77) defective rats are indicated as solid squares (males) and solid circles (females). In order to show the relationship and antecedents of both parents, it has frequently been necessary to insert an animal twice. In such cases an asterisk refers the reader to the list attached to the pedigree chart.

In order to study the mode of inheritance of microphthalmia and kindred eye-defects, a certain procedure was adopted as to matings.

(*a*) In the first place, it was necessary to secure as many rats with eye-defects as possible. Consequently ♂ rat III, 8 (or 21), the first affected animal, and his brother ♂ III, 10 (or 18) were mated as often as possible to different females and especially to rats related to them. These two males are together responsible for 40 ♂♂ and 39 ♀♀ descendants, the greater number of which can be traced to the normal brother.

(*b*) Brother-sister matings were observed whenever possible.

(*c*) Backcrosses, *i.e.* father to daughter, or uncle to niece, were made on several occasions.

(*d*) Defective males and females were repeatedly crossed and frequent changes made.

(*e*) Defectives and recessives were outcrossed with a different strain of rat. As outcrosses, black and white rats of the "hooded" variety (Reading rats) were employed. Not only has microphthalmia not been reported in this variety * but their possession of black eyes made them eminently suitable for the study of a defect which might be associated with albinism only.

The outstanding feature of Table IV is the low incidence of affected obtained from **♂** × **♀** and the high proportion of the total obtained from ♂ × ♀—132 out of 154. It is impossible to estimate the total number of matings of ♂ × ♀ made, as such comprise by far the largest number in a stock which, during the four years under observation, has numbered more than 2000 breeding females.

TABLE IV.—THE OCCURRENCE OF EYE-DEFECT IN RELATION TO THE TYPE OF MATING.

Group.	Type of Mating.	No. of Litters Born to such Mating.	**♂♂**.	**♀♀**.	No. of Litters in which ● ● Present.	Total ● ●/Total oo in Same Litter.
1	**♂** × **♀**	65	3	5	4	8/506
2	**♂** × ♀	122 *	10	3	10	13/903
3	**♀** × ♂	26 †	..	1	1	1/169
4	♂ × ♀	?	72	60	64	132/?
		Total	85	69		

Chart numbers of affected in Groups 1, 2, and 3:

Group 1: VI, 3; VI, 8, 9, 10; VII, 21, 22, 23; XIII, 12.
Group 2: IV, 5; V, 10; VI, 11; VI, 46; VII, 3; IX, 37; IX, 41; IX, 47; X, 48 to 51; XI, 20.
Group 3: VI, 4.

* This figure includes (*a*) 22 matings with outcross females, which produced 74 ♂♂, 79 ♀♀; and (*b*) 21 backcrosses of F_1 ♀♀ from such matings with the defective father, which produced 99 ♂♂, 113 ♀♀, and 1 **♀**. Of the total, 46 matings were between a **♂** and his normal sister; 6 were backcrosses with a defective father.

† Including 9 litters from outcross males; of the remainder, 8 were matings between litter-mates and 3 were backcrosses.

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* "Not a single case has occurred in over 6000 rats (16 generations)."—Personal communication from Dr Kon.

An analysis of the 64 matings which form the last group is given in Table V. From this it is clear that in many cases either one parent or both were genotypically abnormal. In 18 cases either both parents or both grandparents had microphthalmia or other eye-defects, and in 6 others the defect is traced back three, four, or even five generations. In one of the latter—IX, 1 × 2—no microphthalmia occurred in three generations in direct ascent, but in the fourth generation back a ♂ mated with a Reading female (V, 1 × 2). It is of interest that its re-occurrence followed the mating of a ♀ with a Reading male (IX, 1 × 2). Two other normals which gave birth to defectives (XIII, 19, 20, and 21), and which were themselves brother and sister, had no microphthalmia in four generations of ancestors, but in the fifth generation of ascent two defectives (brother and sister—VII, 22 and 23) mated. All of the four generations consisted of brother × sister matings except in the third generation, when a ♀ was outcrossed to a Reading ♂.

TABLE V.—ANALYSIS OF ♂ × ♀ MATINGS OF GROUP 4 IN TABLE 4.

Affected Relations, etc.	of ♂.	of ♀.
Parents were ♂ × ♀	8	1
2nd (*a*) or 3rd (*b*) generation back were ♂ × ♀	(*a*) 7	(*a*) 2
	(*b*) 1	(*b*) 1
4th or 5th generation back were ♂ × ♀	1	3
No defectives in direct ascent	3	3
Mother was ♀	..	2
Father was ♂	3	4
Brothers and/or sisters were ● ●	20	19
Parent had ● ● in another litter	3	
Parent had ● ● litter-mates	..	5
Rat had ● ● in F_1 and F_2 to another mate	7	
No. of matings brother × sister	24	24
,, ,, ♀ × Reading ♂	..	5
,, ,, ♂ × ♀ of younger generation	9	

N.B.—In 6 cases defectives were born to sister of ♀ crossed to the same ♂: total affected born thus was 6 ♂♂, 2 ♀♀.

Hofmann (1912) is of the opinion that the eye-defect was transmitted chiefly through affected females, and states that matings of affected males with normal females produced very few abnormals. The figures given in Groups 2 and 3 in Table IV are contrary to this finding, as also those in Table V; for example, 13 defectives were born to ♂ × ♀, but only 1 to ♀ × ♂, and where ♂ × ♀ matings gave affected it was found that in 16 cases mating occurred between two defective ancestors of the *male* but only in 4 cases of the female. The occurrence of microphthalmia

or other eye-defect among litter-mates of the two parents was, however, shared equally by both parties.

The chief type of mating in Group 4 was between normal brother and sister, viz. 24, and it is remarkable that 47 affected (26 **♂♂**, 21 **♀♀**) should result from such matings when 49 matings between **♂♂** and normal sisters (Table IV, Group 2) produced only 5 **♂♂**, and 42 litters born to brothers and sisters, *both* of which were microphthalmic, contained only 3 **♂♂**, 2 **♀♀** distributed over three litters.

Two other types of mating are noteworthy, namely those, numbering nine, in which defectives occurred after mating between a ♂ and a ♀ of a younger generation not necessarily related; and those in which a normal ♀ crossed with a Reading ♂ produced affected. Of the five pairs in which the latter occurred (VI, 25 × 26; VI, 53 × 54; VII, 26 × 27; VIII, 41 × 42; IX, 1 × 2) two are of special interest, as in one case—VII, 26 × 27—the father of the ♀ also was a Reading ♂, and in the other —VIII, 41 × 42—both the father and the grandfather also were Reading rats.

As King found that the number of microphthalmic rats increased from 3·45 per cent. in the first generation to 43·16 per cent. in the ninth generation, an abstract was made of five generations in which brother × sister matings were observed throughout (Table VI). This shows no such increase as was found in the American stock.

TABLE VI.—SHOWING THE INCIDENCE OF MICROPHTHALMIA IN FOUR GENERATIONS OF BROTHER × SISTER MATINGS.

Type of Mating.	No. of Litters.	♂♂.	♀♀.	**♂♂**.	**♀♀**.	Total Affected to Total Normal.	No. of Affected Litters.
♂ × **♀**	65 *	240	266	3	5	8/506	4
F_1 ♂ × ♀	140	484	538	8	4	12/1022	6
F_2 ♂ × ♀	45	139	149	5	3	8/388	5
F_3 ♂ × ♀	22	84	85	1	..	1/169	1
F_4 ♂ × ♀	15	49	44	1	..	1/93	1

* 44 **♀♀** littered; only 11 **♀♀** of the total affected were either not crossed with **♂♂** or had still-born litters to such **♂♂**.

It has already been stated that rats with microphthalmia and other eye-defects were, on the whole, equal to the rest of the stock in vitality, fertility, and longevity. The female that gave birth to such defectives, however, was generally below the average fertility in that she rarely littered again after producing microphthalmics, or else had still-born young. Of 80 female rats which had litters containing defectives, 61

showed this inability to litter again, although the male was changed and mating occurred. The details given in Table VII of litters born to the remaining 19 females show that, within this group, frequently normal litters were born to the same parent as had previously produced microphthalmics. The data in the third column of Table VII, as also in Table VIII, when studied in conjunction with the pedigree chart, demonstrate the erratic nature of the occurrence of microphthalmia in view of its non-appearance after such matings as are described there. That the same parents sometimes produced normal litters not only after a defective litter but also (as is shown in Table VIII) before the birth of microphthalmics is not to be explained away on the grounds of prenatal mortality of the defective rats, as, with few exceptions, the litters born were of average size.

TABLE VII.—LITTERS BORN SUBSEQUENT TO AFFECTED LITTERS.

Female × Same Male.		Female × Reading Male.		Female × Related Male.		Female × Different Unrelated Male.	
♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.
V, 50	4/2	V, 24	2/1	IV, 33 *	10/6	V, 4	3/3
VI, 39	1 ♂/1 ♀	V, 12	3/1	VII, 10 †	1/5	V, 23	1/1
V, 37	−/1 ♀ −/1 ♀	V, 41	0/6	VII, 10 ‡	6/5		
		VI, 26	1/0 4/5	VII, 10 ‡	1/2		
V, 22	3/1						
VII, 20	3/2 5/3	VI, 2	4/6 4/3	VI, 2 §	3/6		
...		...		VIII, 15 ‖	5/1		
IX, 40	4/2 4/3	.	...	VII, 24 ¶	11/13		
			...	VII, 24 ‡	7/6		
IX, 29	0/1	IX, 29	3/3	...		IX, 29	5/8
XI, 18	1/1						
Total: 11 litters born subsequently to same female × *same male*, yet only 1 ♂, 2 ♀♀; *cf.* 8 ♂♂, 3 ♀♀ previously.		*Total:* 8 normal litters born subsequently to same female × *Reading* ♂.		*Total:* 11 normal litters born to same female × related males (see foot).		*Total:* 3 normal litters born to same female × different unrelated ♂.	

N.B.—The following had previous litters (see Table VIII):—

Column 1: IX, 40. Column 2: V, 24; VI, 26. Column 3: VI, 2; VIII, 15.

* ♂ Uncle. † ♂ Father. ‡ ♂ Son. § ♂ Son. ‖ Brother. ¶ ♂ Father (4 litters).

Reproductive Cycle of the Female Parent.—The rareness with which rats which had once given birth to microphthalmics again became pregnant suggested that such animals might possess or develop an abnormal reproductive cycle. Vaginal smears of 18 such females were taken daily for periods varying from 3 to 5 weeks commencing a few months after the

TABLE VIII.—NORMAL LITTERS BORN BEFORE DEFECTIVE LITTERS.

Female × Same Male as gave Defectives later.		Female × Brother or other Relation whereas Defectives born to Unrelated ♂.		Female × ♂ whereas ● ● born to Unrelated ♂.		Female × Different Brother ♂ from the one which gave ● ●		Others.	
♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.	♀.	Litter: ♂/♀.
VIII, 10	4/7	V, 24 * ♀	3/1	V, 6 ♀ × bro. ♂	7/3	V, 52	2/3	IV, 30 § (× unrel. ♂)	7/2
VIII, 15 × bro.	(i) 2/1 (ii) 8/4	VII, 26†	(i) 3/3 (ii) 7/6	V, 8 §	2/7	VI, 28	5/3	VI, 2 ‖ (× bro.)	6/1
IX, 28	(i) 2/4 (ii) 7/5	XI, 12	(i) 6/2 (ii) 5/4			IX, 44	3/3	VIII, 41 (i) × Read. ♂; (ii) × unrel. ♂;	(i) 7/6 (ii) 4/1
IX, 45 × bro.	(i) 4/5 (ii) 4/7	X, 5	(i) 7/2 (ii) 3/4						
IX, 40 × bro. ♂	5/3	XI, 5	3/2						
XI, 13	4/3	X, 14 ‡	4/3						
XI, 21 × bro.	3/5	XI, 18	3/2						
VIII, 30 × ♂	(i) 3/4 (ii) 4/7	VIII, 30 *	4/5						
		VI, 26 † (*N.B.*–6 still-born)	0/2						

N.B.—The following had subsequent litters (see Table VII):—
Column 1: VIII, 15; IX, 40. Column 2: V, 24; VI, 26. Column 5: VI, 2.

* Defective litter born to unrelated ♂ not ♂.
† " " " Reading ♂.
‡ × father.
§ Defectives born to cousin ♂.
‖ Defective born to father (*i.e.* backcross).

birth of the affected litter and before the rat herself was a year old. Only two rats had normal cycles; the remainder had cycles of one of two types: (*a*) long periods of diœstrus followed by occasional appearances of large epithelial cells about to lose their nucleus (*i.e.* the pro-œstrous smear), but an absence of the normal œstrous condition; 10 rats, + 2 which came in heat twice but otherwise belonged to this type, formed this group. (*b*) A persistence of the pro-œstrous type of smear with œstrus occurring rarely if at all (6 rats). The absence of a condition common to all female parents of defectives, as well as the lack of a rigid cycle in rats which are in all respects normal, makes one hesitate to associate microphthalmia with a maternal reproductive maladjustment, yet the data as to the subsequent infertility of the female parent of such animals are sufficiently striking to suggest that a physiological basis may be partly responsible for the condition.

TABLE IX.—SKULL MEASUREMENTS OF MICROPHTHALMIC AND NORMAL RATS COMPARED.

Rat.	Length of Cranium.	Zygomatic Width.	Length of Nasal Bone.	Fronto-occipital Length.	Squamosal Distance.	Height of Cranium.	Eyes.
♂♂	mm.	mm.	mm.	mm.	mm.	mm.	g.
VI, 33 (368 g.)	48·5	25	19·5	31	16·25	13	0·45
VI, 41 (281 g.)	45	24	18	29	16	12	0·1
VI, 38 (337 g.)	46·25	25	19	30·75	16	12·25	0·340
III, 8 and 21 (365 g.)	48·25	25	19·25	31	16·25	12·25	0·2
V, 2 (300 g.)	47·5	25	19·0	30·8	17	13	0·142
VI, 6 and 32 (300 g.)	46	23·5	17	31·5	15·5	12	0·2
VI, 30 (332 g.)	47·5	25	18·5	31·5	16	12	0·2
VI, 40 (397 g.)	48	25	18·5	32	16	12·5	0·280
VI, 46 (338 g.)	47	24	18·5	30·5	15·5	12	0·270
♂♂ (controls)							
IV, 15 (440 g.)	48·5	26	19·5	31	16	12·25	0·5
♂ 246 g.	44·5	23·75	17	29·75	16·5	11·5	0·340
♂ 310 g.	45	24	18	30	15·5	11·5	0·375
♂ 346 g.	47	25	18	33	16	12·5	0·380
♂ 383 g.	47·5	25	19·75	30	15·75	12·5	0·360
♀♀							
VI, 14 and 24 (264 g.)	44·25	22·25	17	29	15	11·5	0·150
VI, 9 and 22 (192 g.)	42·5	21·5	16·5	28	14	10·5	0·220
VI, 34 (185 g.)	42·5	21·5	16·5	28·25	15·5	12	0·087
♀♀ (controls)							
V, 43 (226 g.)	45·25	23·5	17	30	16	11·8	
♀ 181 g.	40·5	21	15	28	15·25	11·2	0·280
♀ 185 g.	42·25	21	16	28	15·25	11·5	0·285
♀ 197 g.	43	22	16·5	28·5	14·75	11·25	0·300
♀ 210 g.	43·5	22·5	16·2	28	15·8	11·4	0·300

Measurements of Skulls and Eyes.—Since one frequently obtained the impression that the skull of the microphthalmic rat was narrower and more tapering than that of the normal control, measurements were taken of the skulls of defective rats of both sexes and were compared with normals and with the data obtained by Hatai (1907) and reported by Donaldson (1915). The skulls were macerated, bleached, and dried at room temperature. All the rats were adult, and controls were selected

of approximately the same body-weight as the microphthalmics. The measurements were made with vernier calipers and followed those taken by Hatai, except in the determination of the fronto-occipital length. This has been measured from the tip of the frontal bone to the end of the occipital bone, without adjustments. The measurements given in Table IX reveal little difference between affected and control rats of either sex and accord with Hatai's data, indicating that the impression of narrowness was apparent and not real. The difference in the weights of the eyes as between microphthalmics and controls is, on the other hand, very marked (fifth column of Table IX). Two of the defective males (VI, 33 and VI, 38), although blind, had eyes of normal size; hence in these two rats the eyes are of normal weight.

It may be mentioned here that Dr Biggart of the Pathology Department, Edinburgh University, failed to find any abnormality in the pituitary-hypothalamic region of microphthalmic rats submitted to him for examination.

Cytological Study.—Eight ♂ rats (III, 8; V, 2; VI, 6; VI, 30; VI, 33; VI, 40; VI, 41; VI, 46) and ♂ IV, 15 were investigated cytologically; mitosis and meiosis were studied and the chromosome complements were analysed. No irregular chromosome behaviour or structure

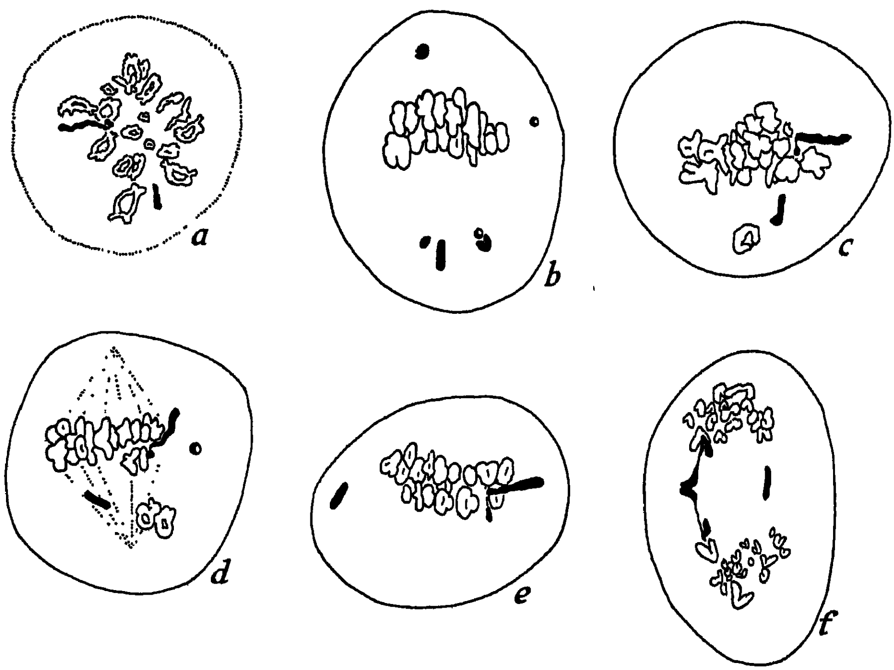


Fig. 1.—Chromosome Pictures of ♂ Rat V, 2.
(Drawn by Dr P. C. Koller.)

was encountered in the animals under observation, except in ♂ V, 2, which was microphthalmic and epileptic. In this individual occasionally an unpaired chromosome was seen during meiotic prophase (fig. 1, *a*), metaphase (fig. 1, *c*, *d*, *e*), and anaphase (fig. 1, *f*). In some instances one smaller and another longer chromosome were observed lying off the equatorial plate (fig. 1, *b*). Such behaviour strongly suggests that in rat V, 2 one chromosome pair is heterozygous for a structural change (translocation, inversion, deletion, or duplication), and it is this which is responsible for the "univalent" and fragment chromosome.

In the other individuals the chromosome complement and chromosome behaviour are normal, indicating that microphthalmia is a genetic abnormality caused by genetic factors rather than brought about by structural changes in the chromosomes themselves. The structurally abnormal chromosome in ♂ V, 2 may have arisen independently and have no causal connection with the morphological abnormality, though it may be responsible for the exaggeration of microphthalmia with epilepsy.

Discussion.

From the preceding data it is manifest that microphthalmia in rats was not associated exclusively with albinism, as more than a dozen affected rats were black and white animals and exhibited microphthalmos, buthalmos, and anophthalmos.

It is also clear that, since it was impossible to predict the character of the progeny from any given type of mating, the mode of its inheritance cannot be a straightforward one. Although the data cannot be taken to show definitely the existence of a genetic factor or factors responsible for the abnormality, they are consistent with the possibility of a dominant factor with a poor expression which is conditioned by genetic and physiological modifiers. The fact that the defect appears in outcrosses suggests that a dominant is involved.

Summary.

1. The incidence of microphthalmia and other eye-defects has been examined throughout fourteen generations of Wistar albino rats, and the results of various types of mating have been given; defectives are rarely born to defective parents.

2. The vitality, growth, and fertility of affected rats is normal, but a tendency to sterility in the female parent of such animals is noted.

3. Associated anomalies are reported.

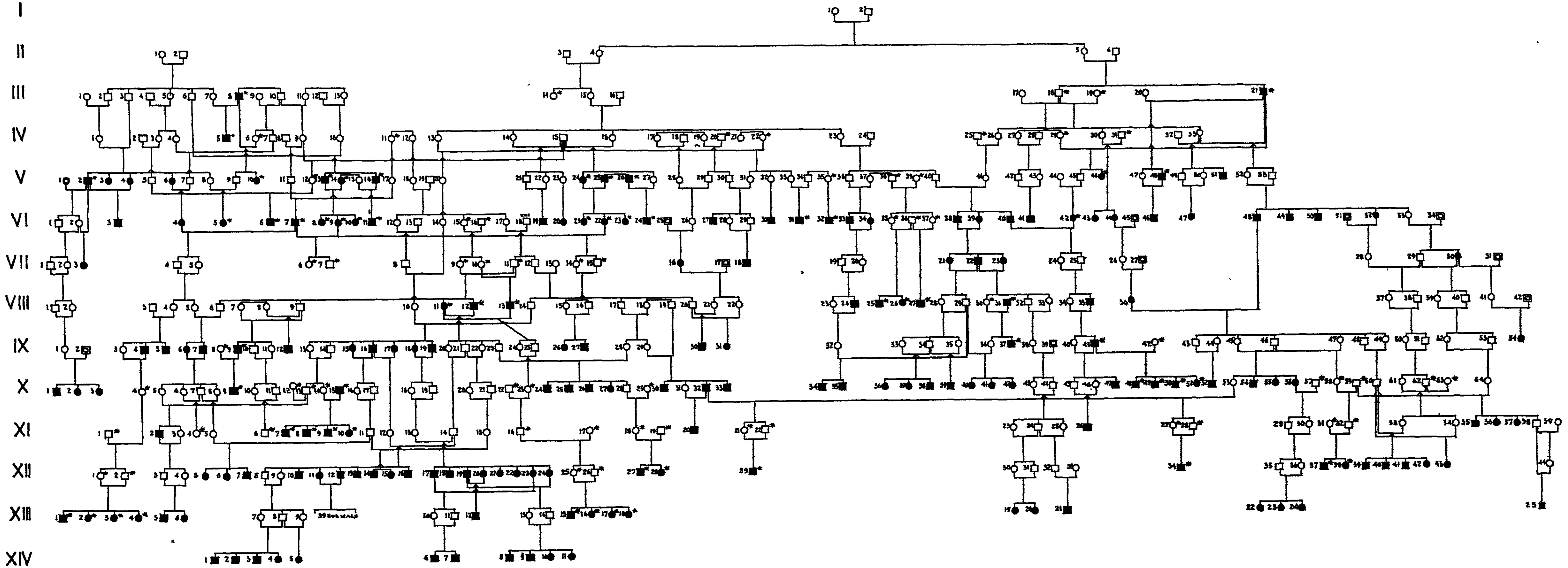
4. It is probable that microphthalmia possesses both a genetic and a physiological basis.

♂ ♀ Duplicate Numbers on Pedigree Chart.

III,	8	= 21.
IV,	5	= V, 48.
V,	10	= 46.
	13	= 25.
	14	= 24.
	16	= 26.
	48	= IV, 5.
VI,	5	= 42.
	6	= 32.
	7	= 31.
	8	= 21.
	9	= 22.
	10	= 23.
	11	= 24.
VIII,	11	= 26.
	12	= 25.
	13	= 27.
	31	= IX, 9.
IX,	9	= VIII, 31.
	37	= X, 9.
	41	= X, 15.
X,	9	= IX, 37.
	15	= IX, 41.
	48	= XI, 7.
	49	= XI, 8.
	50	= XI, 9.
	51	= XI, 10.
XI,	7	= X, 48.
	8	= X, 49.
	9	= X, 50.
	10	= X, 51.
XII,	27	= 37.
	28	= 38.
	29	= 34.
XIII,	1	= 15.
	2	= 16.
	3	= 17.
	4	= 18.

♂ ♀ Duplicate Numbers on Pedigree Chart.

III,	14	= 19.
	18	= IV, 20.
	19 cousin of 18 and	= 14.
IV,	6	= 29.
	11	= 22.
	20	= III, 18.
	25 cousin of V, 2.	
	31 cousin of 30.	
V,	2 cousin of IV, 25.	
	12	= 35.
	38	= VI, 16.
	39	= VI, 15.
VI,	15 and 16	= V, 39 and 38.
	36 cousin of 35 and 37 and	= VII, 11, 9, 10.
VII,	6 and 7	= 14 and 15.
	9, 11, 10	= VI, 35, 36, 37.
VIII,	30	= IX, 8.
IX,	8	= VIII, 30.
	42	= X, 14.
X,	4	= XI, 17.
	12 and 13	= 58 and 59.
	14	= IX, 42.
	22 and 23	= 62 and 63.
	29	= XI, 4.
	57	= XI, 6.
	58 and 59	= 12 and 13.
XI,	1	= 16.
	4	= X, 29.
	6	= X, 57.
	16	= 1.
	17	= X, 4.
	18 and 19	= 31 and 32.
	21 and 22	= 27 and 28.
XII,	1 and 2	= 25 and 26.



Pedigree Chart showing the Incidence of Microphthalmia in Rats.

A. M. Hain.

◎ ⊡ = Outcrosses, *i.e.* Hooded Rats.

* = Rats which appear twice on the Chart; see lists.

Chart.

Acknowledgments.

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VI.—Ovarian Rhythm in *Drosophila*. By H. P. Donald, Ph.D., and Rowena Lamy, Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D. (With Five Text-figures and Four Graphs.)

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CONTENTS.

	PAGE
Introduction	78
Methods	79
Results	80
(1) Counts at 12-hour Intervals	80
(*a*) *Pseudo-obscura*	81
(*b*) *Funebris*	83
(*c*) *Melanogaster*	85
(2) Behaviour of Females of Low Fecundity	85
(3) Dissection of Ovaries	89
(4) The Relation between Number of Egg-strings and Number of Eggs	91
(5) Influence of Day and Night on Laying	92
Discussion	93
Summary	95
Acknowledgments	95
References to Literature	96

Introduction.

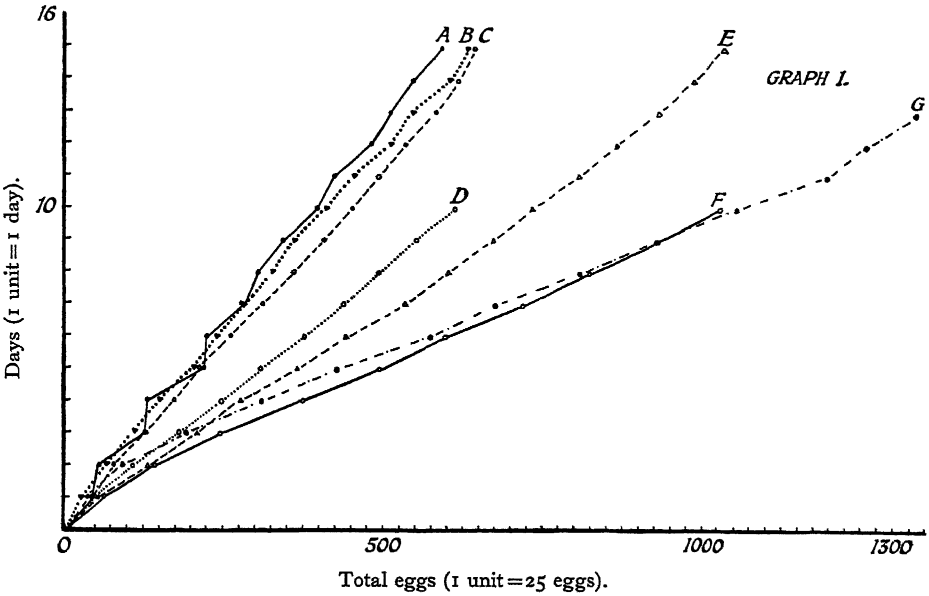
While observations were being carried out on the fecundity of certain experimental *Drosophila pseudo-obscura* females, it became apparent that there was a regular alternation of periods of rapid laying with periods of no laying. Since an irregularity in rate of oviposition in this species had been noticed by other investigators, and since, under similar circumstances, *Drosophila melanogaster* lays approximately the same number of eggs each day, it was decided to investigate more closely the possibility that differences in laying habit existed among the various species.

General reviews of the literature dealing with this subject are to be found in papers by Adolf (1920), Sturtevant (1921), Hanson and Ferris (1929), Alpatov (1932), and in the Genetics of *Drosophila* (Morgan, Bridges and Sturtevant, 1925). There are two papers dealing with fecundity in *D. pseudo-obscura*. Shapiro (1932), who used practically the same methods as those of the present experiments, but who gives no details of his results with *D. pseudo-obscura*, found that Race A females began to lay about two days after emergence, and Race B females about three days after. He noted also that in comparison with *D. melanogaster* females, *D. pseudo-obscura* females laid their eggs very irregularly, but nevertheless their egg-laying curves could be represented by the same equation as could those of *D. melanogaster* females. Dobzhansky (1935),

who used a different technique which involved the counting of eggs from groups of five females, also drew attention to the mode of laying which he found irregular as compared with the even day-to-day production of the female *D. melanogaster*, and suggested that an ovarian rhythm might be responsible.

Methods.

The susceptibility of *Drosophila* to changes in the external and internal environment has been well established by various authors, so



GRAPH 1.—Fecundity of *Drosophila* ♀♀ observed in various experiments. Curves *A* and *B* were obtained by the authors in the course of another investigation.

A, average of $10\,\frac{y\;Px}{v\;Rn}$ *D. pseudo-obscura* ♀♀.

B, average of $10\,\frac{y}{v}$ *D. pseudo-obscura* ♀♀.

C, Data of Hanson and Ferris (1929), *D. melanogaster* ♀♀.
D, average of 7 *D. melanogaster* ♀♀, wild type.
E, average of 11 *D. pseudo-obscura* ♀♀, wild type.
F, average of 15 *D. funebris* ♀♀, wild type.
G, Data of Shapiro (1932), *D. melanogaster* ♀.

that experiments connected with fecundity have to be very carefully controlled to ensure uniform conditions in all cultures.

The females to be observed were raised at 23·5 ± ·5° C. and were kept at this temperature throughout all the experiments. The paper spoon technique was used for collecting the eggs, and all operations were carried out in the room where the flies were kept to avoid changes

of temperature. The females were put singly into vials with a spoon bearing a slice of cornmeal agar food. With each female were placed two males. If either of these was lost or died it was replaced as soon as possible. Although Pearl (1932) has shown the importance of density of population in connection with fecundity in *Drosophila*, this factor is not considered to have played any considerable rôle under these conditions. The time during which any female could be accompanied by only one male was very short, and also, on Pearl's "collision of molecules" analogy, the influence of changes of density would be of smaller magnitude where the flies are as slow moving as *D. pseudo-obscura* or *D. funebris*.

In order to measure the accuracy of counting the eggs, a standard deviation for any one count was arrived at by repeating a group of counts 10 times. The values obtained lay between ± 1·0 and ± 2·3 for counts varying from 60 to 115 eggs.

The published records of fecundity in *D. pseudo-obscura* do not permit the comparison of actual numbers of eggs laid which might provide some idea of the effectiveness of the various techniques employed. Dobzhansky appears to have obtained about 250–380 eggs in 16 days from his masses of five Race A females at 25° C. From Shapiro's graphs a single *D. melanogaster* female heterozygous for vestigial laid about 1600 eggs in that time. Hanson and Ferris obtained about 650 eggs from single white-eyed *D. melanogaster* females, while in the present experiments the number varied according to species and conditions. The relations are brought out in graph 1.

RESULTS.

Individual fecundity records from daily counts of eggs from females used for various experimental purposes showed marked differences in the day-to-day production similar to those observed by Dobzhansky and Shapiro. The alternation of high and low counts was so conspicuous that it took the form of a regular periodicity or rhythm, and although deviations from it were frequent, this rhythm appeared to be fundamental either to the flies or to the method of culturing them, for it was always recovered after a lapse. It was realised that the deviations might be only apparent and due to an unsuitable interval between counts, or to unfavourable variations in the technique, and so experiments were designed to obtain further information concerning the rhythm.

(1) *Counts at Intervals of 12 Hours.*

A series of counts was made at 9 a.m. and 9 p.m. each day with females which had been raised under optimal larval conditions, and selected

immediately after hatching. The following numbers and species were used:—

9 *D. pseudo-obscura* ♀♀ (Nakusp strain from Dr Dobzhansky).
9 *D. pseudo-obscura* ♀♀ (Big Bear strain from Dr Dobzhansky).
7 *D. melanogaster* ♀♀ (Oregon K stock from Professor Timoféeff-Ressovsky).
15 *D. funebris* ♀♀ (from Professor Timoféeff-Ressovsky).

The level of fecundity obtained in this experiment is shown by curves D, E, F of graph 1 which have been constructed from the averages of those females which survived to the end of the tenth day. Samples of the actual data are given in Table I.

(a) *Pseudo-obscura.*—Of the 18 *pseudo-obscura* females, five ceased to lay after a week or ten days for no apparent reason. When they were dissected their ovaries were found to contain about 150 ripe eggs per fly. Two other females were accidentally lost. The records of these seven females were in no way different from those of the remaining eleven whose eggs were counted for twenty-two days. The individual graphs of the latter show a remarkable similarity and exhibit several characteristics which appear to be fundamental to the laying habit. A typical record is shown graphically in fig. 1. The 12-hour intervals between observations, while being purely arbitrary units for measuring the characteristics of the rhythm, were short enough to demonstrate fairly clearly the periods of rest and activity.

When the results were plotted on graph paper, the periodicity was quite obvious. Also, it seemed very likely that females usually laid most of their eggs in periods of less than 12 hours, for there were many occasions on which large counts of eggs were preceded and followed by counts of nil. Subsequent experiments showed that 6 hours would be adequate to lay all the eggs becoming mature at one time, so that this supposition is probably justified although the flies are so sensitive to their environment that they often retain their eggs, or lay them at a rate slower than normal with a consequent lengthening of a particular laying period.

In order to discover whether or not there were any definite characteristics of form of this rhythm, it was subdivided into the most obvious parts which were the individual "waves" of which it was made up. This was, of course, a rather arbitrary process because the real limits of the laying periods were not known, but the results, when averaged, were quite suggestive. A measure of the length of the waves was arrived at by expressing in terms of 12-hour units of time the interval between the central points of the successive hollows of the rhythm. This gave

TABLE I.—SAMPLES OF DATA FROM COUNTS MADE AT 9 A.M. AND 9 P.M.

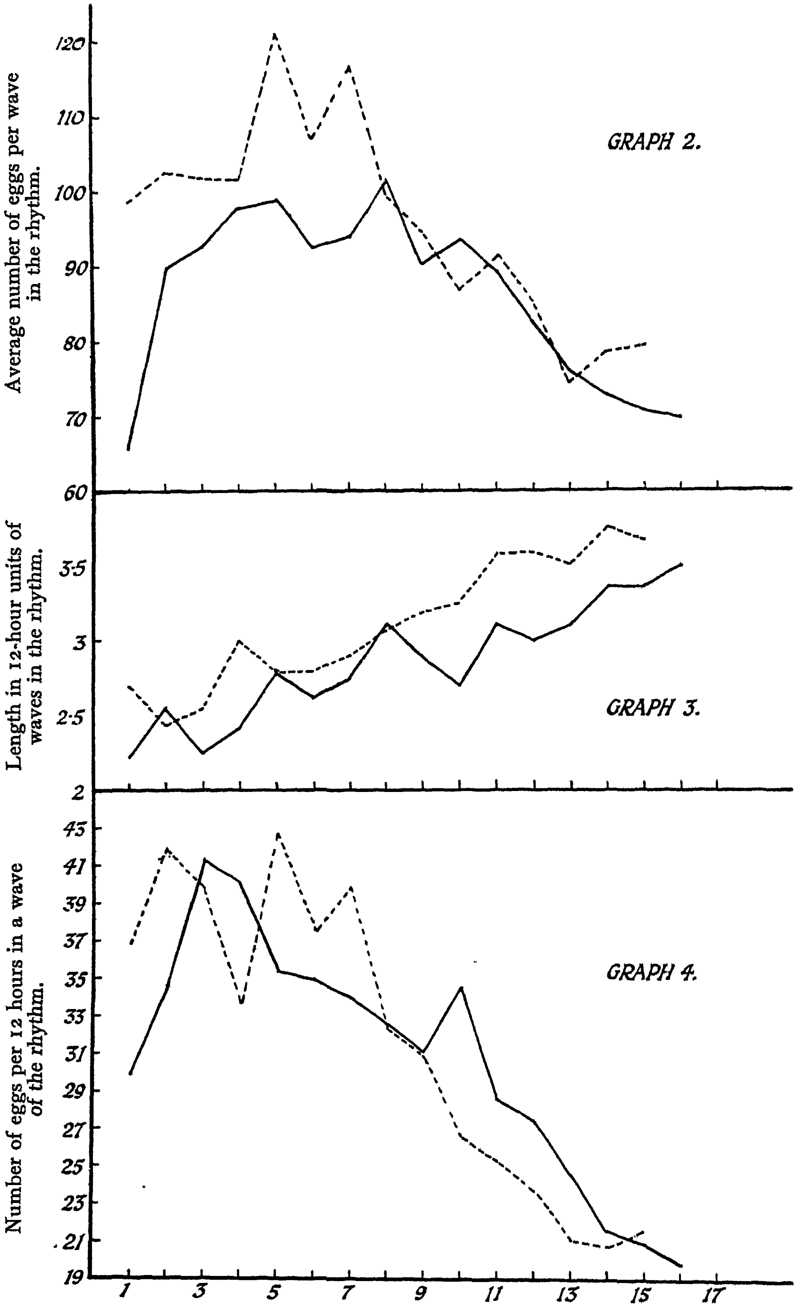
Fly.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.
D. pseudo-obscura (Nakusp).																																
12	..	20	95	0	56	36	4	105	0	27	104	0	50	70	0	124	4	3	118	0	4	111	0	12	101	0	0	31	65	3	46	71
13	..	22	84	0	60	38	12	53	33	61	63	0	63	53	0	26	70	0	21	82	0	1	77	0	1	66	0	5	48	0	1	1
14	..	..	47	57	0	106	0	33	74	17	37	22	0	130	33	0	114	23	0	120	0	4	80	0	0	109	0	0	22	71	0	14
15	..	..	..	38	57	0	85	18	3	105	0	18	90	0	3	102	0	3	72	0	0	107	0	15	56	25	0	12	80	0	0	80
17	..	..	..	35	54	7	104	0	44	65	0	58	40	0	100	21	22	86	0	42	75	0	37	69	1	49	46	4	72	27	0	52
D. melanogaster.																																
19	..	..	..	..	23	66	49	40	53	49	31	49	28	47	55	34	42	36	34	30	42	47	42									
20	56	52	33	43	24	50	30	32	23	46	31	47	27	37	44	41	21	25	38	24	38	23	36									
21	10	33	25	28	42	23	42	26	17	42	34	43	10	12	24	27	19	4	9	12	22	20	24									
24	30	39	22	29	30	28	27	32	7	40	20	28	23	29	21	14	27	4	30	14	22	12	12									
25	4	39	15	54	16	39	34	41	16	37	21	55	18	44	15	36	16	42	8	34	15	32	33									
D. funebris.																																
66	..	13	18	69	30	77	31	83	40	54	58	72	63	38	66	68	37	92	28	87	47	41	49									
67	..	7	13	9	0	66	29	59	53	35	36	56	39	65	71	48	44	58	30	68	11	57	16									
68	..	..	6	36	39	54	39	77	40	65	46	69	47	68	52	63	48	56	53	47	56	59	32									
69	..	10	37	41	54	64	35	82	51	67	55	64	53	72	61	59	50	70	57	52	42	46	52									
70	..	17	14	58	25	62	40	60	53	46	46	62	31	63	17	61	46	59	31	63	36	61	30									

the average length of the successive waves for which the total number of eggs was known, and thus it was possible to watch the changes in these two quantities as wave followed wave and the flies grew older. These changes have been plotted in graphical form in graphs 2 and 3. Graph 2 shows that there was an initial increase and then a decrease in the number of eggs per wave. Such changes would be found whether the intervals were short enough to demonstrate the rhythm or not.

Change in the length of the successive waves appeared to follow a different course. As expressed in graph 3, the change was one of continuous increase from the beginning. The average length of the first waves was about $2\frac{1}{2}$ times 12 hours, and this increased gradually to about $3\frac{3}{4}$ times 12 hours at the seventeenth wave. In spite of the smallness of the numbers of females used and the fact that at the end the nine original females of each strain had diminished to five and six, the linear character of the two curves cannot be mistaken. In graph 4 are shown the changes which took place in the number of eggs laid per 12 hours within each wave. In general shape this resembles the change in number of eggs per wave, but the decrease begins earlier. It is merely the expression of the interaction of the two previously mentioned changes. Although this is a convenient way of expressing the activity of the flies in terms of their fecundity, it has to be remembered that the eggs are probably laid very rapidly at times, and slowly or not at all at other times, and an average per 12 hours does not signify a rate of laying of so many per hour.

Following a suspicion that the alternation of the periods of laying and rest might be connected with a preference for laying at particular periods of the day, further 12-hourly counts were made in exactly the same way except that the changing of the spoons was carried out at midday and midnight instead of at 9 a.m. and 9 p.m. As far as the rhythm was concerned, the results were the same; the alternation of periods of laying and of rest were as clearly marked as before, and the flies appeared to lay without reference to the time. This confirms the observation to be made from fig. 1 that the laying periods occur during both day and night, so that a day and night effect can be discarded as a possible cause of the rhythm. *D. melanogaster* flies were included in this experiment and also gave the same results as before. *D. funebris* females were omitted, which was unfortunate because these are the flies which appear to be most strongly affected by daylight or its absence.

(b) *Funebris.*—The fecundity of the *funebris* females was higher than that of the *pseudo-obscura* females. Their day-to-day production showed some variability, but, unlike the *D. pseudo-obscura* females, they did not show a single clear period of inactivity. On the other hand, one constant



Sequence of "waves" in the rhythm.

Nakusp ♀♀=dotted line. Big Bear ♀♀=continuous line.

GRAPHS 2, 3 and 4.

feature in the performance of all was that the evening count (giving the number of eggs laid during the day) was almost invariably higher than the morning count. Fig. 2 shows an individual graph in which this effect is seen, the first eggs having been laid overnight. As these counts were made in the month of August, it is possible that the effect might be less striking than that obtained from the same observations made during the winter months.

(c) *Melanogaster*.—The egg production of the *melanogaster* females also varied considerably from count to count. The number of eggs laid during the day was usually greater than the number laid during the night, but the difference was not statistically significant. It would appear from this that *melanogaster* females oviposit at a comparatively constant rate throughout the 24 hours. Fig. 3 shows graphically the typical 12-hourly production of an individual fly.

(2) *Behaviour of Females of Low Fecundity.*

A preliminary experiment designed to explore the possibilities of making more frequent daily counts was carried out with ten females of each of the three species. From 9 a.m. to 9 p.m. counts were made at 3-hourly intervals so as to determine with greater accuracy if possible the limits of the laying and rest periods. The results obtained were from this point of view not satisfactory, and it is not proposed to deal with them in detail at this time, but they revealed certain features which are of interest in connection with the preceding data. A typical sample of the observations made will suffice to show that the observed periodicity was not due to mere chance (Table II).

The *D. pseudo-obscura* females were of the same Nakusp stock as that used before, but were taken from cultures which were slightly overcrowded. They showed a lower fecundity, which may have been due to the interruptions caused by changing the spoons or to the effects of less favourable conditions during the larval stage. Whatever the cause, the peaks of production and length of rhythm shown (when the day counts were added up and used as 12-hour counts) were comparable with those previously obtained, but were characteristic of much older females. Fig. 4 shows the behaviour of a typical female of this group.

These results suggested that the environmental conditions of the larvæ were responsible for the lowered fecundity and not the interruptions. A further thirteen females were therefore taken from an old culture bottle in which the food had deteriorated. The flies hatching out were small, owing to semi-starvation and premature pupation of the larvæ, but apparently healthy. The stock (3B) was an old Texas wild-type line,

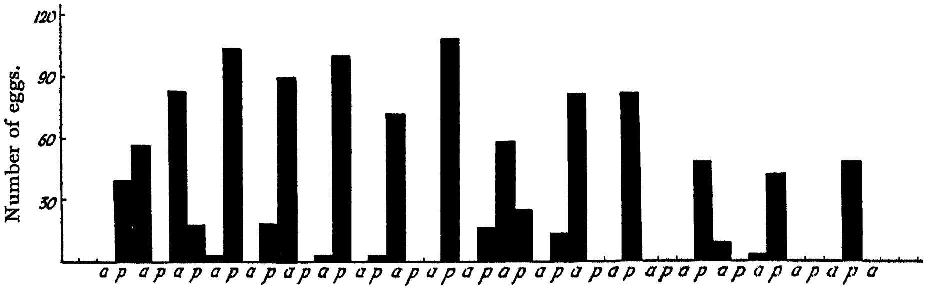


FIG. 1.—Record of fecundity of a *D. pseudo-obscura* ♀.
a = count at 9 a.m. *p* = count at 9 p.m.

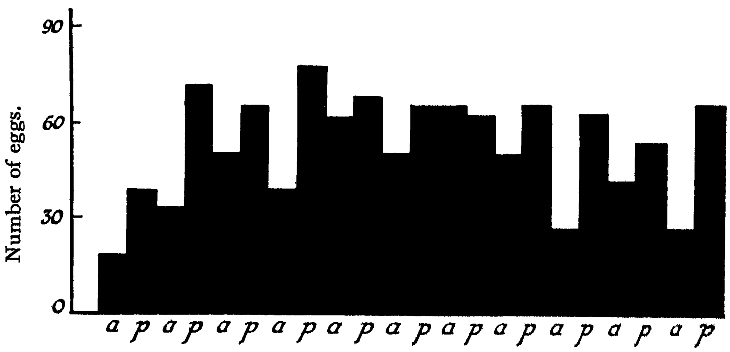


FIG. 2.—Fecundity record of a *D. funebris* ♀.

FIG. 3.—Fecundity record of a *D. melanogaster* ♀.

FIG. 4.—Fecundity record of a *D. pseudo-obscura* ♀, in which the p.m. count (*p*) represents the total eggs laid within 4 three-hour periods between 9 a.m. and 9 p.m.

TABLE II.—THREE-HOURLY COUNTS OF *D. pseudo-obscura* ♀♀ (NAKUSP).

FIRST COUNT AT 9 A.M. AND LAST AT 9 P.M.

Date.	21/9.					22/9.					23/9.					24/9.					25/9.					26/9.					27/9.				
	1.	2.	3.	4.	5.	1.	2.	3.	4.	5.	1.	2.	3.	4.	5.	1.	2.	3.	4.	5.	1.	2.	3.	4.	5.	1.	2.	3.	4.	5.	1.	2.	3.	4.	5.
No. 21	49	0	0	0	0	4	11	30	21	15	0	0	0	0	0	1	0	8	3	31	0	0	0	0	0	0	0	6	16	35	5	0	0	0	0
No. 22	0	4	0	20	57	0	0	0	0	0	33	7	6	36	15	0	0	0	0	4	9	0	20	2	29	0	0	0	0	0	0	0	19	30	13
No. 23	0	0	0	19	13	52	0	0	0	0	0	0	1	19	56	15	0	0	0	0	0	0	0	5	63	4	0	0	0	0	0	0	21	11	23

inbred without selection for several years, and was known to carry an inversion on the X-chromosome. The egg production of these females was noted twice a day at 9 a.m. and 9 p.m. for ten days. They did not begin to lay until the 4th–5th day after hatching, and, in the course of ten days, exhibited a very low fecundity. In fig. 5 is shown the record of

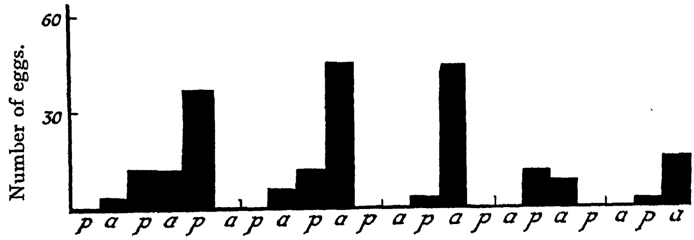


FIG. 5.—Fecundity record of a *D. pseudo-obscura* ♀ hatched from an old culture.

the fly of this group which laid most eggs. Guyénot in 1912 raised females under poor nutritive conditions and observed the same reduction in fecundity.

For purposes of comparison the condensed data obtained from these 3B flies are given in Table III, together with the corresponding figures for the first four waves estimated roughly from the curves for the Nakusp strain given in graphs 2 and 3.

TABLE III.—AVERAGE NUMBER OF EGGS PER WAVE AND AVERAGE LENGTH OF WAVE (IN UNITS OF 12 HOURS) OF FEMALES OF LOW FECUNDITY, TOGETHER WITH THE CORRESPONDING VALUES EXPECTED FOR THE NAKUSP STRAIN.

Wave . . .	1		2		3		4	
Number of ♀♀ .	13		13		12		6	
	Eggs.	Length.	Eggs.	Length.	Eggs.	Length.	Eggs.	Length.
	52·4	3·4	44·5	3·7	32·2	4·0	24·0	3·9
Approx. values for Nakusp females	95	2·3	100	2·5	105	2·6	110	2·7

This table shows that, compared with the previous results, the number of eggs is small and the length of the waves of the rhythm greater than normal. Also, there is a continuous drop in the number of eggs instead of a preliminary rise followed by a fall. By reference to the foregoing

graphs it will be seen that not until attaining the age of about 16 days would a Nakusp or Big Bear female require 3·4 times 12 hours for a wave of the rhythm and would be laying at that time about 90 eggs instead of the 52 observed here. These flies hatching from poor cultures exhibited the laying characteristics of normal flies two to three weeks old. It is probable, of course, that there are genetical differences between the Nakusp and 3B strains, but a considerable experience of the latter lends no support to the view that the fecundity under similar conditions might be strongly dissimilar to that of the Nakusp flies. It may be suggested, therefore, that environmental factors may operate in larval stages to produce effects on ovarian rhythm which may be defined as premature senility in the adult fly following on premature development in the larva.

From the 3-hourly counts another point of interest emerged. When due allowance is made for the relatively low fecundity of the females used, it seems fairly certain that a *D. pseudo-obscura* female could lay all the eggs of one matured batch in about 6 hours. The maximum number of eggs laid by any female in any 3-hour period from 9 a.m. to 9 p.m. was 66, 32, 49, for *D. pseudo-obscura*, *melanogaster*, and *funebris* respectively. Considering the relative fecundities of these species as shown in graph 1, these figures are probably significant. The total production of the *funebris* group was twice that of the *pseudo-obscura* group in this experiment, but the largest single counts came from the latter. Further, a study of the individual records shows clearly that oviposition in the latter was far more concentrated in time, the main counts being comparatively large and few, while those of the former were smaller and spread over the four intervals. The fecundity of the *melanogaster* group was slightly below that of the *pseudo-obscura*, and their counts tended, like those of *funebris*, to be small and frequent.

(3) *Dissection of Ovaries.*

The next step was to determine whether or not differences could be detected in the condition of the ovaries corresponding to the differences in laying habit. This was done by observing the number of eggs laid for several days and then dissecting the females immediately after a count. The number and condition of the eggs in the ovaries could then be correlated with the immediately preceding batch of eggs laid or with an absence of eggs, as the case might be. Some of the results for *D. pseudo-obscura* are given in Table IV, which indicates how a period of rapid laying leaves the ovaries without any mature eggs, and how mature eggs accumulate during a period of non-laying. Owing to the fact that eggs were found at all stages of development in *D. melanogaster*, it was

not feasible to make accurate counts of the mature eggs and to construct a similar table for this species.

The general conclusion that there is a distinct difference in the functioning of the ovaries of the two species is supported. The egg-strings of *D. melanogaster* contain eggs gradually diminishing in size towards the tip of the egg-string, as figured by Nonidez (1920). This leads to continuous maturation and deposition of eggs. In *D. pseudo-obscura*, on the other hand, the eggs ripen in batches. When immature eggs are present they are divided into groups, each of which is at a different stage of development: hence a large number of eggs become ready for laying at one time. Table IV shows that females such as E6, 9, 12 and 22, which have paused for a day after laying many eggs, have large numbers of mature eggs in their abdomen, while those, such as E10, 17, 19 and 24, which were dissected immediately after a high count, contain no ripe eggs. Female number 13 had evidently been interrupted before laying all her ripe eggs, or else had retained some; number 14 was about half-way towards the next laying period; numbers 18 and 21 were presumably abnormal; number 23 was getting rather old.

TABLE IV.—NUMBER OF EGGS FOUND ON DISSECTION OF LAYING *D. pseudo-obscura* FEMALES; OVARIES SEPARATELY WHERE POSSIBLE.

No.	Eggs Laid on Last 4 Days of Life.				Eggs in Abdomen on Dissection.		Remarks.
E6	48,	3,	56,	0	32,	31	Eggs mature.
7	0,	0,	0,	0	116		Eggs mature.
8	0,	7,	9,	3	25,	21	
9	12,	20,	58,	0	41,	38	
10	25,	27,	0,	62	0,	0	Very young eggs only.
11	0,	0,	75,	1	38,	38	
12	74,	7,	111,	0	45,	44	Some not quite mature.
13	0,	19,	19,	74	34,	29	Mature; also many half-formed; clearly 2 stages.
14	0,	23,	35,	78	42,	34	About half-formed.
15	40,	18,	31,	51	14,	10	Approx.; many immature.
16	0,	0,	62,	0	50,	56	Not quite mature.
17	66,	62,	27,	63	0,	0	Very young eggs only.
18	0,	0,	0,	0	22,	32	Various stages of maturity.
19	0,	7,	13,	95	0,	0	Very young eggs only.
20	0,	0,	0,	33	36		Not quite mature.
21	0,	0,	20,	17	4,	0	Ovaries small; eggs very young.
22	57,	50,	37,	4	32,	29	
23	0,	0,	0,	0	30,	18	
24	28,	15,	0,	79	0,	0	Only very young eggs.

(4) *The Relation between Number of Egg-strings and Number of Eggs.*

Saveliev (1928) noted that by varying the standard of nutrition in the larval stages he could alter the number of egg-strings in the adult fly (from 15–40 in *D. melanogaster*) and that, as the number of egg-strings was reduced, so was the fecundity. Since a record has been kept of the number of egg-strings of most of the flies concerned in the foregoing experiments, it is possible to test the inference that fecundity is low because the number of egg-strings is small. The major portion of the appropriate data is given in Table V.

TABLE V.—RELATION BETWEEN NUMBER OF EGG-STRINGS AND TOTAL NUMBER OF EGGS.

Species.	Number of Egg-strings.	Number of Counts.	Total Eggs.	Average Number of Eggs per Count.	Average Number of Eggs per Egg-string per Count.
D. pseudo-obscura .	49	44	1359	30·89	·63
	50	46	1340	29·13	·58
	55	46	1205	26·20	·48
	55	46	1605	34·89	·63
	57	45	1550	34·44	·60
	57	44	1288	29·27	·51
	57	44	1444	32·82	·58
	59	46	1376	29·91	·51
	59	46	1181	25·67	·44
D. melanogaster .	51	21	486	23·14	·45
	51	20	429	21·45	·42
	54	23	664	28·87	·53
	56	23	821	35·70	·64
	57	23	540	23·48	·41
D. funebris . .	52	22	1161	52·77	1·01
	54	22	870	39·55	·73
	54	20	1052	52·60	·97
	55	23	1333	57·96	1·05
	55	22	977	44·41	·81
	56	20	877	43·85	·78
	56	22	1174	53·36	·95
	56	22	1111	50·50	·90
	56	22	1036	47·09	·84
	57	23	1220	53·04	·93
	57	22	1098	49·91	·88
	58	22	981	44·59	·77
	59	22	1064	48·36	·82

On this data, the inference cannot be considered sound. Within any of the comparable groups of females there is no regular increase in fecundity with increase in the number of egg-strings, so that the relation between them, if any, must be sought with the aid of other types of experiment.

In passing, attention may be directed to several interesting points about the numbers of egg-strings. The mean number of egg-strings was remarkably constant for the three species: they were 52·6, 51·9, 50·6, for *pseudo-obscura*, *melanogaster* and *funebris* respectively. The efficiency, so to speak, of egg-strings apparently varies among species. The resemblance between ovaries was also very close, the difference in number of egg-strings in any pair being rarely more than 2. One unusual *pseudo-obscura* female possessed three ovaries, or rather one normal and one branched ovary, the numbers of egg-strings being 24 and (26 + 19). The fecundity was not correspondingly high.

(5) *Influence of Day and Night on Laying.*

Although a day-and-night effect cannot be regarded as the cause of the rhythm in *D. pseudo-obscura*, the effect may nevertheless exist. Examination of the records obtained in the foregoing experiments leads to the suspicion that there is a real effect, but a satisfactory demonstration will have to come from differently designed tests. A comparison

TABLE VI.

The ratio $\frac{\text{Total eggs at p.m. count}}{\text{Total eggs at a.m. count}}$.

Species.	9 o'clock.			12 o'clock.		
	Number of Counts.	Ratio.	Number of Flies.	Number of Counts.	Ratio.	Number of Flies.
D. melanogaster .	22	1·26	7–5	12	1·4	7
	first 12	1·39	7			
	last 10	1·05	5			
D. funebris . .	22	1·3	14–13			
D. pseudo-obscura—						
(*a*) Nakusp .	44	1·03	9–6	16	·76	18–13
	first 22	·94	9–7			
(*b*) Big Bear .	44	1·52	9–5			
	first 22	1·36	9–5			
(*c*) 3B . .	20	1·7	15–13			

of the numbers of eggs laid in the two halves of the day has been made by calculating a p/a ratio; that is, by dividing the total at p.m. counts (representing total eggs laid during the 12 hours previous to each p.m. count) by the total at a.m. counts. This has led to the data in Table VI.

It is a disadvantage of this concise method of presenting the data that the variations in performance are obscured. The periodicity in laying shown by the *pseudo-obscura* flies makes it difficult or impossible to decide whether the ratios obtained are the result of the chance coincidence of periods of rapid laying with either a.m. or p.m. counts or are due to a real preference for particular hours of the day for laying. The Nakusp flies may be suspected on this evidence of having a preference for the hours before midday. The *funebris* flies, as exemplified by the record shown in fig. 2, have given a regularly higher number of eggs at the p.m. count, and the evidence of individual records is strongly in favour of a real day-and-night effect. The *melanogaster* flies, which showed similar ratios, were by no means so regular in their performance. The same conclusion, therefore, cannot yet be drawn concerning them.

Discussion.

There is abundant evidence in the entomological literature concerning the oviposition of various insects, but, as a rule, it is difficult or impossible to dissociate the egg-laying habits from the mode of nutrition. Back and Pemberton (1914) found that the melon fly laid up to 30 eggs in batches at intervals of 3–30 days, but it is uncertain whether this periodicity was the result of the conditions under which laying took place or whether it was a specific habit of the flies.

The present experiments have provided adequate substantiation of the fact emerging from previous experiments that females which are kept singly do not lay at a constant rate and show that under closely controlled culture conditions *D. pseudo-obscura* females show a periodicity in laying which does not appear in the daily egg-records of *D. melanogaster* and *D. funebris*. Along with this difference exists the possibility, as yet not thoroughly investigated, that there is a further one connected with a preference by the latter species for laying in daylight.

Apart from the interest attaching to the actual differences established between *D. pseudo-obscura* and the other two species and the light they may shed on the ecological conditions associated with the differentiation of the various species, an opportunity arises of extending the analysis of the factors involved in the process of egg-laying. Assuming a perfect standardization of environmental and hereditary conditions which, though difficult to approach, seems at least possible, the egg-production of a fly at any moment appears to be a function of the number of egg primordia formed at the apices of the egg-strings, and of the rate at which these primordia develop into eggs ready for fertilisation and laying, the former determining the number of eggs to be laid at any one time, and

the latter determining the length of the period separating the bursts of oviposition. The evidence for the separate existence of these two factors in egg production arises from two sources. Firstly, the history of the behaviour of the average *D. pseudo-obscura* ♀ (graphs 2, 3, 4) indicates that change in the length of the rhythm is linear, whilst change in the number of eggs laid conforms to the usual curve of production which rises for a time and then falls more gradually. It has to be remembered, however, that the data on which these graphs were based were not wholly without reproach as a result of the necessarily arbitrary separation of the units of the rhythm consequent on the use of 12-hour intervals between counts. It would be surprising, nevertheless, if further work were to show that the apparent linear change in length of rhythm shown by the two distinct groups of females used was in reality properly represented by a mirror image of the graph of number of eggs per wave of the rhythm which would be expected if length of wave were a function of number of eggs per wave. There is nothing in the individual records to suggest that the average result could be so far from the truth. The second line of evidence interpreted in favour of two separate processes is derived from the study of flies of low fecundity induced by poor food conditions. In these flies the length of the first few waves of the rhythm was typical of normal females about 2 weeks old, whilst rate of egg-laying was lower than that of any of the flies whose egg production was observed for 3 weeks. This is interpreted to mean that larval malnutrition has affected one process more than the other. Further investigation of this hypothesis is contemplated.

The part played by the number of egg-strings in determining either rate of egg production or total number of eggs has not been important. The number of egg-strings has been counted for most of the females used in the later experiments, and no correlation between the number obtained and any aspect of egg production has been observable. The conclusion is drawn, therefore, that under the conditions of the present experiments, the number of egg-strings did not limit the number of eggs produced. Table V shows that the number of egg-strings in the three species used were much alike, although there are distinct differences in the rate of egg-laying as is shown by graph 1. It is true that the number of egg-strings in the 3B females was much below average, as was the number of eggs laid, but there is nothing to show that the two were directly connected. The alternative explanation, namely, that the metabolism of the flies was impaired by their larval treatment, is equally possible and more in accord with the results. It might be maintained, therefore, that the conditions which would reduce the number of egg-strings would at the same time decrease the capacity of the flies for the

metabolic processes involved in egg formation. In Pearl's terminology (1928), this idea could be expressed by saying that the vitality of the fly is the limiting factor in egg formation. The truth of this hypothesis would not exclude the possibility that occasions could arise on which the number of egg-strings was the limiting factor, but such a situation does not appear to have been experienced.

The discovery of an ovarian rhythm in *Drosophila* creates an interesting problem. It is well known that such rhythms in mammals are conditioned by hormones, and the inference that hormones exist in insects is not without support (Fraenkel, 1935, Wigglesworth, 1934). Whilst the immediate reason for the difference between *D. melanogaster* and *D. pseudo-obscura* may be found in the conditions imposed by the natural habitats of the two species, there must be some internal difference responsible for the dissimilar behaviour of the ovaries under the same conditions. The question, therefore, arises as to whether the functioning of the ovaries is autonomously controlled or is subject to the influence of internal agents acting externally to the ovaries.

Summary.

1. The previously observed periodicity of oviposition in *D. pseudo-obscura* has been found to be due to an ovarian rhythm which does not occur in *D. melanogaster* or *D. funebris*.

2. This rhythm has been found to alter its shape with increasing age of the females, and from this fact it has been deduced that fecundity is affected by two distinct processes. One of these is thought to be the number of egg primordia formed at the apices of the egg-strings, and the other the rate of development of the eggs in the egg-strings.

3. From a study of records of females of different levels of fecundity, it is concluded that, under most conditions, not number of egg-strings but metabolism is the controlling factor of fecundity.

4. Dissection of ovaries substantiates the idea of an ovarian rhythm. Unlike the ovaries of *D. melanogaster*, those of *D. pseudo-obscura* contain eggs at distinctly different stages of development, so that mature eggs become ready for laying in batches.

5. The data indicate that *D. funebris* females distinctly prefer to lay during the day-time, but they are not conclusive with respect to the other species.

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VII.—On the Geometry of Dirac's Equations and their Expression in Tensor Form. By Professor H. S. Ruse,

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(MS. received October 12, 1936. Read January 11, 1937.)

CONTENTS.

	PAGE
1. The Space S_3	98
2. Transformations of Co-ordinates	101
3. Parametric Equations for the Fundamental Quadric	103
4. Spinors and Spin Transformations	105
5. The Algebra of the Fundamental Tensors and Spinors	110
6. Covariant Differentiation of Spinors	112
7. Dirac's Equations	114
8. The Geometry of Dirac's Equations	115
9. First Tensor Form of Dirac's Equations	120
10. Second Tensor Form of Dirac's Equations	123
11. The 6-vector of Electric and Magnetic Moment	124
12. The Momentum-Energy Tensor	126
13. References to Literature	127

THE purpose of the present paper is to give as simple an account as possible of the general-relativity theory of two-component spinors, and to investigate its geometrical and analytical consequences. The work was suggested by courses of lectures given at Edinburgh in 1932 and 1935 by Professor E. T. Whittaker, who, on the basis of the special-relativity spinor theory of van der Waerden (1929), obtained the completely tensorized form of Dirac's equations given by him in a recent paper (1937).

§§ 1, 2 of the present work deal with preliminary matters which, though elementary, are needed as an explanation of the notation and methods of the subsequent sections; while §§ 3–7 are devoted to a geometrical development of the spinor theory, divested as far as possible of the mathematically interesting but physically unnecessary generalities which have occupied some earlier writers. An apology is perhaps needed for the length of this part of the paper, since it contains little that is essentially new, and, indeed, follows very closely the exposition given by Veblen in his paper on the two-component theory (1933). But for simplicity a number of modifications have been made in his treatment: for example, the whole theory is explained in terms of the 3-dimensional section of the null cone to which he refers only briefly, spin transformations are restricted to be of determinant unity, and the introduction of an auxiliary gauge-variable x^0 has been avoided. Since an explanation of

these modifications is necessary, and since an extensive quotation of results would in any case be needed for the purposes of the subsequent work, a development of the subject *ab initio* seems to be the only effective mode of presentation. Furthermore, it has appeared desirable to make a certain change of emphasis, so that greater attention is given to the transformation theory and less to that of covariant differentiation.

In § 8 appears an examination of the particular geometrical figure, namely a skew quadrilateral upon a quadric, that underlies the algebra of the Dirac theory. No detailed study of this figure in relation to the Dirac equations has yet been made, although its existence has been recognized by Schouten (1933, p. 107) and independently by Mr W. L. Edge, who commented upon it during the second of the courses of lectures mentioned above. The results of this section are closely related to those of an earlier paper on a similar subject (Ruse, 1936).

The work of §§ 9–12 is mainly analytical, although further references to geometry are made in §§ 9 and 11. It is shown that all the principal formulæ of the Dirac theory can be expressed in a tensor form not explicitly involving spinors, but depending instead upon four null vectors and two conjugate complex scalars. Two of the null vectors are real and the others are complex conjugates, and the four are shown in § 11 to define the principal null directions of the 6-vector of electric and magnetic moment. A derivation of Whittaker's tensor form of Dirac's equations is given in § 9. The formulæ of the tensorized theory are somewhat complicated and perhaps of no great practical value, but the work is presented because it may be of some intrinsic interest and because it throws light upon the inner relationships of the vectors and tensors of the Dirac theory.

I am grateful to Mr C. J. Seelye for giving me the benefit of his extensive knowledge of the literature of the Dirac theory, and for allowing me access to manuscripts which contain, in a different notation, a general-relativity formulation of the results given in Professor Whittaker's lectures.

§ 1. The Space S_8.

Let

$$ds^2 = g_{ij}dx^i dx^j, \qquad (i, j = 1, 2, 3, 4),$$

where the quadratic form is of signature $(- - - +)$, define the metric of the space-time V_4 of general relativity. Throughout the paper we take the velocity of light c in vacuo to be equal to unity.

The components X^i of a contravariant vector at any particular point O, (x^i), of V_4 may be regarded as the co-ordinates of a point in the 4-dimensional tangent space T_4 at O, in which space the quadratic form

$g_{ij}X^iX^j$ gives a euclidean measure of distance (Veblen and Whitehead, 1932). Any algebraic relation involving current co-ordinates X^i represents a locus in T_4, so that, for example, $\phi_i X^i = \text{const.}$ is the equation of a hyperplane or *3-flat*; and, if in such an equation we replace X^i by X^i/X^0 and multiply the resulting relation by $(X^0)^n$, where n is its degree, we obtain the equation of the locus in homogeneous co-ordinates (X^0, X^i). If we then omit the terms involving X^0, we get the equation of the projection of the locus from O on the 3-flat S_3 at infinity, the equation of which is $X^0=0$. Thus *the components X^i of a contravariant vector at O may be regarded as the homogeneous co-ordinates of a point in S_3.* It must, however, be observed that a vector ρX^i whose components are a scalar multiple of those of the original vector represents the same point of S_3, which means, in other words, that a point of S_3 corresponds not to a single point of T_4 (or, otherwise expressed, to a unique vector of V_4), but to the infinity of points ρX^i which lie on the line joining O to the given point in S_3. Nevertheless a *normalized* set of co-ordinates of a point in S_3, that is, a set of homogeneous co-ordinates with a definitely chosen factor of proportionality ρ, do define a unique point of T_4 and hence a unique vector of V_4. With these facts in mind we may interpret contravariant vectors and other tensors in terms of the 3-dimensional space.

A covariant vector ϕ_i at O defines in S_3 an ordinary (2-dimensional) plane of homogeneous co-ordinates of ϕ_i and equation $\phi_i X^i=0$. A symmetric tensor, say g_{ij}, represents in S_3 the quadric surface whose point-equation is

$$g_{ij}X^iX^j=0, \qquad (1.1)$$

and whose tangential equation is

$$g^{ij}X_iX_j=0,$$

where the X_i are current tangential co-ordinates. The quadric (1.1) is, in fact, the section by the 3-flat S_3 of the null cone of T_4, and any point upon this quadric corresponds to a null vector of V_4. The raising and lowering of suffixes by means of the fundamental tensor is represented in S_3 by the taking of poles and polars with respect to the quadric. Thus the pole of a plane ϕ_i has co-ordinates $\phi^i=g^{ij}\phi_j$, and the polar plane of a point X^i has co-ordinates $X_i=g_{ij}X^j$. Two points X^i, Y^i are conjugate with respect to the quadric if $g_{ij}X^iX^j=0$, and two planes ϕ_i, ψ_i are conjugate if $g^{ij}\phi_i\psi_j=0$.

Let ϵ^{ijkl} and ϵ_{ijkl} be the fundamental skew tensors of V_4 defined by

$$\epsilon^{1234}=1/\sqrt{(-g)}, \qquad \epsilon_{1234}=\sqrt{(-g)}.$$

These, it may be noticed, are not obtainable from one another by a simple

raising and lowering of suffixes, but by a raising and lowering of suffixes with a change of sign. Further, let X^i, Y^i be two points in S_3, and let ϕ_i, ψ_i be any two distinct planes through the line joining them. Then the numbers

$$p^{ij}=X^iY^j-Y^iX^j, \qquad \overset{\circ}{p}_{ij}=\phi_i\psi_j-\psi_i\phi_j, \quad . \quad . \quad . \quad (1.2)$$

which in the original space-time are the components of skew-symmetric tensors of a special type, are dual sets of Plücker co-ordinates of the straight line joining the points, and are connected by the relation

$$\rho p^{ij}=\tfrac{1}{2}\epsilon^{ijkl}\overset{\circ}{p}_{kl}, \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

where ρ is a factor depending upon the normalizations of the vectors involved. The co-ordinates satisfy an identity which may be written in the equivalent forms

$$\tfrac{1}{2}\epsilon_{ijkl}p^{ij}p^{kl}\equiv 0, \qquad \tfrac{1}{2}\epsilon^{ijkl}\overset{\circ}{p}_{ij}\overset{\circ}{p}_{kl}\equiv 0, \qquad p^{ij}\overset{\circ}{p}_{ij}\equiv 0. \quad . \quad . \quad (1.4)$$

If Z^i is any point, then the plane through it and the line p^{ij} has co-ordinates $\overset{\circ}{p}_{ij}Z^j$. The point lies on the line if $\overset{\circ}{p}_{ij}Z^j=0$. Similarly a plane χ_i meets the line in the point of co-ordinates $p^{ij}\chi_j$, and passes through it if $p^{ij}\chi_j=0$.

Any skew-symmetric tensor $\overset{\circ}{Q}_{ij}$ of V_4 represents in S_3 a *linear complex* (Baker, 1923; or Sommerville, 1934), that is, the totality of lines whose Plücker co-ordinates satisfy the linear equation

$$\overset{\circ}{Q}_{ij}p^{ij}=0,$$

the dual co-ordinates of the same complex being $Q^{ij}\equiv\rho\epsilon^{ijkl}\overset{\circ}{Q}_{kl}$. Through any point Z^i of S_3 passes a pencil of lines of the complex lying in the plane of co-ordinates

$$\overset{\circ}{Q}_{ij}Z^j, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5)$$

and in any plane χ_i lies a pencil of lines of the complex, the vertex of this pencil having co-ordinates

$$Q^{ij}\chi_j. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.6)$$

The totality of points, lines and planes is a *focal system* or *null system.* The plane (1.5), obtained, so to speak, by lowering the suffix of Z^i by means of the tensor $\overset{\circ}{Q}_{ij}$, is the *polar plane* of the point Z^i with respect to the focal system, and (1.6) is similarly the *pole* of the plane χ_i with respect to the same focal system.

If the co-ordinates $\overset{\circ}{Q}_{ij}$, Q^{ij} of the complex satisfy the identity (1.4), it is called *special.* In this case the $\overset{\circ}{Q}_{ij}$ and Q^{ij} are the Plücker co-ordinates of a line, the *axis* or *directrix* of the complex, which consists of all the lines meeting this axis.

The lines common to two complexes, Q°_{ij} and S°_{ij}, say, form a *linear congruence*. Their Plücker co-ordinates satisfy the two equations $Q^{\circ}_{ij}p^{ij}=0$ and $S^{\circ}_{ij}p^{ij}=0$, and hence also the equation $(\lambda Q^{\circ}_{ij}+\mu S^{\circ}_{ij})p^{ij}=0$ for all λ, μ. Thus the lines of the congruence belong to each member of the *pencil of complexes* $\lambda Q^{\circ}_{ij}+\mu S^{\circ}_{ij}$; and because the substitution of $\lambda Q^{ij}+\mu S^{ij}$ for p^{ij} in (1.4) gives a quadratic equation for λ/μ, there are in general two special complexes belonging to the pencil. The axes of these are the *directrices* of the congruence, which consists of all the lines meeting them.

Although it is not strictly relevant to what follows, we may here notice that a natural interpretation of the covariant Riemann-Christoffel tensor $R_{ij,\,kl}$ is that it represents in S_3 the quadratic complex of equation $R_{ij,\,kl}p^{ij}p^{kl}=0$. The contravariant tensor $R^{ij,\,kl}$ then represents the quadratic complex $R^{ij,\,kl}p^{\circ}_{ij}p^{\circ}_{kl}=0$ polar to it with respect to the quadric g_{ij} (*cf.* Struik, 1927–28; Lamson, 1930).

It must be observed that, although we are confining our attention to the S_3 associated with a particular point O, (x^i), of V_4, any properly covariantive algebraic relation established for tensors at O will hold at all points of V_4 at which those tensors (or rather tensor-fields) are defined.

§ 2. Transformations of Co-ordinates.

Let X^i, ϕ_i be arbitrary contravariant and covariant vectors respectively in the space-time V_4. Then under a transformation of co-ordinates

$$x^i=x^i(x'^j) \qquad (2.1)$$

these vectors become

$$X'^i=U^i_jX^j, \qquad \phi'_i=u^j_i\phi_j, \qquad (2.2)$$

where $U^i_j\equiv \partial x'^i/\partial x^j$ and $u^j_i\equiv \partial x^j/\partial x'^i$. Interpreted in the S_3 belonging to the particular point (x^i) of V_4, the linear equations (2.2) represent a change from one homogeneous co-ordinate system to another; that is, the point co-ordinates X^i become X'^i, and the plane co-ordinates ϕ_i become ϕ'_i.

Now let $h^i_a\equiv(h^i_1, h^i_2, h^i_3, h^i_4)$ be the vectors of a real orthogonal ennuple defined at all points of V_4. We use a, b, c, d for "scalar" or, as we shall call them, *ennuplet* * suffixes, and $i, j, k, \ldots$ for tensor suffixes. Then, as is well known (Eisenhart, 1926),

$$g^{ab}h^i_ah^j_b=g^{ij}, \qquad g_{ij}h^i_ah^j_b=g_{ab}, \qquad (2.3)$$

* The term "scalar" is inappropriate for present purposes because the ennuple will be subject to rotations.

where the constant coefficients g^{ab}, g_{ab} are defined by

$$g_{ab} \equiv \begin{pmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & +1 \end{pmatrix} \equiv g^{ab}.$$

These coefficients take the place of the "indicators" e_a which are frequently employed in connection with orthogonal ennuples, and are notationally distinguished from the tensors g_{ij}, g^{ij} by having ennuplet instead of tensor suffixes. They may, if desired, be used to raise and lower ennuplet suffixes, so that, for example, $g^{ab}h^i_b$ may be written h^{ai}.

Let (h^a_i) be the matrix reciprocal to (h^i_a). Then

$$h^a_i h^j_a = \delta^j_i, \qquad h^a_i h^i_b = \delta^a_b, \qquad h^a_i = g^{ab} g_{ij} h^j_b,$$

and

$$g_{ab} h^a_i h^b_j = g_{ij}, \qquad g^{ij} h^a_i h^b_j = g^{ab}. \quad . \quad . \quad . \quad . \quad (2.4)$$

The $h^a_i (a = 1, \ldots, 4)$ are four real covariant vectors.

Resolving the arbitrary vectors X^i, ϕ_i at every point in the directions of the ennuple, we obtain their *ennuplet components*

$$X^a = h^a_i X^i, \qquad \phi_a = h^i_a \phi_i, \quad . \quad . \quad . \quad . \quad (2.5)$$

which behave like scalars under the co-ordinate transformation (2.1).

But now suppose the ennuple subject to real rotations. That is, suppose we define a new orthogonal ennuple by the equations

$$'h^i_a = l^b_a h^i_b, \qquad 'h^a_i = L^a_b h^b_i, \quad . \quad . \quad . \quad . \quad (2.6)$$

where the l^a_b and L^a_b are real scalar functions of the x^i and have reciprocal matrices, so that $l^a_b L^b_c = \delta^a_c$. Since the $'h$'s form an orthogonal ennuple they satisfy (2.3) and (2.4), from which it at once follows that

$$\left.\begin{array}{ll} g_{ab} l^a_c l^b_d = g_{cd}, & g^{cd} l^a_c l^b_d = g^{ab}; \\ g_{ab} L^a_c L^b_d = g_{cd}, & g^{cd} L^a_c L^b_d = g^{ab}; \\ L^a_b = g^{ac} g_{bd} l^d_c; & \det | L^a_b | = \pm 1. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2.7)$$

Then the components of the vectors X^i, ϕ_i referred to the new ennuple are

$$'X^a = {}'h^a_i X^i, \qquad '\phi_a = {}'h^i_a \phi_i,$$

whence by (2.6),

$$'X^a = L^a_b X^b, \qquad '\phi_a = l^b_a \phi_b. \quad . \quad . \quad . \quad . \quad (2.8)$$

The transformation (2.8) from one set of ennuplet components to another, where the real coefficients L^a_b, l^a_b satisfy (2.7), is a *Lorentz transformation*.

By (2.3) and (2.4) the ennuplet components of the tensors g_{ij}, g^{ij} are,

at all points of space-time, the constants g_{ab}, g^{ab}. By (2.4) and (2.5) we get

$$g_{ij}X^iX^j = g_{ab}X^aX^b,$$

so that in the ennuplet system the equation of the quadric (1.1), in the S_3 associated with any point (x^i) of V_4, becomes

$$g_{ab}X^aX^b \equiv -(X^1)^2-(X^2)^2-(X^3)^2+(X^4)^2=0. \qquad (2.9)$$

The quadratic expression on the left of this equation retains its form under the Lorentz transformation (2.8), since, by (2.7),

$$g_{ab}{}'X^{a\prime}X^b = g_{ab}X^aX^b. \qquad (2.10)$$

In the threefold space S_3 belonging to a given point (x^i) of space-time, equations (2.5), like (2.2), define a transformation from one homogeneous co-ordinate system (X^i, ϕ_i) to another, (X^a, ϕ_a). In general this transformation is *non-holonomic*; that is, the coefficients h^a_i, h^i_a, unlike the U^i_j, u^j_i of (2.2), cannot in general be expressed as partial derivatives of one set of co-ordinates of V_4 with respect to another. The Lorentz transformation (2.8) may likewise be interpreted in S_3 as a transformation of homogeneous co-ordinates, or, perhaps more significantly, as defining a (1, 1) correspondence between points X^a and points $'X^a$ (*i.e.* a collineation) which, by (2.10), leaves the quadric (2.9) invariant.

The 3-flat S_3 is a projective space because its transformation-group is defined by (2.2) and (2.5).

§ 3. Parametric Equations for the Fundamental Quadric.

We continue to consider the projective 3-flat S_3 at infinity in the tangent space T_4 at the point (x^i) of the underlying space-time V_4. In an ennuplet co-ordinate system X^a the equation of the fundamental quadric in S_3 has the form (2.9), namely,

$$g_{ab}X^aX^b \equiv -(X^1)^2-(X^2)^2-(X^3)^2+(X^4)^2=0. \qquad (3.1)$$

This may be written

$$-(X^1+iX^2)(X^1-iX^2)=(X^3+X^4)(X^3-X^4),$$

from which we obtain the equations of the two systems of generators in the usual way. Thus for each value of the parameter p, the equations

$$\left.\begin{aligned}-(X^1+iX^2)&=p(X^3-X^4),\\ X^1-iX^2&=p^{-1}(X^3+X^4)\end{aligned}\right\} \qquad (3.2)$$

represent a pair of planes intersecting in a generator of the one system,

while the equations

$$\left.\begin{aligned} -(X^1 - iX^2) &= q(X^3 - X^4), \\ X^1 + iX^2 &= q^{-1}(X^3 + X^4) \end{aligned}\right\} \qquad (3.3)$$

similarly give a generator of the other system.

If p and q are each given a definite value, real or complex, a generator of each system is specified, and hence their point of intersection. Consequently if we solve (3.2) and (3.3) for the X's, we obtain parametric equations for the quadric. But before doing so we introduce homogeneous parameters (λ^1, λ^2), (μ^{1*}, μ^{2*}) by putting

$$p = \lambda^1/\lambda^2, \qquad q = \mu^{1*}/\mu^{2*},$$

where the asterisk denotes the complex conjugate. (It is for future convenience that the μ's are regarded as conjugates of other complex numbers rather than, so to speak, as complex in their own right.) We then obtain, on solving for the X's and choosing a suitable factor of proportionality, the normalized parametric equations

$$\left.\begin{aligned} X^1 &= (\lambda^1\mu^{2*} + \lambda^2\mu^{1*})/\sqrt{2}, & X^2 &= i(-\lambda^1\mu^{2*} + \lambda^2\mu^{1*})/\sqrt{2}, \\ X^3 &= (\lambda^1\mu^{1*} - \lambda^2\mu^{2*})/\sqrt{2}, & X^4 &= (\lambda^1\mu^{1*} + \lambda^2\mu^{2*})/\sqrt{2}. \end{aligned}\right\} \qquad (3.4)$$

These give

$$\left.\begin{aligned} \lambda^1\mu^{1*} &= (X^3 + X^4)/\sqrt{2}, & \lambda^1\mu^{2*} &= (X^1 + iX^2)/\sqrt{2}, \\ \lambda^2\mu^{1*} &= (X^1 - iX^2)/\sqrt{2}, & \lambda^2\mu^{2*} &= -(X^3 - X^4)/\sqrt{2}, \end{aligned}\right\} \qquad (3.5)$$

from which it follows that the conditions for the X's to be real are

$$\mu^1 = \rho\lambda^1, \qquad \mu^2 = \rho\lambda^2,$$

where ρ is an arbitrary real number. Normalizing by taking $\rho = 1$, we find that the points on the quadric with real co-ordinates X^a are given by

$$\left.\begin{aligned} X^1 &= (\lambda^1\lambda^{2*} + \lambda^2\lambda^{1*})/\sqrt{2}, & X^2 &= i(-\lambda^1\lambda^{2*} + \lambda^2\lambda^{1*})/\sqrt{2}, \\ X^3 &= (\lambda^1\lambda^{1*} - \lambda^2\lambda^{2*})/\sqrt{2}, & X^4 &= (\lambda^1\lambda^{1*} + \lambda^2\lambda^{2*})/\sqrt{2}, \end{aligned}\right\} \qquad (3.6)$$

or equivalently by

$$\left.\begin{aligned} \lambda^1\lambda^{1*} &= (X^3 + X^4)/\sqrt{2}, & \lambda^1\lambda^{2*} &= (X^1 + iX^2)/\sqrt{2}, \\ \lambda^2\lambda^{1*} &= (X^1 - iX^2)/\sqrt{2}, & \lambda^2\lambda^{2*} &= -(X^3 - X^4)/\sqrt{2}. \end{aligned}\right\} \qquad (3.7)$$

Equations (3.6) may be written

$$X^a = g^a_{AB*}\lambda^A\lambda^{B*}, \qquad (a = 1, 2, 3, 4), \qquad (3.8)$$

where the capital-letter suffixes take the values 1, 2 only, and, when repeated, sum over that range. The asterisk following the second suffix in g^a_{AB*} indicates that it will always be associated in summations with complex conjugates. The coefficients g^a_{AB*} are given by

$$\left.\begin{aligned} g^{1}_{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, & g^{2}_{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\\ g^{3}_{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, & g^{4}_{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}\right\} \quad . \quad . \quad (3.9)$$

Equations (3.7) may be written similarly,

$$\lambda^{A}\lambda^{B*}=g_{a}^{AB^*}X^{a}, \quad . \quad . \quad . \quad . \quad . \quad (3.10)$$

where

$$\left.\begin{aligned} g_{1}^{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, & g_{2}^{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\\ g_{3}^{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, & g_{4}^{AB^*}&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}\right\} \quad . \quad . \quad (3.11)$$

We note that the point of S_3 defined by (3.4), which we now write

$$X^{a}=g^{a}_{AB^*}\lambda^{A}\mu^{B*}, \quad . \quad . \quad . \quad . \quad . \quad (3.12)$$

also lies on the quadric, although the X's on the left of this equation are complex unless μ^{A} is a real multiple of λ^{A}. The equation inverse to (3.12) is

$$\lambda^{A}\mu^{B*}=g_{a}^{AB^*}X^{a}. \quad . \quad . \quad . \quad . \quad . \quad (3.13)$$

§ 4. Spinors and Spin Transformations.

The pair of numbers λ^{A}, then, specifies a generator of the quadric in S_3, and the conjugate pair λ^{A*} specifies a generator of the other system. These intersect in the real point whose normalized co-ordinates are given by (3.8), namely,

$$X^{a}=g^{a}_{AB^*}\lambda^{A}\lambda^{B*}. \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

A linear transformation

$$\lambda'^{A}=T^{A}_{B}\lambda^{B}, \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

where the T's are any real or complex numbers of non-zero determinant, may be regarded as defining a transformation of a generator λ^{A} into a generator λ'^{A} of the same system, or, alternatively, as defining a change of parameters for the quadric. Taking the former point of view first, consider the point $'X^{a}$ in which the generator λ'^{A} intersects the generator λ'^{A*}; that is, the point

$$'X^{a}=g^{a}_{AB^*}\lambda'^{A}\lambda'^{B*}. \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

Then by (4.2),

$$\begin{aligned}'X^{a}&=g^{a}_{AB^*}T^{A}_{C}T^{B*}_{D}\lambda^{C}\lambda^{D*}\\ &=L^{a}_{b}X^{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4)\end{aligned}$$

by (3.10), where

$$L^{a}_{b}=g^{a}_{AB^*}T^{A}_{C}T^{B*}_{D}g_{b}^{CD^*}.$$

Since the points $'X^{a}$ and X^{a} are both real and both upon the quadric,

L_b^a defines a real transformation of the quadric into itself; and it is easy to prove that, if the determinant $|T_B^A|$ is equal to unity, then L_b^a is in fact a Lorentz transformation of determinant $+1$ (*cf.* Weyl, 1931, p. 147). Thus the group of complex unimodular transformations T_B^A is isomorphic with the group of real Lorentz transformations of positive determinant, a fact upon which van der Waerden based his original theory of spinors (v. d. Waerden, 1929; see also Whittaker, 1937).

Hereafter, therefore, we assume † that

$$T \equiv \det |T_B^A| = 1. \qquad (4.5)$$

Let (t_B^A) denote the matrix reciprocal to T_B^A, so that

$$t_C^A T_B^C = \delta_B^A, \qquad (4.6)$$

where δ_B^A is a Kronecker symbol, and

$$t \equiv \det |t_B^A| = 1. \qquad (4.7)$$

Now consider the second point of view, in which (4.2) represents a change of parameters for the quadric. Then λ'^A defines not a new point $'X^a$, but the original point X^a, and we have in fact

$$X^a = g'^a_{AB*}\lambda'^A\lambda'^{B*}, \qquad (4.8)$$

where

$$g'^a_{AB*} = g^a_{CD*}t_A^C t_B^{D*}. \qquad (4.9)$$

By (4.4), (4.8), and (4.3),

$$g^a_{AB*} = L_b^a g'^b_{AB*}. \qquad (4.10)$$

This second point of view will be adopted below, and we may therefore say that the transformation (4.2), which is called a *spin transformation* or *change of spin frame*, induces the transformation (4.9) in the coefficients g^a_{AB*}. Any pair of numbers λ^A transforming according to (4.2) is called a *contravariant spin-vector* or *spinor of rank* 1, and a pair of numbers σ_A which under the spin transformation (4.2) become

$$\sigma'_A = t_A^B\sigma_B$$

is called a *covariant spin-vector* or *spinor of rank* 1.

Spinors of higher ranks are similarly defined according to their law of transformation. Thus a mixed spinor, χ_B^A say, transforms like $\lambda^A\sigma_B$, and the coefficients g^a_{AB*} themselves (and likewise g_a^{AB*}) are to be regarded as the components of a spinor (but having one ennuplet as well as the two

† I think it preferable to make this assumption at the beginning rather than to follow other authors in allowing T_B^A to have non-unit determinant and then to make the transformations effectively unimodular by introducing "weighted" spinors, even though this involves a certain loss of mathematical generality.

spin suffixes) which transforms according to (4.9). The matrices (T^A_B) and (t^A_B) are used respectively in connection with ordinary contravariant and covariant spin suffixes, and their complex conjugates with "starred" suffixes.

One point needs to be noticed: a change of spin frame induces a transformation of all spinors simultaneously. Consequently when every generator λ^A of the one system on the quadric in S_3 is transformed (according to our former point of view) into a generator λ'^A of the same system, every generator μ^{A*} of the other system is transformed into a generator $\mu'^{A*}=T^{A*}_B\mu^{B*}$ of the same system. So a real Lorentz transformation of determinant unity corresponds to a simultaneous transformation of each regulus of the quadric into itself. By using the later letters P, Q, . . . of the alphabet in place of the starred suffixes A*, B*, . . ., Veblen (1933) indicates a way of constructing the theory so as to allow for the independent transformation of each regulus into itself; but such a generalization of the theory is unnecessary for the purposes of this paper.

The transformation-theory may therefore be summarized as follows: We are concerned with two kinds of geometric object, namely, ennuplet tensors and spinors. The former are subject to real unimodular Lorentz transformations defined by

$$'X^a=L^a_bX^b,\qquad '\phi_a=l^b_a\phi_b, \qquad (4.11)$$

and the latter to complex unimodular spin transformations defined by

$$\lambda'^A=T^A_B\lambda^B,\qquad \mu'_A=t^B_A\mu_B. \qquad (4.12)$$

The connection between ennuplet tensors and spinors is provided by the "tensor-spinors" g^a_{AB*}, g^{AB*}_a. Under the transformation (4.11) these become

$$'g^a_{AB*}=L^a_bg^b_{AB*},\qquad 'g^{AB*}_a=l^b_ag^{AB*}_b, \qquad (4.13)$$

and under the transformation (4.12) they become

$$g'^a_{AB*}=g^a_{CD*}t^C_At^{D*}_B,\qquad g'^{AB*}_a=g^{CD*}_aT^A_CT^{B*}_D. \qquad (4.14)$$

Under (4.11) and (4.12) simultaneously they transform into

$$\bar{g}^a_{AB*}=L^a_bg^b_{CD*}t^C_At^{D*}_B,\qquad \bar{g}^{AB*}_a=l^b_ag^{CD*}_bT^A_CT^{B*}_D. \qquad (4.15)$$

Thus *Lorentz transformations and spin transformations are made independently of one another*, this being a consequence of our choice of the second of the two possible geometrical interpretations of a spin transformation. But it must be emphasized that, having subjected the g's to a Lorentz transformation (4.13) or to a spin transformation (4.14), we can always find a spin or Lorentz transformation respectively which transforms

them back into themselves. That is, given the L^a_b and l^b_a in (4.15), we can always find t^A_B and T^A_B so that

$$\bar{g}_{AB^*} \equiv g^a_{AB^*}, \qquad \bar{g}^{AB^*}_a \equiv g^{AB^*}_a, \quad . \quad . \quad . \quad (4.16)$$

and *vice versa*. For from (4.14) and (4.15) we get

$$\begin{aligned} \bar{g}_{AB^*} &= L^a_b g'^b_{AB^*}, \\ &= g^a_{AB^*} \end{aligned}$$

by (4.10), provided that L^a_b is the Lorentz transformation induced by the spin transformation T^A_B when the latter is regarded as defining a displacement of generators instead of a change of parameters. Equations (4.16) are analogous to, and in a sense are a disguised form of, the equation

$$\bar{g}_{ab} \equiv g_{ab}, \qquad \text{where} \qquad \bar{g}_{ab} = l^c_a l^d_b g_{cd},$$

which by (2.7) is satisfied by the fundamental ennuplet tensor.

The covariant spinor ϵ_{AB} and the contravariant spinor ϵ^{AB} defined in any spin frame by

$$\epsilon_{AB} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \epsilon^{AB}$$

have the same numerical components in all spin frames. For example, under the spin transformation (4.12), ϵ_{AB} becomes

$$\begin{aligned} \epsilon'_{AB} &= \epsilon_{CD} t^C_A t^D_B \\ &\equiv t\epsilon_{AB} \\ &= \epsilon_{AB} \qquad \text{by (4.7)}. \end{aligned}$$

We have hitherto been considering the 3-flat S_3 at infinity in the tangent space T_4 at a particular point (x^i) of V_4, and have employed in T_4 a particular co-ordinate system in which the equation of the null cone has the simple form (3.1). But we now regard the spinors λ^A, μ_A, etc., and the spin transformations T^A_B, as defined at all points of the underlying space V_4, so that their components are complex functions of the real variables x^i that behave like scalars for transformations of co-ordinates $x \rightarrow x'$. Moreover, we no longer employ the particular ennuplet reference-system X^a in T_4 or S_3, but restore the original system X^i by the transformation inverse to (2.5); and further, as explained above, we allow the coefficients $g^a_{AB^*}$ to undergo spin and co-ordinate transformations, so that an equation such as (4.1) becomes, in its most general form,

$$X^i = g^i_{AB^*}\lambda^A\lambda^{B^*}. \quad . \quad . \quad . \quad . \quad . \quad (4.17)$$

The $g^i_{AB^*}$ are functions of the x^i, but are such that there exist non-holonomic transformations h^a_i and spin transformations t^A_B which convert

them into the constant numerical matrices (3.9). Thus

$$g^i_{AB*} = h^i_a g^a_{CD*} t^C_A t^{D*}_B, \quad . \quad . \quad . \quad . \quad . \quad (4.18)$$

where the g's on the right are the constants (3.9) and the other quantities are functions of the (x^i). Since, as explained above, there exists a Lorentz transformation l^a_b equivalent to a given spin transformation, (4.18) may also be written

$$\begin{aligned} g^i_{AB*} &= h^i_a l^a_b g^b_{AB*} \\ &= {}'h^i_b g^b_{AB*}, \quad \text{say}, \quad . \quad . \quad . \quad . \quad (4.19) \end{aligned}$$

where the $'h$'s are the components of a new ennuple and the g's on the right are the same constants as those on the right of (4.18). Similar formulæ hold for the reciprocal coefficients g^{AB*}_i.

We note incidentally that any properly covariantive relation established for a special reference-system in which the g-spinors have the constant values (3.9), (3.11), holds for all reference-systems.

Until now we have implicitly assumed that, although we are interpreting the components of spinors and ordinary vectors as homogeneous co-ordinates of geometrical objects, any symbol such as X^i or λ^A represents a set of definite (normalized) numbers or functions of the x^i. This we continue to do, but we now remark that any spin-vector λ^A may be subjected to a *gauge transformation*—that is, to a transformation of the type

$$''\lambda^A = e^{i\theta}\lambda^A, \quad . \quad . \quad . \quad . \quad . \quad (4.20)$$

where θ is a real scalar function of the x^i, without affecting the real null vector X^i which, according to the fundamental equation (4.17), is defined by the spinor. Any quantity such as X^i whose components are functions of one or more given spinors will be called a *gauge scalar* if it is unaltered when the spinors undergo given gauge transformations, and a *gauge covariant* if its components transform into the same multiple of themselves. Thus the spinor $H^{AB} \equiv \lambda^A\lambda^B$ is a gauge covariant under the transformation (4.20), since it becomes $''H^{AB} = e^{2i\theta}H^{AB}$.

It is to be noted that the fundamental g- and ϵ-spinors are not subject to gauge transformations. For this reason we choose to regard the latter as on a different footing from spin and co-ordinate transformations, and to be, so to speak, merely a peculiarity of spin-vectors. This point of view is perhaps not altogether logical, but it has the advantage of lending simplicity to the mathematical theory, particularly in connection with the covariant differentiation of spinors (§ 6 below). Geometrically, the possibility of performing gauge transformations may be regarded as a consequence of the homogeneity of the parameterization of the fundamental quadric in S_3, and physically, it may be thought of as the source

of the important *principle of gauge invariance* (Weyl, 1931, pp. 100, 213, 220).

Hereafter, therefore, a symbol representing a spin-vector or a function of spin-vectors will be understood to represent a uniquely defined function or set of functions of the variables x^i; but the possibility of performing gauge transformations will be continuously kept in view.

§ 5. The Algebra of the Fundamental Tensors and Spinors.

We have, then, the relations

$$X^i = g^i_{AB*}\lambda^A\lambda^{B*}, \qquad (5.1)$$

and inversely

$$\lambda^A\lambda^{B*} = g_i^{AB*}X^i, \qquad (5.2)$$

connecting a given spinor λ^A, referred to any spin frame, with a real null vector X^i of space-time. In the special reference-system of § 3 the matrices g^a_{AB*}, g_a^{AB*} are hermitian; hence

$$g^i_{AB*} = (g^i_{BA*})^*, \qquad g_i^{AB*} = (g_i^{BA*})^* \qquad (5.3)$$

in all systems. Again, substituting from (5.2) in (5.1) and *vice versa*, we obtain

$$g^i_{AB*}g_j^{AB*} = \delta^i_j \qquad (5.4)$$

and

$$g^i_{AB*}g_i^{CD*} = \delta^C_A\delta^D_B. \qquad (5.5)$$

Consistency of notation really requires the writing of δ^{D*}_{B*} for the second Kronecker symbol on the right of (5.5), but there is no need to use two notations for the same symbol if we allow the Kronecker delta (and similarly the other real numerically invariant spinors ϵ_{AB}, ϵ^{AB}) to be associated in summations with starred as well as with ordinary spin suffixes.

Tensor suffixes are raised and lowered in the usual way by means of the metric tensor g_{ij}. Spin suffixes are raised and lowered by means of the skew-symmetrical ϵ^{AB}, ϵ_{AB}. Thus, given a contravariant spinor λ^A, then λ_A is defined to mean

$$\lambda_A \equiv \lambda^E\epsilon_{EA}, \qquad (5.6)$$

and, given a covariant spinor μ_A, we define μ^A by

$$\mu^A \equiv \epsilon^{AE}\mu_E. \qquad (5.7)$$

It is to be noted that the summation is with respect to the first suffix of the ϵ-spinor in (5.6), and with respect to the second in (5.7). The two formulæ are consistent because

$$\epsilon^{AE}\epsilon_{BE} = -\epsilon^{AE}\epsilon_{EB} = \delta^A_B. \qquad (5.8)$$

It is important to notice that

$$\lambda_A\mu^A = -\lambda^A\mu_A, \qquad \lambda_A\lambda^A = \epsilon_{AB}\lambda^A\lambda^B \equiv 0. \quad . \quad . \quad . \quad (5.9)$$

The following identities, which are easily proved for the special reference-system of § 3, are of fundamental importance:—

$$g_i^{AB^*} = \epsilon^{AE}\epsilon^{BF}g_{ij}g^j_{EF^*}, \qquad g^i_{AB^*} = g^{ij}g_j^{EF^*}\epsilon_{EA}\epsilon_{FB}, \quad . \quad . \quad (5.10)$$

$$\epsilon^{EF}(g^i_{AE^*}g^j_{BF^*} + g^j_{AE^*}g^i_{BF^*}) = \epsilon_{AB}g^{ij}. \quad . \quad . \quad . \quad (5.11)$$

Formulæ (5.10) state that $g_i^{AB^*}$ and $g^i_{AB^*}$, which have hitherto appeared as the coefficients of the inverse transformations (5.1) and (5.2), are obtainable from one another by raising and lowering suffixes according to the above rules. We shall write g^{AB^*i} for $g^{ij}g_j^{AB^*}$, and so on, but to avoid further complications of notation we shall not raise and lower single spin suffixes of the g-spinors: thus we write ϵ^{EF} explicitly in (5.11) rather than use the notation $g_B{}^{E^*j}$ for $\epsilon^{EF}g^j_{BF^*}$.

Equations (5.11) may be written in other forms. For example, taking the complex conjugate and using (5.3), we obtain

$$\epsilon^{EF}(g^i_{EA^*}g^j_{FB^*} + g^j_{EA^*}g^i_{FB^*}) = \epsilon_{AB}g^{ij}, \quad . \quad . \quad . \quad (5.12)$$

and a further form of (5.11) is

$$g_{AE^*i}g_j^{BE^*} + g_{AE^*j}g_i^{BE^*} = \delta^B_A g_{ij}. \quad . \quad . \quad . \quad . \quad (5.13)$$

From (5.5) we get

$$g^i_{AB^*}g_{CD^*i} = \epsilon_{AC}\epsilon_{BD}. \quad . \quad . \quad . \quad . \quad (5.14)$$

We shall have occasion to use the "tensor-spinor" S^{ij}_{AB} defined by

$$S^{ij}_{AB} \equiv \epsilon^{EF}(g^i_{AE^*}g^j_{BF^*} - g^j_{AE^*}g^i_{BF^*}). \quad . \quad . \quad . \quad (5.15)$$

Evidently this is skew in i, j and symmetric in A, B—that is,

$$S^{ij}_{AB} = -S^{ji}_{AB} = S^{ij}_{BA}. \quad . \quad . \quad . \quad . \quad . \quad (5.16)$$

In the special reference-system of § 3 the S's are given by

$$S^{12}_{AB} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad S^{13}_{AB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S^{14}_{AB} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$S^{23}_{AB} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad S^{24}_{AB} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$S^{34}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

from which we get, for all reference-systems,

$$\tfrac{1}{2}\epsilon_{ijkl}S^{ij}_{AB}S^{kl}_{CD} = 4i(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}), \quad . \quad . \quad . \quad (5.17)$$

$$\tfrac{1}{2}\epsilon_{ijkl}S^{ij}_{AB}S^{kl*}_{CD} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.18)$$

$$S^{ij}_{AB} = \tfrac{1}{2}i\epsilon^{ijkl}S_{ABkl}, \quad . \quad . \quad . \quad . \quad . \quad (5.19)$$

where

$$S_{ABkl} = g_{km}g_{ln}S^{mn}_{AB}.$$

From (5.11) and (5.15) we obtain the important formula

$$\epsilon^{\mathrm{EF}} g^{i}_{\mathrm{AE}*} g^{j}_{\mathrm{BF}*} = \tfrac{1}{2}(\epsilon_{\mathrm{AB}} g^{ij} + \mathrm{S}^{ij}_{\mathrm{AB}}), \quad . \quad . \quad . \quad . \quad (5.20)$$

and from (4.19), (5.4) and (2.4) we quickly get

$$g^{k}_{\mathrm{AB}*} \frac{\partial g^{\mathrm{AB}*}_{k}}{\partial x^{i}} = -g^{\mathrm{AB}*}_{k} \frac{\partial g^{k}_{\mathrm{AB}*}}{\partial x^{i}} = \frac{\partial}{\partial x^{i}} \log \sqrt{(-g)}. \quad . \quad . \quad (5.21)$$

The following simple identity will be useful: If $(\mathrm{M}^{\mathrm{A}}_{\mathrm{B}})$ is any matrix of the second order, then

$$\mathrm{M}^{\mathrm{E}}_{\mathrm{A}} \epsilon_{\mathrm{EB}} + \epsilon_{\mathrm{AE}} \mathrm{M}^{\mathrm{E}}_{\mathrm{B}} \equiv \mathrm{M}^{\mathrm{E}}_{\mathrm{E}} \epsilon_{\mathrm{AB}}. \quad . \quad . \quad . \quad . \quad (5.22)$$

The proof consists of writing the left-hand side in full.

§ 6. Covariant Differentiation of Spinors.

Let Γ^{i}_{jk} denote the Christoffel symbol formed from the g_{ij}. Then by a well-known formula of tensor analysis,

$$\Gamma^{k}_{ki} = \frac{\partial}{\partial x^{i}} \log \sqrt{(-g)}. \quad . \quad . \quad . \quad . \quad (6.1)$$

We now define the *spin connection* $\Gamma^{\mathrm{A}}_{\mathrm{B}i}$ to be

$$\Gamma^{\mathrm{A}}_{\mathrm{B}i} = \tfrac{1}{2} g^{\mathrm{AF}*}_{k} \left(\frac{\partial g^{k}_{\mathrm{BF}*}}{\partial x^{i}} + \Gamma^{k}_{li} g^{l}_{\mathrm{BF}*} \right). \quad . \quad . \quad . \quad . \quad (6.2)$$

Under a co-ordinate transformation $x \rightarrow x'$ and a spin transformation $\mathrm{T}^{\mathrm{A}}_{\mathrm{B}}$, this becomes

$$\bar{\Gamma}^{\mathrm{A}}_{\mathrm{B}i} = \left(\Gamma^{\mathrm{E}}_{\mathrm{F}k} t^{\mathrm{F}}_{\mathrm{B}} + \frac{\partial t^{\mathrm{E}}_{\mathrm{B}}}{\partial x^{k}} \right) \mathrm{T}^{\mathrm{A}}_{\mathrm{E}} \frac{\partial x^{k}}{\partial x'^{i}} ; \quad . \quad . \quad . \quad . \quad (6.3)$$

it is unaffected by a gauge transformation.

Covariant derivatives of spinors and "tensor-spinors" are formed according to the usual rules: the Christoffel symbols are used with tensor suffixes, the spin connection $\Gamma^{\mathrm{A}}_{\mathrm{B}i}$ with ordinary spin suffixes, and its conjugate $\Gamma^{\mathrm{A}*}_{\mathrm{B}i}$ with starred suffixes and with the complex conjugates of spinors; a plus sign goes with contravariant suffixes and a minus sign with covariant. Thus, to take a typical example, the covariant derivative of $g^{i}_{\mathrm{AB}*}$ is

$$g^{i}_{\mathrm{AB}*,j} \equiv \frac{\partial g^{i}_{\mathrm{AB}*}}{\partial x^{j}} + \Gamma^{i}_{kj} g^{k}_{\mathrm{AB}*} - \Gamma^{\mathrm{E}}_{\mathrm{A}j} g^{i}_{\mathrm{EB}*} - \Gamma^{\mathrm{E}*}_{\mathrm{B}j} g^{i}_{\mathrm{AE}*}. \quad . \quad . \quad (6.4)$$

It is evident that the operations of covariant differentiation and of taking the complex conjugate are commutative. That is, if $\mathrm{P}\, \vdots\, \vdots$ is any spinor, where the dots represent suffixes of any types, we have

$$(\mathrm{P}\, \vdots\, \vdots_{\,,j})^{*} = (\mathrm{P}\, \vdots\, \vdots^{\,*})_{,j}. \quad . \quad . \quad . \quad . \quad (6.5)$$

It follows at once from (6.3) that the covariant derivative of a spinor transforms into the covariant derivative of the transformed spinor under a change of co-ordinates or of spin frame. But it is not true that the covariant derivative of a spin-vector, as here defined, is a gauge covariant. For if

$$''\lambda^{A} = e^{i\theta}\lambda^{A}, \qquad . \quad . \quad . \quad . \quad . \quad (6.6)$$

where θ is a scalar, then

$$\begin{aligned} ''\lambda^{A}_{,j} &= e^{i\theta}\lambda^{A}_{,j} + ie^{i\theta}\lambda^{A}\frac{\partial\theta}{\partial x^{j}} \qquad . \quad . \quad . \quad . \quad (6.7) \\ &\neq e^{i\theta}\lambda^{A}_{,j} \end{aligned}$$

in general. But from the operation Δ_j of covariant differentiation we can construct the gauge-covariant operation D_j defined by

$$D_j \equiv \Delta_j - i\phi_j, \qquad . \quad . \quad . \quad . \quad . \quad (6.8)$$

where ϕ_j is any covariant vector which, by definition, undergoes the transformation

$$''\phi_j = \phi_j + \frac{\partial\theta}{\partial x^{j}} \qquad . \quad . \quad . \quad . \quad . \quad (6.9)$$

when the spin-vector is transformed according to (6.6). For we then have

$$\begin{aligned} ''D_j\,''\lambda^{A} &= (\Delta_j - i\,''\phi_j)\,''\lambda^{A} \\ &= e^{i\theta}D_j\lambda^{A} \qquad . \quad . \quad . \quad . \quad . \quad (6.10) \end{aligned}$$

by (6.7) and (6.9). We shall continue to use an extra suffix, preceded by a comma, to denote the covariant differentiation Δ_j.

The spin connection (6.2) is not of course the most general possible one, since any set of functions Γ^{A}_{Bi} with the law of transformation (6.3) would serve. But it is the one which is most appropriate for use in association with the Christoffel symbols and with the fundamental g- and ϵ-spinors; for *the covariant derivatives of the fundamental spinors* $g^{i}_{AB^*}$, $g^{AB^*}_{i}$, ϵ_{AB}, ϵ^{AB}, *with respect to the spin connection* (6.2) *and the Christoffel symbols* Γ^{i}_{jk}, *are all identically zero*. Thus, for example,

$$\begin{aligned} \epsilon_{AB,i} &= \frac{\partial}{\partial x^{i}}\epsilon_{AB} - \Gamma^{E}_{Ai}\epsilon_{EB} - \epsilon_{AE}\Gamma^{E}_{Bi} \\ &= -(\Gamma^{E}_{Ai}\epsilon_{EB} + \epsilon_{AE}\Gamma^{E}_{Bi}) \end{aligned}$$

since the ϵ's are constants, whence, by (5.22),

$$\epsilon_{AB,i} = -\Gamma^{E}_{Ei}\epsilon_{AB}.$$

But from (6.2), (5.21) and (5.4) we get

$$\begin{aligned} \Gamma^{E}_{Ei} &= \tfrac{1}{2}\left\{-\frac{\partial}{\partial x^{i}}\log\sqrt{(-g)} + \Gamma^{k}_{ki}\right\}, \\ &= 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.11) \end{aligned}$$

by (6.1), so that

and similarly
$$\left.\begin{aligned}\epsilon_{AB,i}&=0,\\ \epsilon^{AB}{}_{,i}&=0.\end{aligned}\right\} \qquad (6.12)$$

The fact that the covariant derivatives of the g-spinors are zero follows by a straightforward calculation, and it is easy to prove that Γ^{A}_{Bi}, as defined by (6.2), is the only spin connection with respect to which all the fundamental spinors have zero covariant derivatives. From (6.12) it follows that the operation of raising or lowering a spin suffix, like that of raising or lowering a tensor suffix, is commutative with the operation of covariant differentiation.

§ 7. DIRAC'S EQUATIONS.

Let h denote $1/2\pi$ times Planck's constant, and, as above, take $c=1$. Then putting x^1, x^2, x^3, x^4 for x, y, z, t, we may write the special-relativity Dirac equations in the form † (Weyl, 1931, p. 213; v. d. Waerden, 1929, p. 106)

$$\left.\begin{aligned}\varpi_1\Psi_2-i\varpi_2\Psi_2+\varpi_3\Psi_1+\varpi_4\Psi_1+ik\Psi_3&=0,\\ \varpi_1\Psi_1+i\varpi_2\Psi_1-\varpi_3\Psi_2+\varpi_4\Psi_2+ik\Psi_4&=0,\end{aligned}\right\} \qquad (7.1)$$

$$\left.\begin{aligned}-\varpi_1\Psi_3-i\varpi_2\Psi_3+\varpi_3\Psi_4+\varpi_4\Psi_4+ik\Psi_2&=0,\\ \varpi_1\Psi_4-i\varpi_2\Psi_4+\varpi_3\Psi_3-\varpi_4\Psi_3-ik\Psi_1&=0,\end{aligned}\right\} \qquad (7.2)$$

where $\varpi_i\equiv\dfrac{\partial}{\partial x^i}-\dfrac{ie}{h}a_i$, $(i=1, 2, 3, 4)$, and $k=\dfrac{m}{h}$. Here $(-a_1, -a_2, -a_3)$ is the vector potential and a_4 is the scalar potential of the external field.

Put

$$\Psi_1\equiv\psi^{1*}\equiv\psi_2{}^*,\qquad \Psi_2\equiv\psi^{2*}\equiv-\psi_1{}^*, \qquad (7.3)$$

$$\Psi_3\equiv-\chi^2\equiv\chi_1,\qquad \Psi_4\equiv\chi^1\equiv\chi_2, \qquad (7.4)$$

so that $\psi^{A*}=\epsilon^{AB}\psi_B{}^*$ and $\chi^A=\epsilon^{AB}\chi_B$ in accordance with (5.7). Then if we multiply (7.1) and (7.2) by $1/\sqrt{2}$ and put

$$\kappa=k/\sqrt{2}=m/h\sqrt{2}, \qquad (7.5)$$

it is at once evident from (3.9) that (7.1) and (7.2) may be written respectively

$$g^a_{AB*}\varpi_a\psi^{B*}+i\kappa\chi_A=0,\qquad (A=1, 2),$$
$$g^a_{BA*}\varpi_a\chi^B-i\kappa\psi_A{}^*=0,\qquad (A=1, 2).$$

The extension of these equations to general relativity is now immediate. The operator ϖ_j becomes $\Delta_j-i\phi_j$, where Δ_j is the operator of covariant

† Dirac's ψ's are given in terms of the Ψ's of (7.1) and (7.2) by $(\psi_1, \ldots, \psi_4)=2^{-\frac{1}{2}}(\Psi_1+\Psi_3, \Psi_2+\Psi_4, \Psi_1-\Psi_3, \Psi_2-\Psi_4)$. The second of equations (7.2) is usually written with the sign of each term altered, and is usually placed before the first.

differentiation and

$$\phi_j = ea_j/h, \qquad (7.6)$$

so that Dirac's equations for general relativity are

$$g^j_{\mathrm{AE}*}(\psi^{\mathrm{E}*}_{,j} - i\phi_j\psi^{\mathrm{E}*}) + i\kappa\chi_{\mathrm{A}} = 0, \qquad (7.7)$$

$$g^j_{\mathrm{EA}*}(\chi^{\mathrm{E}}_{,j} - i\phi_j\chi^{\mathrm{E}}) - i\kappa\psi_{\mathrm{A}}{}^* = 0. \qquad (7.8)$$

These have a rather more symmetrical appearance if we take the complex conjugate of (7.8), noting that κ and ϕ_i are real and using (5.3). We then obtain the pair of spinor equations

$$g^j_{\mathrm{AE}*}(\psi^{\mathrm{E}*}_{,j} - i\phi_j\psi^{\mathrm{E}*}) + i\kappa\chi_{\mathrm{A}} = 0, \qquad (7.9)$$

$$g^i_{\mathrm{AE}*}(\chi^{\mathrm{E}*}_{,j} + i\phi_j\chi^{\mathrm{E}*}) + i\kappa\psi_{\mathrm{A}} = 0, \qquad (7.10)$$

which are equivalent to the general-relativity Dirac equations.

If in (7.9) and (7.10) we replace

$$\psi^{\mathrm{A}*} \text{ by } e^{i\theta}\psi^{\mathrm{A}*}, \qquad \chi^{\mathrm{A}} \text{ by } e^{i\theta}\chi^{\mathrm{A}}, \qquad \phi_j \text{ by } \phi_j + \frac{\partial\theta}{\partial x^i} \qquad (7.11)$$

(and therefore ψ^{A} by $e^{-i\theta}\psi^{\mathrm{A}}$, ψ_{A} by $e^{-i\theta}\psi_{\mathrm{A}}$, etc.), where θ is any real scalar, the left-hand sides of the equations are multiplied respectively by $e^{i\theta}$ and $e^{-i\theta}$. Hereafter the phrase *gauge transformation* will always refer to the triple transformation (7.11), and a quantity which is transformed into itself but for a factor $e^{ni\theta}$ will be called a *gauge covariant of degree n* (*gauge scalar* if $n=0$). Thus the left-hand sides of (7.9) and (7.10) are gauge covariants of degrees $+1$ and -1 respectively, and the equations themselves are "gauge invariant" (Weyl). The replacement of ϕ_j by $\phi_j + \partial\theta/\partial x^j$ is of course legitimate, since it merely involves adding a gradient $\frac{h}{e}\frac{\partial\theta}{\partial x^j}$ to the potential 4-vector a_i.

§ 8. The Geometry of Dirac's Equations.

We have seen that a spinor λ^{A} defines a real null vector

$$\mathrm{X}^i = g^i_{\mathrm{AB}*}\lambda^{\mathrm{A}}\lambda^{\mathrm{B}*},$$

and that, at any point (x^i) of the space-time V_4, the components of this null vector may be interpreted as the homogeneous co-ordinates of a point upon a quadric in the S_3 at infinity in the tangent space T_4 at (x^i).

Now Dirac's equations involve two spinors ψ^{A}, χ^{A}, and these define upon the quadric the two real points

$$\xi^i = g^i_{\mathrm{AB}*}\psi^{\mathrm{A}}\psi^{\mathrm{B}*}, \qquad (8.1)$$

$$\eta^i = g^i_{\mathrm{AB}*}\chi^{\mathrm{A}}\chi^{\mathrm{B}*}, \qquad (8.2)$$

ξ^i and η^i being null vectors of V_4. Inversely,

$$\psi^A\psi^{B*}=g_i^{AB*}\xi^i, \qquad \chi^A\chi^{B*}=g_i^{AB*}\eta^i. \qquad (8.3)$$

Through each of the points ξ^i, η^i on the quadric pass two generators, one of each system, and these form a skew quadrilateral of which the other opposite vertices have the complex co-ordinates

$$\xi'^i=g^i_{AB*}\psi^A\chi^{B*}, \qquad \eta'^i=g^i_{AB*}\chi^A\psi^{B*}.$$

In V_4 ξ'^i and η'^i are complex null vectors. By (5.3),

$$\eta'^i=\xi'^{i*},$$

so we represent ξ'^i by ζ^i and η'^i by ζ^{i*}; hence

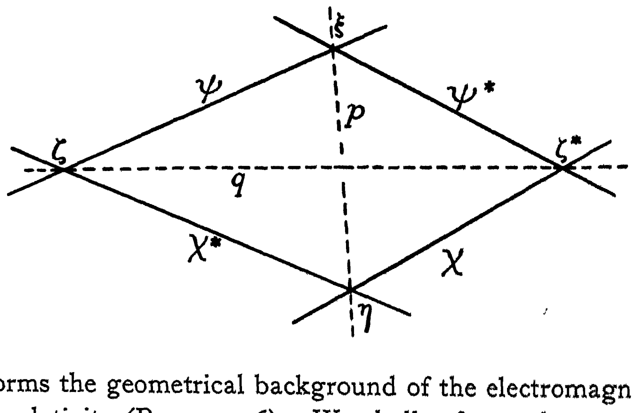
$$\zeta^i=g^i_{AB*}\psi^A\chi^{B*}, \qquad \zeta^{i*}=g^i_{AB*}\chi^A\psi^{B*}, \qquad (8.4)$$

and inversely,

$$\psi^A\chi^{B*}=g_i^{AB*}\zeta^i, \qquad \chi^A\psi^{B*}=g_i^{AB*}\zeta^{i*}. \qquad (8.5)$$

We note that ξ^i, η^i are gauge scalars in the sense of § 7, and that ζ^i, ζ^{i*} are gauge covariants of degrees -2 and $+2$ respectively. ξ^i, η^i, ζ^i, ζ^{i*} are the four null vectors A^i, B^i, $-C^{i*}$, $-C^i$ of Whittaker (1937).

A configuration similar to this, namely, a skew quadrilateral upon a



quadric, forms the geometrical background of the electromagnetic theory in general relativity (Ruse, 1936). We shall refer to the points in S_3 of co-ordinates ξ^i, η^i, ζ^i, ζ^{i*} as *the points* ξ, η, ζ, ζ^*. Also the notation (XY) will be used for the inner product X^iY_i of a contravariant and a covariant vector. Thus

$$(XY)\equiv(YX)\equiv g_{ij}X^iY^j. \qquad (8.6)$$

Since the points ξ, η, ζ, ζ^* all lie upon the quadric g_{ij}, we have

$$(\xi\xi)=0=(\eta\eta)=(\zeta\zeta)=(\zeta^*\zeta^*); \qquad (8.7)$$

and since ζ, ζ^* are conjugate to both ξ and η with respect to the quadric,

we have

$$(\xi\zeta)=0=(\xi\zeta^*)=(\eta\zeta)=(\eta\zeta^*). \quad . \quad . \quad . \quad . \quad . \quad (8.8)$$

But

$$(\xi\eta)\neq 0, \qquad (\zeta\zeta^*)\neq 0.$$

The last three sets of equations, which may, if desired, be established analytically with the help of the formulæ of § 5, mean that each of the null vectors ξ, η, ζ, ζ^* of V_4 is perpendicular to itself and to two of the others, but not to the third.

It is convenient here to introduce the scalar α defined by

$$\alpha \equiv \epsilon_{AB}\psi^A\chi^B \equiv \psi_B\chi^B \equiv -\psi^A\chi_A. \quad . \quad . \quad . \quad (8.9)$$

α and α^*, which are gauge scalars (see (7.11)), are the only two independent quadratic invariants of the Dirac theory which do not involve derivatives (Laporte and Uhlenbeck, 1931*a*, p. 1396). We assume them to be non-zero, and use them in place of the well-known invariants I, J of Darwin (1928*b*, p. 627), which are given by $I=\alpha+\alpha^*$, $J=i(\alpha^*-\alpha)$.

By (8.1) and (8.2),

$$\begin{aligned}(\xi\eta)&=g^i_{AB^*}g_{CD^*i}\psi^A\psi^{B*}\chi^C\chi^{D*}\\&=\epsilon_{AC}\epsilon_{BD}\psi^A\chi^C\psi^{B*}\chi^{D*}, \qquad \text{by (5.14)},\end{aligned}$$

that is,

$$(\xi\eta)=\alpha\alpha^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.10)$$

Similarly

$$(\zeta\zeta^*)=-\alpha\alpha^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.11)$$

Now consider the gauge-scalar 6-vector Q^{ij} defined by

$$Q^{ij}=S^{ij}_{AB}\psi^A\chi^B, \quad . \quad . \quad . \quad . \quad . \quad (8.12)$$

where S^{ij}_{AB} is given by (5.15). Since Q^{ij} is skew in i, j it represents a linear complex in S_3, and from (5.14), (5.8) and (5.9) it quickly follows that

$$Q^{ij}\xi_j=\alpha\xi^i, \qquad Q^{ij}\eta_j=-\alpha\eta^i, \qquad Q^{ij}\zeta_j=\alpha\zeta^i, \qquad Q^{ij}\zeta_j^*=-\alpha\zeta^{i*}. \quad (8.13)$$

Thus ξ^i, η^i, ζ^i, ζ^{i*} are the principal null vectors (*cf.* Ruse, 1936) of the 6-vector Q^{ij}, and α, $-\alpha$ are its principal invariants.

Now in S_3, ξ_i is the polar plane of the point ξ^i with respect to the quadric g_{ij}, and hence, since ξ^i lies upon the quadric, it is the tangent plane $\xi\zeta\zeta^*$ at ξ^i. Also the first of equations (8.13) states that the pole $Q^{ij}\xi_j$ of this plane with respect to the focal system Q^{ij} is the point ξ^i. Therefore the linear complex Q^{ij} contains all the lines in the tangent plane ξ_i that pass through the point ξ^i, and hence in particular contains the generators $\xi\zeta$, $\xi\zeta^*$ of the quadric. A similar argument shows that it also contains the generators $\eta\zeta$, $\eta\zeta^*$ which form the other two sides of the skew quadrilateral. But the four sides of the quadrilateral belong to

the linear congruence $\mathfrak{C}$ having the diagonals $\xi\eta$, $\zeta\zeta^*$ as directrices, and it follows at once that Q^{ij} is a member of the pencil $\mathfrak{P}$ of linear complexes which contain this congruence (*cf.* Ruse, 1936, p. 309).

Further, Q^{ij} represents the complex polar to Q^{ij} with respect to the quadric. But by (5.19), multiplying by $\psi^A\chi^B$, summing for A and B and using (8.12), we get

$$Q^{ij}=\tfrac{1}{2}i\epsilon^{ijkl}Q_{kl}, \qquad (8.14)$$

so that Q^{ij} and Q_{ij} are dual co-ordinates of the same complex. Hence *Q^{ij} is one of the two complexes of the pencil $\mathfrak{P}$ which are self-polar with respect to the quadric g_{ij}.* The other self-polar complex of the pencil is evidently Q^{ij*}, and this is apolar to Q^{ij} since, by (5.18) and (8.12),

$$\tfrac{1}{2}\epsilon_{ijkl}Q^{ij}Q^{kl*}=0.$$

In the space-time V_4 Q^{ij} and Q^{ij*} are self-dual 6-vectors.

Now let

$$p^{ij}=\xi^i\eta^j-\eta^i\xi^j \qquad (8.15)$$

and

$$q^{ij}=i(\zeta^i\zeta^{j*}-\zeta^{i*}\zeta^j). \qquad (8.16)$$

Then since ξ^i and η^i are real, *both p^{ij} and q^{ij} are real*; also they are both gauge-scalars. In S_3 they represent normalized Plücker co-ordinates of the diagonals of the skew quadrilateral; and since these diagonals are the axes of the two special complexes of the pencil $\mathfrak{P}$ to which the complexes Q^{ij} and Q^{ij*} also belong, there exist coefficients λ, μ, ρ, σ such that

$$p^{ij}=\lambda Q^{ij}+\mu Q^{ij*}, \qquad (8.17)$$

$$q^{ij}=\rho Q^{ij}+\sigma Q^{ij*}. \qquad (8.18)$$

To find λ and μ, multiply (8.17) by ξ_j and afterwards by ζ_j, in each case summing for j. From (8.10), (8.7), (8.8), (8.13) and the fact that ξ_j is real, we then get

$$\alpha\alpha^*=\lambda\alpha+\mu\alpha^*, \qquad 0=\lambda\alpha-\mu\alpha^*$$

whence $\lambda=\frac{1}{2}\alpha^*$, $\mu=\frac{1}{2}\alpha$. Thus

$$\xi^i\eta^j-\eta^i\xi^j=p^{ij}=\tfrac{1}{2}(\alpha^*Q^{ij}+\alpha Q^{ij*}), \qquad (8.19)$$

and similarly from (8.18),

$$i(\zeta^i\zeta^{j*}-\zeta^{i*}\zeta^j)=q^{ij}=\tfrac{1}{2}i(-\alpha^*Q^{ij}+\alpha Q^{ij*}). \qquad (8.20)$$

From these equations we obtain

$$Q^{ij}\equiv S^{ij}_{AB}\psi^A\chi^B=(p^{ij}+iq^{ij})/\alpha^*, \qquad Q^{ij*}=(p^{ij}-iq^{ij})/\alpha. \qquad (8.21)$$

Now the diagonals p^{ij}, q^{ij} of the skew quadrilateral are polar to one another with respect to the quadric. But the polar of p^{ij} is obtained by lowering the suffixes i, j by means of the fundamental tensor g_{ij} (*cf.* § 1),

so that q^{ij} and p_{ij} must be dual co-ordinates of the same line. Hence there exists a scalar ρ such that

$$q^{ij}=\tfrac{1}{2}\rho\epsilon^{ijkl}p_{kl}.$$

By (8.19) the right-hand side is equal to $\tfrac{1}{4}\rho\epsilon^{ijkl}(a^*Q_{kl}+aQ^*_{kl})$, and therefore, by (8.14) and (8.20), to ρq^{ij}. Hence $\rho=1$, and we have

$$q^{ij}=\tfrac{1}{2}\epsilon^{ijkl}p_{kl}. \qquad (8.22)$$

To be given the points ξ, η upon the quadric is to be given the generators through them, and hence the points ζ, ζ^*. The co-ordinates of the latter must therefore be expressible as functions of ξ, η, and it is in fact easy to prove from (8.16) that

$$\zeta^i\zeta^{j*}=-\tfrac{1}{2}\{g_{kl}q^{ik}q^{jl}+iaa^*q^{ij}\}/aa^*. \qquad (8.23)$$

From this equation and (8.22) ζ^i may be determined, but for an arbitrary gauge factor, in terms of ξ^i, η^i.

We now consider the 6-vectors ψ^{ij}, χ^{ij} defined by

$$\psi^{ij}=\tfrac{1}{2}S^{ij}_{AB}\psi^A\psi^B, \qquad (8.24)$$

$$\chi^{ij}=\tfrac{1}{2}S^{ij}_{AB}\chi^A\chi^B, \qquad (8.25)$$

which are gauge covariants of degrees -2 and $+2$ respectively. Like Q^{ij}, they represent complexes which are self-polar with respect to the quadric, since by (5.19),

$$\psi^{ij}=\tfrac{1}{2}i\epsilon^{ijkl}\psi_{kl}, \quad \chi^{ij}=\tfrac{1}{2}i\epsilon^{ijkl}\chi_{kl} \qquad (8.26)$$

(*cf.* (8.14) above). Also

$$\tfrac{1}{2}\epsilon_{ijkl}\psi^{ij}\psi^{kl}=\tfrac{1}{8}\epsilon_{ijkl}S^{ij}_{AB}S^{kl}_{CD}\psi^A\psi^B\psi^C\psi^D$$
$$\equiv 0 \text{ by (5.17) and (5.9).}$$

Hence ψ^{ij} represents a special complex, and similarly χ^{ij}. In other words, ψ^{ij} and χ^{ij} are the Plücker co-ordinates of lines which are self-polar with respect to the quadric—that is, they represent generators. Now a simple calculation shows that (*cf.* (8.13))

$$\psi^{ij}\xi_j=0=\psi^{ij}\zeta_j, \qquad \psi^{ij}\eta_j=-a\zeta^i, \qquad (8.27)$$

so that ψ^{ij} is the line of intersection of the planes ξ_i, ζ_i—that is, of the tangent planes to the quadric at the points ξ^i, ζ^i. Therefore it is the generator through these points, and consequently

$$\psi^{ij}=\rho(\xi^i\zeta^j-\zeta^i\xi^j).$$

Multiplying this by η_j, using (8.8), (8.10) and the third of equations (8.27), we get

$$-a\zeta^i=-\rho aa^*\zeta^i,$$

whence $\rho = 1/\alpha^*$. Thus

$$\psi^{ij} \equiv \tfrac{1}{2}S^{ij}_{AB}\psi^A\psi^B = (\xi^i\zeta^j - \zeta^i\xi^j)/\alpha^*. \qquad (8.28)$$

Interchanging ψ^A and χ^A in this formula, we at once obtain

$$\chi^{ij} \equiv \tfrac{1}{2}S^{ij}_{AB}\chi^A\chi^B = -(\eta^i\zeta^{j*} - \zeta^{i*}\eta^j)/\alpha^*, \qquad (8.29)$$

so that χ^{ij} is the side of the skew quadrilateral opposite to ψ^{ij}. The other two sides are ψ^{ij*} (joining ξ, ζ^*) and χ^{ij*} (joining η, ζ).

In the original space-time V_4, ψ^{ij} and χ^{ij} are self-dual 6-vectors whose invariants $\psi^{ij}\psi_{ij}$ and $\chi^{ij}\chi_{ij}$ are zero. The importance of such 6-vectors in the spinor theory has been stressed by Whittaker (1937), who, having obtained equations equivalent to (8.24) and (8.25), pointed out that to a spinor ψ^A corresponds a unique 6-vector ψ^{ij} and *vice versa*.

We conclude this section by observing that the points of co-ordinates

$$j^i = \xi^i + \eta^i, \qquad s^i = \xi^i - \eta^i \qquad (8.30)$$

lie on the diagonal $\xi\eta$ of the skew quadrilateral and are harmonically conjugate with respect to the points ξ, η. By (8.7) and (8.10),

$$j^i j_i = 2\alpha\alpha^* > 0, \qquad s^i s_i = -2\alpha\alpha^* < 0, \qquad j^i s_i = 0.$$

Thus in the original space-time the vector j^i is time-like and s^i is space-like, and the two are perpendicular to one another. The vector

$$J^i = -ej^i\sqrt{2} = -e(\xi^i + \eta^i)\sqrt{2}, \qquad (8.31)$$

where $-e$ is the charge on the electron, is the 4-vector of electric density and electric current density (Darwin, 1928*a*, p. 660; v. Laue, 1933, pp. 96, 97; de Broglie, 1934, p. 225). Its components for the special reference-system of § 3 are easily calculated. For example,

$$\begin{aligned} J^4 &= -e(\psi^1\psi^{1*} + \psi^2\psi^{2*} + \chi^1\chi^{1*} + \chi^2\chi^{2*}), \\ &= -e(\Psi_1\Psi_1^* + \Psi_2\Psi_2^* + \Psi_3\Psi_3^* + \Psi_4\Psi_4^*) \end{aligned}$$

by (7.3) and (7.4).

The vector

$$\sigma^i = hs^i/\sqrt{2} = h(\xi^i - \eta^i)/\sqrt{2} \qquad (8.32)$$

is the *spin 4-vector* (de Broglie, 1934, pp. 219, 220; and see also Laporte and Uhlenbeck, 1931*b*, p. 1552, § 4). I am indebted to Mr C. J. Seelye for calling my attention to this fact.

§ 9. First Tensor Form of Dirac's Equations.

We take Dirac's equations in the form (7.9) and (7.10), namely,

$$\lambda_A \equiv g^i_{AB*}(\psi^{B*}_{,i} - i\phi_i\psi^{B*}) + i\kappa\chi_A = 0, \qquad (9.1)$$

$$\mu_A \equiv g^i_{AB*}(\chi^{B*}_{,i} + i\phi_i\chi^{B*}) + i\kappa\psi_A = 0, \qquad (9.2)$$

where $\kappa = m/h\sqrt{2}$ and $\phi_i = ea_i/h$, a_i being the 4-potential of the external field.

Multiply the former equation by ψ^{A}, summing for A, and take the real part of the resulting equation. By (5.3), (8.9) and the fact that κ and ϕ_i are real, this at once gives

$$\xi^i{}_{,i} = -i\kappa(\alpha^* - \alpha), \qquad (9.3)$$

and (9.2) gives similarly

$$\eta^i{}_{,i} = i\kappa(\alpha^* - \alpha). \qquad (9.4)$$

Adding these equations and using (8.31), we get

$$\text{J}^i{}_{,i} = 0, \qquad (9.5)$$

the usual divergence-equation for the 4-vector of charge and current density.

Now let

$$\Omega^i = g^i_{\text{AB}*}\psi^{\text{A}}\lambda^{\text{B}*}, \qquad (9.6)$$

where $\lambda^{\text{B}} = \epsilon^{\text{BF}}\lambda_{\text{F}}$ and λ_{F} is given by (9.1). Then it is easy to see that, since we are assuming $\alpha \neq 0$, the equation

$$\Omega^i = 0 \qquad (9.7)$$

is true if and only if $\lambda^{\text{B}*} = 0$, and hence if and only if $\lambda_{\text{A}} = 0$. This is practically obvious geometrically (*cf.* the remarks following (9.18) below), and may be proved analytically by using the special reference-system of § 3. Hence (9.7) is a tensor equation equivalent to two of Dirac's. Similarly, if

$$\Lambda^i = g^i_{\text{AB}*}\chi^{\text{A}}\mu^{\text{B}*}, \qquad (9.8)$$

where $\mu^{\text{B}*}$ is given by (9.2), then the relation

$$\Lambda^i = 0 \qquad (9.9)$$

is equivalent to the other pair of Dirac's equations.

Now by (9.1), (9.6) and (5.3),

$$\begin{aligned}\Omega^i &= g^i_{\text{AB}*}\psi^{\text{A}}\epsilon^{\text{BF}}\{g^j_{\text{EF}*}(\psi^{\text{E}}{}_{,j} + i\phi_j\psi^{\text{E}}) - i\kappa\chi^*_{\text{F}}\} \\ &= \tfrac{1}{2}(\epsilon_{\text{AE}}g^{ij} + \text{S}^{ij}_{\text{AE}})(\psi^{\text{A}}\psi^{\text{E}}{}_{,j} + i\phi_j\psi^{\text{A}}\psi^{\text{E}}) - i\kappa g^i_{\text{AB}*}\psi^{\text{A}}\chi^{\text{B}*}\end{aligned}$$

by (5.20) and (5.7). Hence, using (5.9), (8.24), and (8.4), we get

$$\Omega^i = \tfrac{1}{2}g^{ij}\epsilon_{\text{AE}}\psi^{\text{A}}\psi^{\text{E}}{}_{,j} + \tfrac{1}{2}\text{S}^{ij}_{\text{AE}}\psi^{\text{A}}\psi^{\text{E}}{}_{,j} + i\psi^{ij}\phi_j - i\kappa\zeta^i, \qquad (9.10)$$

where ψ^{ij} is given in terms of ξ^i, ζ^i and α by (8.28). Now on account of the symmetry of $\text{S}^{ij}_{\text{AE}}$ in the lower suffixes, and because the covariant derivatives of the fundamental spinors are zero, the second term on the

right is equal to

$$\tfrac{1}{4}(S^{ij}_{AB}\psi^A\psi^B)_{,j}$$
$$=\tfrac{1}{2}\psi^{ij}{}_{,j} \quad . \quad . \quad . \quad . \quad . \quad (9.11)$$

by (8.24).

We now consider the expression $\epsilon_{AB}\psi^A\psi^B{}_{,j}$ appearing in the first term on the right-hand side of (9.10). From the definition (8.9) of a, we have

$$\begin{aligned} a^*.\epsilon_{AB}\psi^A\psi^B{}_{,j} &= \epsilon_{CD}\psi^{C*}\chi^{D*}.\epsilon_{AB}\psi^A\psi^B{}_{,j} \\ &\equiv \epsilon_{AB}\epsilon_{CD}\psi^A\psi^{C*}(\chi^{D*}\psi^B{}_{,j}+\psi^B\chi^{D*}_{,j}) \\ &= \epsilon_{AB}\epsilon_{CD}\psi^A\psi^{C*}(\psi^B\chi^{D*})_{,j}, \end{aligned}$$

the added term in the second line being zero because $\epsilon_{AB}\psi^A\psi^B \equiv 0$. By (8.3) the last expression is equal to

$$\begin{aligned} &\epsilon_{AB}\epsilon_{CD}g^{AC*}_{k}\xi^k(\psi^B\chi^{D*})_{,j} \\ &\qquad = \xi_k g^k_{BD*}(\psi^B\chi^{D*})_{,j} \quad \text{by} \quad (5.10), \\ &\qquad = \xi_k\zeta^k{}_{,j} \end{aligned}$$

by (8.4) and the fact that the covariant derivatives of the g's are zero. Thus

$$\epsilon_{AB}\psi^A\psi^B{}_{,j} = \xi_k\zeta^k{}_{,j}/a^*. \quad . \quad . \quad . \quad . \quad (9.12)$$

Substituting from (9.11) and (9.12) in (9.10), we get

$$\Omega^i = \tfrac{1}{2}\psi^{ij}{}_{,j} + i\psi^{ij}\phi_j + \tfrac{1}{2}g^{ij}(a^{-1})^*\xi_k\zeta^k{}_{,j} - i\kappa\zeta^i. \quad . \quad . \quad (9.13)$$

Similarly

$$\Lambda^i = \tfrac{1}{2}\chi^{ij}{}_{,j} - i\chi^{ij}\phi_j - \tfrac{1}{2}g^{ij}(a^{-1})^*\eta_k\zeta^{k*}{}_{,j} - i\kappa\zeta^{i*}. \quad . \quad . \quad (9.14)$$

Here, by (8.28) and (8.29),

$$\psi^{ij} = (\xi^i\zeta^j - \zeta^i\xi^j)/a^*, \qquad \chi^{ij} = -(\eta^i\zeta^{j*} - \zeta^{i*}\eta^j)/a^*. \quad . \quad . \quad (9.15)$$

We thus have the theorem: *The tensor equations*

$$\Omega^i = 0, \qquad \Lambda^i = 0, \quad . \quad . \quad . \quad (9.16)$$

where Ω^i *and* Λ^i *are defined in terms of the fundamental null vectors* ξ^i, η^i, ζ^i, ζ^{i*} *by* (9.13), (9.14) *and* (9.15), *are completely equivalent to Dirac's equations.*

This theorem (with $\phi_i = 0$) was first obtained by E. T. Whittaker from the special-relativity spinor theory, and was given by him in the second of the courses of lectures mentioned at the beginning of this paper.

It is to be noticed that the four equations of Dirac are now replaced by the eight tensor equations (9.16), but it is easy to show that the latter satisfy the four identities

$$\xi_i\Omega^i = 0, \qquad \zeta_i\Omega^i = 0, \qquad \eta_i\Lambda^i = 0, \qquad \zeta^*_i\Lambda^i = 0, \quad . \quad . \quad (9.17)$$

so that their number is effectively reduced to four. Interpreted geometrically in S_3, Dirac's equations $\Omega^i = 0$, $\Lambda^i = 0$ mean that the points of co-ordinates Ω^i, Λ^i, which have an analytical existence in virtue of their

definition (9.13), (9.14), do not, so to speak, exist physically. Formulæ (9.17) mean that the points lie respectively upon the lines of intersection of the pairs of planes ξ_i, ζ_i and η_i, ζ_i^*—that is, upon the generators $\xi\zeta$ and $\eta\zeta^*$ which form opposite sides of the skew quadrilateral on the fundamental quadric.

We now observe that *the four equations*

$$\Omega^i - \Lambda^i = 0 \quad . \quad . \quad . \quad . \quad . \quad (9.18)$$

are also completely equivalent to those of Dirac. This is almost obvious geometrically, since (9.18) states that the points Ω^i and Λ^i, which lie upon the non-intersecting opposite sides of the skew quadrilateral, are nevertheless the same point because their homogeneous co-ordinates Ω^i and Λ^i are equal. This is absurd unless the points are both non-existent—that is, unless Ω^i and Λ^i are both zero. Hence (9.18) implies (9.16). Analytically, we see by returning to the original definitions (9.6) and (9.8) of Ω^i and Λ^i that (9.18) is the same as

$$g^i_{\text{AB}*}(\psi^{\text{A}}\lambda^{\text{B}*} - \chi^{\text{A}}\mu^{\text{B}*}) = 0. \quad . \quad . \quad . \quad . \quad (9.19)$$

These are four homogeneous linear simultaneous equations for the $\lambda^{\text{A}*}$ and $\mu^{\text{A}*}$, and it is easily shown by transferring to the special reference-system of § 3 that the determinant of the coefficients is $i\alpha^2/\sqrt{(-g)}$, which is not zero by hypothesis. Hence the only solution of the equations is $\lambda^{\text{A}*} = 0$, $\mu^{\text{A}*} = 0$, and these are Dirac's equations.

The last theorem (with $\phi_i = 0$) is given by Whittaker in the paper already quoted. But it may be observed that Ω^i and Λ^i are gauge covariants of degrees -2 and $+2$ respectively, so that (9.18) is not properly gauge invariant in the sense of Weyl. A gauge-invariant tensor equation equivalent to Dirac's can, however, be formed by taking $\text{A}\Omega^i - \Lambda^i = 0$ instead of (9.18), where A is any non-zero function which is a scalar for co-ordinate and spin transformations, and a gauge covariant of degree 4 for the transformation (7.11).

§ 10. Second Tensor Form of Dirac's Equations.

In this section we obtain another set of tensor equations which are equivalent to Dirac's, and which will be of use in § 11.

Again taking Dirac's equations in the form

$$\lambda_{\text{A}} = 0, \qquad \mu_{\text{A}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (10.1)$$

given by (9.1) and (9.2), consider the relations

$$\Theta^i \equiv g^i_{\text{AB}*}(\chi^{\text{A}}\lambda^{\text{B}*} + \psi^{\text{A}}\mu^{\text{B}*}) = 0. \quad . \quad . \quad . \quad (10.2)$$

The determinant of these, regarded as linear simultaneous equations

for the λ's and μ's, has the same value $i\alpha^2/\sqrt{(-g)}$ as that of (9.19), and is therefore non-zero. Consequently their only solution is $\lambda^{A*}=0$, $\mu^{A*}=0$, so that they are equivalent to Dirac's equations: this, again, being geometrically almost obvious. We note that Θ^i is a gauge scalar.

Substituting from (9.1) and (9.2) in (10.2) and using (5.20), (8.1), (8.2), (8.9), (5.16), we quickly find that

$$\Theta^i=\tfrac{1}{2}(S^{ij}_{AB}\psi^A\chi^B)_{,j}-i\alpha\phi^i+\tfrac{1}{2}g^{ij}\epsilon_{AB}(\chi^A\psi^B{}_{,j}+\psi^A\chi^B{}_{,j})-i\kappa(\xi^i+\eta^i).$$

By (8.21) and a calculation similar to that by which (9.12) was obtained, we then get

$$\Theta^i=\tfrac{1}{2}\{(p^{ij}+iq^{ij})/\alpha^*\}_{,j}-i\alpha\phi^i+\tfrac{1}{2}g^{ij}(\alpha^{-1})^*(\xi_k\eta^k{}_{,j}+\zeta^*_k\zeta^k{}_{,j})-i\kappa(\xi^i+\eta^i), \quad (10.3)$$

where p^{ij}, q^{ij} are the dual 6-vectors (8.15) and (8.16), namely,

$$p^{ij}=\xi^i\eta^j-\eta^i\xi^j, \qquad q^{ij}=i(\zeta^i\zeta^{j*}-\zeta^{i*}\zeta^j). \quad . \quad . \quad . \quad (10.4)$$

Hence *the four tensor equations*

$$\Theta^i=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.5)$$

where Θ^i *is given in terms of the fundamental null vectors by* (10.3) *and* (10.4), *are equivalent to those of Dirac.*

It is interesting to notice that equations (10.5) involve the vector $\xi^i+\eta^i$, which by (8.31) is equal to $-J^i/e\sqrt{2}$.

§ 11. The 6-vector of Electric and Magnetic Moment.

By (10.5),

$$\Theta^i-\Theta^{i*}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.1)$$

If we calculate Θ^{i*} from (10.3), noting that ξ^i, η^i, p^{ij}, q^{ij}, and κ $(=m/h\sqrt{2})$ are real, it is evident that the left-hand side of (11.1) contains the term $-2i\kappa(\xi^i+\eta^i)$, or $miJ^i/(eh)$. On solving for J^i we get

$$J^i=V^i+M^{ij}{}_{,j}, \quad . \quad . \quad . \quad . \quad . \quad (11.2)$$

where

$$V^i=\mu i g^{ij}\left\{\left(\frac{1}{\alpha^*}-\frac{1}{\alpha}\right)\xi_k\eta^k{}_{,j}+\frac{1}{\alpha^*}\zeta^*_k\zeta^k{}_{,j}-\frac{1}{\alpha}\zeta_k\zeta^{k*}{}_{,j}\right\}+e^2(\alpha+\alpha^*)a^i/m, \quad . \quad (11.3)$$

$$M^{ij}=\mu i\left\{\frac{1}{\alpha^*}(p^{ij}+iq^{ij})-\frac{1}{\alpha}(p^{ij}-iq^{ij})\right\}. \quad . \quad . \quad . \quad . \quad . \quad (11.4)$$

Here $\mu=eh/(2m)$ and a^i is as usual the 4-potential of the external field.

I am grateful to Mr C. J. Seelye for an independent verification of the last three formulæ, which give in a completely tensorized form the well-known decomposition of the electric current discovered by Gordon and Darwin (see, *e.g.*, Darwin, 1928*b*; or v. Laue, 1933, p. 98). The 4-vector V^i and the 6-vector M^{ij} are obviously both real and both gauge

scalar. An equation equivalent to (11.4) was discovered by Schouten (1931, p. 262, (93)).

Since V^i is a contravariant vector it represents a point in the threefold space S_3, but it appears to have no connection of particular interest with the skew quadrilateral already described. This, no doubt, is because it involves covariant derivatives which can be interpreted only in terms of displacements of the tangent spaces T_4 of the space-time V_4, whereas the quadrilateral was obtained from the algebraic relationships between tensors at a particular point of V_4.

The tensor M^{ij} is skew in i, j, and therefore represents a linear complex in S_3. Moreover, from (11.4) and (8.21) we obtain

$$M^{ij} = \mu i(Q^{ij} - Q^{ij*}), \qquad (11.5)$$

or equivalently,

$$M^{ij} = \mu i(S^{ij}_{AB}\psi^A\chi^B - S^{ij*}_{AB}\psi^{A*}\chi^{B*}), \qquad (11.6)$$

a formula which gives M^{ij} in terms of the original spinors ψ^A, χ^B (*cf.* Laporte and Uhlenbeck, 1931*a*, pp. 1394–5). Now we saw in § 8 that Q^{ij} and Q^{ij*} represent members of the pencil $\mathfrak{P}$ of linear complexes which contain all the lines, including the sides of the skew quadrilateral, that belong to the linear congruence $\mathfrak{C}$ having the diagonals p^{ij}, q^{ij} as directrices. Consequently from (11.5), or equally well from (11.4), it follows that the complex M^{ij} is also a member of the pencil $\mathfrak{P}$. Its dual co-ordinates are

$$M^{\circ}_{ij} = \tfrac{1}{2}\epsilon_{ijkl}M^{kl}$$

or

$$M^{\circ}_{ij} = -\mu(Q_{ij} + Q^{*}_{ij}) \qquad (11.7)$$

by (11.5) and (8.14).

It follows from (11.5) that

$$M^{ij}\xi_j = \mu i(Q^{ij}\xi_j - Q^{ij*}\xi_j)$$
$$= -\mu i(\alpha^* - \alpha) \qquad \text{by (8.13)},$$

whence, putting $\lambda = \mu i(\alpha^* - \alpha)$ and lowering the suffix i, we get

Similarly
$$\left.\begin{aligned} (M_{ij} + \lambda g_{ij})\xi^j &= 0. \\ (M_{ij} - \lambda g_{ij})\eta^j &= 0, \\ (M_{ij} + \lambda' g_{ij})\zeta^j &= 0, \\ (M_{ij} - \lambda' g_{ij})\zeta^{j*} &= 0, \end{aligned}\right\} \qquad (11.8)$$

where $\lambda' = -\mu i(\alpha + \alpha^*)$. But $\alpha + \alpha^*$ and $i(\alpha^* - \alpha)$ are respectively the invariants I, J of Darwin, so that

$$\lambda = \mu J, \qquad \lambda' = -\mu i I,$$

and from (11.8) we have the theorem:

The four null directions defined by ξ^i, η^i, ζ^i, ζ^{i*} are the principal

directions of the 6-vector M^{ij} of electric and magnetic moment, and μJ, $-\mu i I$ are its principal invariants.

Equations (11.8) may be compared with equations (4.7) of the earlier paper already quoted (Ruse, 1936). Geometrically they mean that the polar planes of the four points ξ, η, ζ, ζ^* with respect to the focal system M_{ij} are respectively the same as their polar planes with respect to the quadric g_{ij}, and are therefore the tangent planes to the quadric at the four points.

§ 12. The Momentum-Energy Tensor.

We conclude by giving expressions for the momentum-energy tensor of the Dirac theory in terms of the fundamental null vectors.

Let

$$W^i_j \equiv g^i_{AB^*}\psi^A(\psi^{B^*}_{,j} - i\phi_j\psi^{B^*}) - g^i_{AB^*}\chi^A(\chi^{B^*}_{,j} + i\phi_j\chi^{B^*}), \quad . \quad . \quad (12.1)$$

where, as usual, $\phi_i = ea_i/h$. Then the material part of the momentum-energy tensor is

$$T_{ij} = \tfrac{1}{2}(t_{ij} + t_{ji}), \quad . \quad . \quad . \quad . \quad . \quad (12.2)$$

where t_{ij} is defined by

$$t^i_j = -ih(W^i_j - W^{i*}_j)/\sqrt{2}, \quad . \quad . \quad . \quad . \quad (12.3)$$

(*cf.* Infeld and v. d. Waerden, 1933, p. 23; Weyl, 1931, p. 218). It can be proved by using Dirac's equations (7.9) and (7.10) that

$$T^{ik}{}_{,k} = F^{ik}J_k, \quad . \quad . \quad . \quad . \quad . \quad (12.4)$$

where $F_{ij} = a_{i,j} - a_{j,i}$ is the 6-vector of electric and magnetic force.

By (8.1) and (8.2),

$$W^i_j = g^i_{AB^*}(\psi^A\psi^{B^*}_{,j} - \chi^A\chi^{B^*}_{,j}) - i(\xi^i + \eta^i)\phi_j. \quad . \quad . \quad . \quad (12.5)$$

Now it is easy to prove by methods similar to those of §§ 9, 10 that

$$g^i_{AB^*}\psi^A\psi^{B^*}_{,j} = \frac{1}{2\alpha\alpha^*}\zeta^i\xi_k\zeta^{k*}_{,j} + \frac{1}{2\alpha^*}\zeta_k\psi^{ki*}_{,j}, \quad . \quad . \quad . \quad (12.6)$$

$$g^i_{AB^*}\chi^A\chi^{B^*}_{,j} = \frac{1}{2\alpha\alpha^*}\zeta^{i*}\eta_k\zeta^k_{,j} - \frac{1}{2\alpha^*}\zeta^*_k\chi^{ki}_{,j}, \quad . \quad . \quad . \quad (12.7)$$

where ψ^{ki} and χ^{ki} are given by (9.15). From these equations and (12.2), (12.3), (12.5), we obtain the momentum-energy tensor T_{ij} in terms of the null vectors ξ^i, η^i, ζ^i, ζ^{i*}. The formula thus obtained may be reduced to the simpler form

$$\begin{aligned} T_{ij} = \frac{hi\sqrt{2}}{4\alpha\alpha^*}\{&\xi_i(\zeta_k\zeta^{k*}_{,j} + \alpha^*\alpha_{,j}) + \eta_i(\zeta_k\zeta^{k*}_{,j} + \alpha\alpha^*_{,j}) \\ &+ (\zeta_i\zeta^*_k - \zeta^*_i\zeta_k)(\xi^k + \eta^k)_{,j} \\ &+ \text{same with } i, j \text{ interchanged}\}, \quad . \quad . \quad (12.8) \end{aligned}$$

a result which is due to Mr C. J. Seelye.

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VIII.—On the Ciliary Currents on the Gills of Some *Tellinacea* (Lamellibranchiata). By Alastair Graham, M.A., B.Sc.

(From the Department of Zoology, Birkbeck College, University of London.) *Communicated by* CHARLES H. O'DONOGHUE, D.Sc.
(With Two Figures.)

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I HAVE previously (Graham, 1934) noted the fact that in the bivalve mollusc *Solecurtus scopula* the structure of the outer gills is peculiar in that they possess no groove along the free edge as do the inner and the great majority of lamellibranch gills. In *Solecurtus* (fig. 1) the gills are rather long and narrow, the outer shorter than the inner on each side, leaving a small triangular area of the latter exposed at the anterior end as well as a narrow strip along the ventral margin. The outer gill is provided with a supra-axial extension reaching to the dorsal part of the body on each side. The lamellæ are plicate, and the filaments lying on the summits of the folds on the outer gill project at the free edge further than do those at the bottom of the folds, so that the entire gill has a wavy or scalloped edge. It was of interest to investigate the effect of this departure from the normal structure of the gill on the ciliary mechanisms found there.

One valve and the underlying pallial fold were removed from a living specimen of *Solecurtus scopula* so as to expose the gills, and the ciliary currents were examined with the binocular microscope with the help of a suspension of carmine particles. The results of the examination are shown in fig. 1. On the inner gills (i.g.) the currents beat towards the free edge, along which an orally directed current is maintained. On the supra-axial extension of the outer lamella (s.) and on the outer lamella of the outer gill itself (o.g.) the ciliary currents beat all towards the free edge of the gill; on the inner lamella the current runs away from the free edge towards the ctenidial axis. Particles are therefore carried ventrally from the attachment of the supra-axial extension towards the free edge, are swept round that on to the surface of the inner lamella, and thence travel dorsally again towards the axis of the gill. Should the inner and the outer gills be in contact with each other, many of the particles passing down the outer lamella of the outer gill

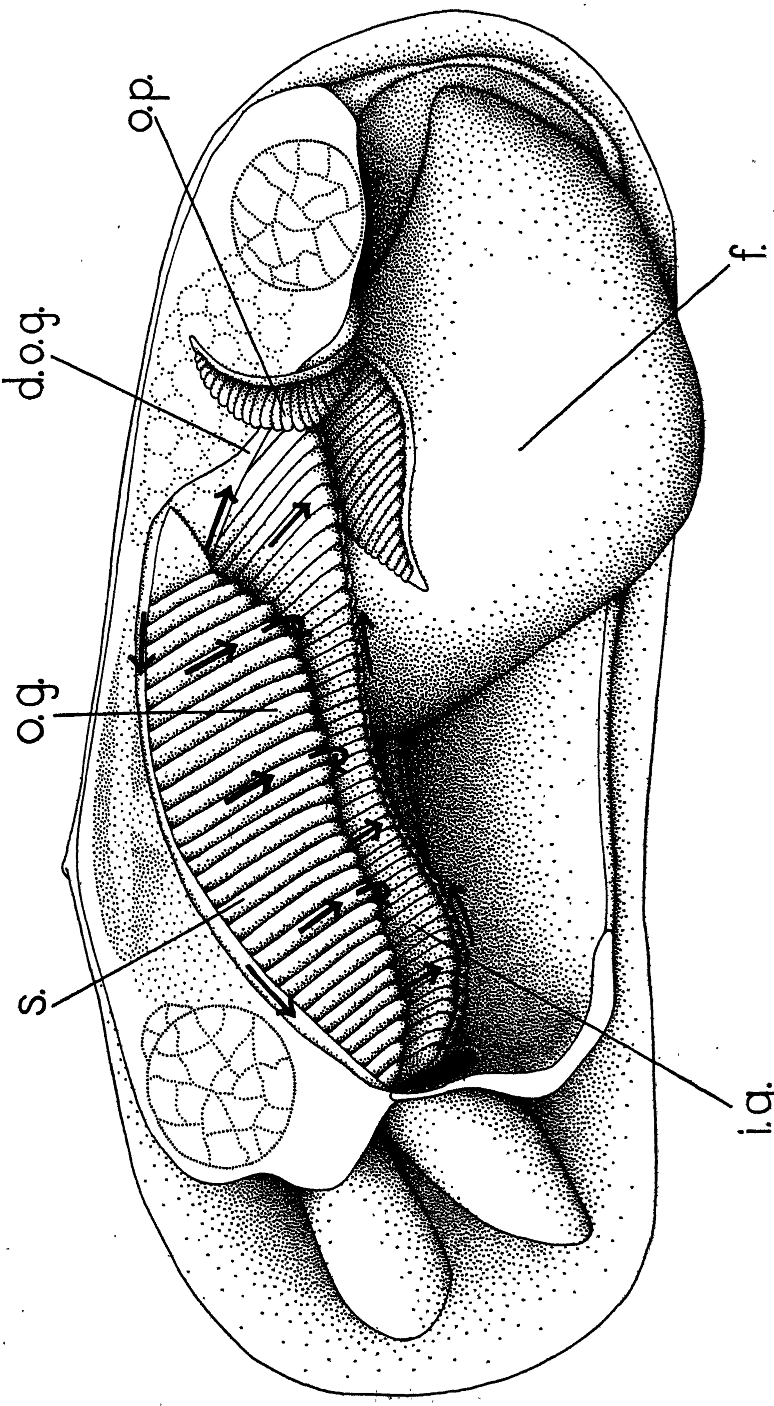

FIG. 1.—*Solecurtus scopula.* The right valve and mantle fold have been removed. Arrows show the direction of the ciliary currents. d.o.g.: distal oral groove; f.: foot; i.g.: inner gill; o.g.: outer lamella of the outer gill; o.p.: outer palp; s.: supra-axial extension of the outer lamella of the outer gill. (× 5.)

never undergo the journey dorsally to the ctenidial axis, but are transferred directly on to the outer lamella of the inner gill.

An orally directed current runs along the ctenidial axis and bears particles forwards into the distal oral groove (d.o.g.), which runs from the point where the outer gill ends along the dorsal margin of the inner gill to the beginning of the labial palps. Along the line of the attachment of the supra-axial extension to the mantle there is a backwardly directed ciliary stream.

As the argument which I developed previously (Graham, 1934) involved the relationship of *Solecurtus* and the Tellinacea, I have also examined the ciliary mechanisms on the gills of certain members of this group. Of these, *Macoma secta* has been previously described by Kellogg (1915), and the ciliary tracts of *Tellina crassa*, which I have examined, and which are shown in fig. 2, are exactly similar. In these molluscs the outer gill (o.g.) is usually described (Ridewood, 1903) as completely supra-axial, being bent dorsally so that the free edge is attached to the body near the dorsal limit of the mantle cavity, and the originally ventrally directed filaments now run dorsally. In the anterior two-thirds all trace of the outer lamella has vanished, but at the posterior end Ridewood describes some ventrally directed filaments, median to the rest of the gill, which he believes to be part of the outer lamella.

On the inner gill (i.g.) the ciliary currents beat towards the free margin on both lamellæ. On the outer gill (o.g.) all currents beat towards the ctenidial axis (c.t.), along which is situated a strong anteriorly beating tract of cilia. This leads, as in *Solecurtus*, to the distal oral groove (d.o.g.) along the anterior edge of the inner gill, and thence to the palps. A second anteriorly directed tract runs on the body along the dorsal edge of the outer gill and flows into the distal oral groove.

A similar arrangement of the ciliary tracts has been found to occur in the genera *Scrobicularia* and *Gari*.

It will therefore be seen that these lamellibranchs all agree in possessing a peculiar arrangement of the ciliary mechanisms on the outer gills. This may be taken as a further argument, this time of a physiological character, for their close kinship. Of the genera which fall within the group, the only one which I have found described as possessing a different type of ciliation is the genus *Tagelus*, in which *T. californianus* is shown by Kellogg (1915) to have the normal lamellibranch type on the outer gills.

Such observations suggest the possibility of drawing some conclusion regarding the homologies of the outer gill of the Tellinacea. According to Ridewood (1903), the outer gill of *Solecurtus* is the homologue of the

outer gill of other lamellibranchs, and it is provided with a short supra-axial extension of the outer lamella. In *Donax*, and amongst other groups of lamellibranchs, in *Tapes* and *Venus*, the outer gill has the normal orientation in the mantle cavity but has a somewhat better

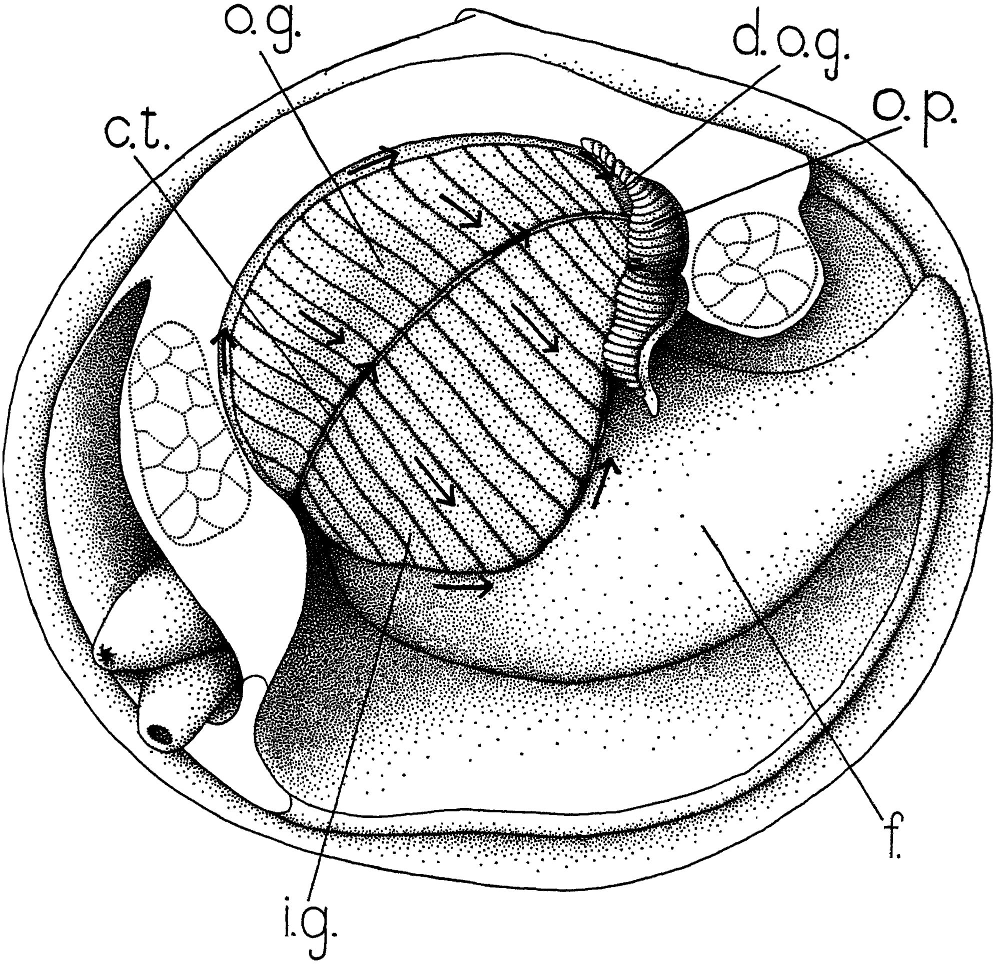

FIG. 2.—*Tellina crassa*. The right valve and mantle fold have been removed. Arrows show the direction of the ciliary currents. c.t.: ctenidial axis; d.o.g.: distal oral groove; f.: foot; i.g.: inner gill; o.g.: outer gill; o.p.: outer palp. (× 4.)

developed supra-axial extension; in *Cardium* the supra-axial extension is larger still, the reduplicated part of the gill being reduced. In *Tellina*, *Scrobicularia*, some members of the Anatinacea, and in *Tridacna*, the outer gill has rotated upwards and lies in an entirely supra-axial position, with the original inner lamella now lying lateral to the original outer lamella, although in most cases this has either completely or in greater

part disappeared. In a later paragraph of the same paper Ridewood offers an alternative explanation of these differing arrangements of the outer gills, which was later strongly supported by Pelseneer (1911). On this view the outer gills of *Scrobicularia* and of *Tellina* are taken to be merely the supra-axial extensions of the outer lamellæ, the rest of the gills having been completely lost by shortening of the filaments and the confluence of the terminal groove with the ctenidial axis.

If it can be assumed that on the primitive eulamellibranch gill the frontal cilia on the filaments beat towards the terminal groove on the free margin, and that the direction of this frontal beat has remained unchanged, then it should be possible to decide between these two alternative interpretations of the nature of the outer gill. As has been shown above, the cilia on the outer gill of *Tellina*, *Macoma*, and of *Scrobicularia* beat ventrally towards the ctenidial axis. On Ridewood's first interpretation this involves a reversal of the direction of the ciliary beat; on Pelseneer's it is exactly what should occur. The existence in the posterior third of the gill of *Tellina*, as Ridewood showed, of filaments running ventrally, and interpreted by him as the remains of the reversed outer lamella, does not necessarily indicate any such thing, but is merely a folding of the gill tissue in the only part of the pallial cavity where there is sufficient space for such reduplication to exist.

Starting from this point, the outer gill of *Solecurtus* may be explained as an extension of the doubling process from the posterior part so as to involve the entire length of the gill. This leads to the conclusion that the outer gill of *Solecurtus* is not homologous with the outer gill of other lamellibranchs, but is a new structure formed from the supra-axial extension of the outer lamella. A parallel evolutionary process would appear to have occurred in such members of the Cardiacea and the Anatinacea as have their outer gills arranged in this fashion. I have examined *Cardium edule*, and find there a modification of the ciliary tracts on the outer gill similar to what has been described for *Solecurtus*, while Kellogg (1915) reports similarly on *C. corbis* and *Chama exogyra* among the Cardiacea, and on *Mytilimeria nuttallii* among the Anatinacea. Yonge (1936) describes the ciliation on the gills of *Tridacna* as being similar to what has been said of *Tellina*, and therefore similar to what is found in *Cardium*—a physiological support for the relationship between these forms which he argues on structural grounds. On the other hand, however, this does not apply to *Venus*, in which the ciliary tracts on the outer gill on both lamellæ lead into the terminal groove.

It appears doubtful, nevertheless, if this argument can be carried to its logical conclusion, owing to the questionable validity of one of the

premises: the assumption that the direction of the ciliary beat has remained unchanged. It is difficult to imagine how a reversal in the direction of the beat of the frontal cilia could possibly have occurred without a sudden mutation or a period of chaotic upset in the feeding of the animal, especially if the cilia possess the complex structure described by Carter (1924) for the cilia of the gill of *Mytilus*. That such a reversal may actually have taken place, however, appears to be shown by the observations of Wallengren (1905) on *Anodonta*, on the outer gill of which all currents beat away from the free edge, and of Kellogg (1915) on *Unio*, on the outer gill of which the same arrangement is found. The Unionidæ being freshwater members of a different order, in which the outer gills are at times used for the purpose of protecting developing larvæ, may well have specialised along distinct lines. From this we must conclude either that the outer gills of eulamellibranchs are not all strictly homologous, or that there is no constancy of direction in the beat of the frontal cilia, which may be altered to suit the changing conditions under which the animal lives. Caution must be exercised in using arguments based either on the structure or the functions of the gills as indicating relationships between lamellibranchs unless they are supported by arguments based on less plastic systems.

SUMMARY.

The ciliary mechanisms on the gills of *Solecurtus scopula*, *Tellina crassa*, *Scrobicularia plana*, and *Gari tellinella* are described, and are found to be similar in arrangement, supporting the idea of a relationship between the bivalves.

The homologies of the outer gills of the Tellinacea are discussed.

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IX.—The Gravitational Field of a Distribution of Particles Rotating about an Axis of Symmetry.

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§ 1. INTRODUCTION.

THE generalisation to the stationary case of the solution given by Weyl (1918) of Einstein's gravitational equations in a statical axially symmetric universe, has been attempted by various writers. It has not, however, so far been possible to reduce the equations to linear form and thus to find the most general solution. A set of special solutions has been obtained by Lewis (1932), valid in a region free of matter, which contain Weyl's solution as a particular case. They depend upon an arbitrary solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{\partial^2 V}{\partial z^2}+\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

In the first part of the present paper the gravitational equations are considered in the interior of an axially symmetric distribution of particles rotating with constant angular velocity about its axis of symmetry. Solutions of the equations are found depending upon an arbitrary solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

In the second part the field of an infinite rotating cylinder is considered, and the internal solution obtained by the method of the present paper is associated with the external solution given by Lewis. The boundary conditions, namely the continuity of the coefficients of the fundamental form and their first derivatives across the surface of the cylinder, determine uniquely the constants occurring in Lewis's solution in terms of the physical dimensions of the system. It appears that there are two essentially different types of external field of a rotating cylinder, according as the radius of the cylinder is less or greater than a certain critical value. In the first case, the geodesic planes normal to the axis of symmetry are infinite and tend to euclidean planes at infinity; in the second case, these planes are finite and closed. The external field of a rotating

cylinder given by Lewis corresponds to the case where the radius is less than the critical value.

§ 2. Definition of Axial Symmetry.

It is customary to define stationary axially symmetric space-time to be such that by a suitable choice of co-ordinates the fundamental form assumes a certain simple expression. As however, in what follows, we shall allow ourselves to be guided by geometrical intuition in defining the energy tensor of a rotating system of particles, it may be more consistent to give a geometrical definition of axial symmetry, and to deduce the particularisation of the fundamental form from our definition.

Stationary axially symmetric space-time we define as follows. The universe contains a privileged observer O, whose world-line is a time-like geodesic g. The observer O separates space-time into space and time by referring to the 3-spaces S formed by all geodesics at O normal to g as space and to some suitably chosen parameter t defining his position on g as time. The universe is said to be stationary if with the passing of time the observer O detects no change in the intrinsic geometry of the space S. If the parameter t is used as one of the co-ordinates, it follows that the coefficients of the fundamental form must be independent of t. We now say that, in addition to being stationary, the universe is axially symmetric if at any instant there exists in S a privileged geodesic a, passing through O, which is such that at any point of it all directions in S normal to a are intrinsically indistinguishable. We have then at every point of g a privileged geodesic a normal to it. We shall now show that it follows from the definition of axial symmetry that the unit tangent vectors at g to the geodesics a are parallel in Levi-Civita's sense. For suppose this is not the case; then selecting the S, passing through the point O on g, we obtain a cone of directions at O by propagating the unit tangent vectors to the geodesics a parallelly along g to O. Considering the geodesics in S defined by this cone of directions, we obtain a surface in S, containing the privileged geodesic a of S. But this surface would define, at all points of a, privileged directions in S normal to a, namely those tangent to the surface, which contradicts the assumption of axial symmetry.

We may now choose a system of co-ordinates as follows. At an arbitrary point of g we select two unit vectors in S which, with the unit tangent vector to a, form an orthogonal triad. We can use the triad to set up in this S a system of geodesic polar co-ordinates, r being the length of the geodesic joining an arbitrary point to O, θ the angle between the tangent to this geodesic and the tangent to a at O, and ϕ the azimuthal

angle. Propagating the triad parallelly along g, we have defined a triad in every S and r, θ, ϕ, t, can now be used as co-ordinates of space-time. From the definition of axial symmetry it follows at once that the coefficients of the fundamental form must be independent of ϕ. We can show that in the present system of co-ordinates the t-lines will be normal to the r-lines. This is seen by noting that the t-line of a point, the r, θ, ϕ co-ordinates of which are kept fixed, is described by propagating the geodesic radius vector, which is normal to g, parallelly along it, and it is well known that its extremity then describes a curve which is at all points normal to the geodesic radius vector. It can further be shown to follow from the definition of axial symmetry that the t-lines are normal to the θ-lines. Consider the surface Σ in S formed by all the points geodesically equidistant from O, and let this surface intersect a in a point P. The directions of the t-lines at points of S can be projected on to S. Suppose this is done at all points of S lying on Σ. Then since the t-lines are normal to the r-lines the projections will lie in Σ, forming a congruence on Σ. Now, since all directions at P in Σ must be equivalent, this congruence must lie symmetrically about P, and hence must cut the θ-lines orthogonally. It follows that the t-lines are normal to the θ-lines. From these last two results it follows that the product terms $drdt$ and $d\theta dt$ are absent in the fundamental form, which will then be of the form

$$ds^2 = dr^2 + Ad\theta^2 + Bd\phi^2 + Cd\phi dt + Ddt^2$$

where the coefficients are functions or r and θ only.

§ 3. Calculation of Ricci Tensor.

For analytical purposes the co-ordinate system established in the preceding section is not the most convenient. We therefore apply the transformation

$$(3.1) \qquad x^1 = x^1(r, \theta), \qquad x^2 = x^2(r, \theta), \qquad x^3 = \phi, \qquad x^4 = t,$$

thus obtaining the slightly more general form

$$(3.2) \qquad ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{mn}dx^m dx^n, \qquad \begin{pmatrix} \alpha, \beta = 1, 2 \\ m, n = 3, 4 \end{pmatrix}$$

where the $g_{\alpha\beta}$ and the g_{mn} are functions of the co-ordinates x^1 and x^2 only. In what follows, unless the contrary be explicitly stated, it will be understood that Greek indices are to have the range 1, 2 and Roman indices the range 3, 4. The summation convention is adhered to, with the understanding that repeated Greek and Roman indices are to be summed through the ranges 1, 2 and 3, 4 respectively.

We note that the transformation (3.1) can always be chosen so that $g_{11} = g_{22}$ and $g_{12} = 0$. If we suppose this to have been done, we may write (3.2)

$$(3.3) \quad ds^2 = e^{2\psi}(dx^1dx^1 + dx^2dx^2) + l\,dx^3dx^3 + 2m\,dx^3dx^4 - f\,dx^4dx^4.$$

It will be convenient first of all to obtain the expressions for the components of the Ricci tensor of the form (3.2), in tensor form with regard to transformations of the type (3.1). We notice that with respect to this group of transformations the functions g_{mn}, g^{mn}, D, where D is defined by

$$D^2 = -\,|\,g_{mn}\,|,$$

transform like invariants, and that their partial derivatives transform like vectors in the (x^1, x^2)-surfaces. We denote partial differentiation by small Greek suffixes. These suffixes can then be raised and lowered in the usual way with respect to the form

$$(3.4) \quad d\sigma^2 = g_{\alpha\beta}dx^\alpha dx^\beta.$$

Covariant derivation with respect to (3.4) we denote by a comma preceding the suffix, or alternatively by the symbol $\frac{\delta}{\delta x^\alpha}$. For the Christoffel symbols of the form (3.2), containing the indices 3 or 4, we have the formulæ

$$(3.5) \quad \begin{cases} \begin{Bmatrix} l \\ mn \end{Bmatrix} = 0, & \begin{Bmatrix} m \\ \alpha\beta \end{Bmatrix} = 0, \quad \begin{Bmatrix} \alpha \\ \beta m \end{Bmatrix} = 0, \\ \begin{Bmatrix} m \\ \alpha n \end{Bmatrix} = \frac{1}{2}g^{ms}g_{ns\alpha}, & \begin{Bmatrix} \alpha \\ mn \end{Bmatrix} = -\frac{1}{2}g^{\alpha}_{mn}. \end{cases}$$

Substituting these values in the usual expressions for the components of the Ricci tensor in terms of the Christoffel symbols, we obtain, after some simplification, the following formulæ for the non-vanishing components

$$(3.6) \quad \begin{cases} R^{\alpha}_{\beta} = K^{\alpha}_{\beta} + D^{-1}D^{\alpha}_{,\beta} - D^{-1}D^{\alpha}D_{\beta} - \frac{1}{4}g^{\alpha}_{mn}g^{mn}_{\beta}, \\ R^{m}_{n} = \frac{1}{2}D^{-1}\frac{\delta}{\delta x^\alpha}(Dg^{ms}g^{\alpha}_{ns}), \end{cases}$$

where K^{α}_{β} is the Ricci tensor of (3.4). Writing these formulæ explicitly for the form (3.3), we obtain

$$(3.7) \quad \begin{cases} \sqrt{(-g)}R^{\alpha}_{\beta} = D\Delta\psi\delta^{\alpha}_{\beta} + D_{\alpha,\beta} - \frac{1}{4}D^{-1}(l_\alpha f_\beta + l_\beta f_\alpha + 2m_\alpha m_\beta), \\ \sqrt{(-g)}R^{3}_{3} = \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left[\frac{fl_\alpha + mm_\alpha}{D}\right], \\ \sqrt{(-g)}R^{3}_{4} = \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left[\frac{fm_\alpha - mf_\alpha}{D}\right], \\ \sqrt{(-g)}R^{4}_{3} = \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left[\frac{ml_\alpha - lm_\alpha}{D}\right], \\ \sqrt{(-g)}R^{4}_{4} = \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left[\frac{lf_\alpha + mm_\alpha}{D}\right], \\ \sqrt{(-g)}(R^{3}_{3} + R^{4}_{4}) = \Delta D, \end{cases}$$

where Δ is the ordinary Laplacian operator in the variables x^1 and x^2.

§ 4. The Energy Tensor.

We consider now the energy tensor of a distribution of particles rotating round the axis of symmetry. We suppose that from the point of view of the observer O of § 2, the particles are describing the ϕ-lines in the space S. It follows then that the first two components of the unit tangent vector to their world-lines vanish. We furthermore suppose that the particles are describing their paths without mutual interaction, the world-line of each particle being a geodesic in space-time. The energy tensor of such a system of particles is of the form

$$T^r_s = \mu\lambda^r\lambda_s, \qquad (r, s = 1, 2, 3, 4)$$

where μ is the density of the particles and λ^r is the unit tangent vector to their world-lines. In order to satisfy the condition of axial symmetry, it is of course necessary that the density μ be a function of x^1 and x^2 only. In the present case we have $\lambda^1 = \lambda^2 = 0$. If we write $\Omega = \lambda^3/\lambda^4$, we find for the components of the unit tangent vector

$$\lambda^3 = \frac{\Omega}{\sqrt{(f - 2\Omega m - \Omega^2 l)}},$$
$$\lambda^4 = \frac{1}{\sqrt{(f - 2\Omega m - \Omega^2 l)}}.$$

We obtain for the non-vanishing components of the energy tensor the expressions

$$(4.1) \qquad \left\{\begin{aligned} T^3_3 &= \mu\frac{\Omega^2 l + \Omega m}{f - 2\Omega m - \Omega^2 l}, & T^3_4 &= \mu\frac{\Omega^2 m - \Omega f}{f - 2\Omega m - \Omega^2 l}, \\ T^4_3 &= \mu\frac{\Omega l + m}{f - 2\Omega m - \Omega^2 l}, & T^4_4 &= \mu\frac{\Omega m - f}{f - 2\Omega m - \Omega^2 l}. \end{aligned}\right.$$

The vanishing of the divergence of the energy tensor gives the equations

$$(4.2) \qquad T^{rs}_{\cdot\cdot,s} = \frac{\partial\mu}{\partial x^s}\lambda^r\lambda^s + \mu\lambda^r_{\cdot,s}\lambda^s + \mu\lambda^r\lambda^s_{\cdot,s} = 0. \quad (r, s = 1, 2, 3, 4).$$

Since $\lambda^a = 0$, and $\dfrac{\partial\mu}{\partial x^m} = 0$, the first term in (4.2) disappears. Using the fact that $\lambda^a = \lambda_a = 0$ and that λ^m is a function of x^1 and x^2 only, we find that

$$(4.3) \qquad \left\{\begin{aligned} \lambda^m_{,n} &= \lambda^a_{,\beta} = 0, \\ \lambda^m_{,a} &= \frac{\partial\lambda^m}{\partial x^a} + \begin{Bmatrix} m \\ an \end{Bmatrix}\lambda^n, \\ \lambda^a_{,m} &= \begin{Bmatrix} a \\ mn \end{Bmatrix}\lambda^n. \end{aligned}\right.$$

Substituting from (4.3) in (4.2), the latter reduce to the two equations

$$\mu\left\{ {\alpha \atop mn} \right\}\lambda^m\lambda^n = 0 \tag{4.4}$$

which, using (3.5), may be written

$$\mu(f_\alpha - 2\Omega m_\alpha - \Omega^2 l_\alpha) = 0. \tag{4.5}$$

We now suppose that Ω is a constant, and that $\mu \neq 0$, then the equations (4.5) can be integrated at once, giving

$$f - 2\Omega m - \Omega^2 l = \text{constant}. \tag{4.6}$$

It is important to notice that the equation (4.6) holds only when $\mu \neq 0$. We now adopt a new system of co-ordinates with the transformation

$$\bar{x}^1 = x^1, \qquad \bar{x}^2 = x^2, \qquad \bar{x}^3 = x^3 - \Omega x^4, \qquad \bar{x}^4 = x^4. \tag{4.7}$$

This transformation constitutes a change to a rotating system of reference, relative to which the matter is at rest. If the bars be omitted, which may be done without danger of ambiguity, the fundamental form becomes

$$ds^2 = e^{2\psi}(dx^1dx^1 + dx^2dx^2) + Ldx^3dx^3 + 2Mdx^3dx^4 - Fdx^4dx^4, \tag{4.8}$$

where

$$\begin{cases} L = l, \\ M = m + \Omega l, \\ F = f - 2\Omega m - \Omega^2 l. \end{cases} \tag{4.9}$$

If we denote the components of the energy tensor in the new system of co-ordinates by accented letters, we find

$$\begin{cases} 'T^3_3 = T^3_3 - \Omega T^4_3, \\ 'T^3_4 = \Omega T^3_3 + T^3_4 - \Omega^2 T^4_3 - \Omega T^4_4, \\ 'T^4_3 = T^4_3, \\ 'T^4_4 = \Omega T^4_3 + T^4_4, \end{cases} \tag{4.10}$$

the remaining components being zero as before. If we now substitute from (4.1), using (4.9), we find that the quantity Ω disappears from the expressions, and we obtain, finally,

$$\begin{cases} 'T^3_3 = 0, & 'T^3_4 = 0, \\ 'T^4_3 = \mu\dfrac{M}{F}, & 'T^4_4 = -\mu. \end{cases} \tag{4.11}$$

The equation (4.6), yielded by the vanishing of the divergence of the energy tensor, becomes in the present system of co-ordinates

$$F = \text{constant}. \tag{4.12}$$

§ 5. The Gravitational Equations.

Since the fundamental form (4.8) is of exactly the same type as (3.3), we obtain the expressions for the components of the Ricci tensor in the system of co-ordinates of § 4, by writing F, L, M, for f, l, m, in (3.7). We write the gravitational equations in the form

$$R^i_j = -\kappa(T^i_j - \tfrac{1}{2}T\delta^i_j), \qquad (i, j = 1, 2, 3, 4) \tag{5.1}$$

where we may, without ambiguity, omit the accents relating to the present system of co-ordinates. We have, from (4.11),

$$T = T^4_4 = -\mu,$$

and hence we deduce from (5.1),

$$R^3_3 + R^4_4 = 0,$$

and this gives, by (3.7),

$$\Delta D = 0. \tag{5.2}$$

On the basis of this equation we may proceed with the introduction of Weyl's canonical co-ordinates, the application of which to the stationary case was first noted by Lewis (*loc. cit.*). We define a transformation

$$r = D, \qquad z = D', \tag{5.3}$$

where D' is a function of x^1 and x^2 such that

$$D + iD' = f(x^1 + ix^2),$$

which, if (5.2) is satisfied, is always possible. This transformation leaves the fundamental form of the (x^1, x^2)-surfaces in the isothermal form, and hence occasions no change in the expressions for the Ricci tensor. We suppose the transformation to the co-ordinates r, z to have been effected, but we will retain the indicial notation whenever convenient, suffixes 1 and 2 referring to differentiation with respect to r and z respectively. We may then put $D = r$ in all formulæ thus far calculated. For $D_{\alpha, \beta}$ we have

$$D_{1,1} = -\psi_1, \qquad D_{1,2} = -\psi_2, \qquad D_{2,2} = \psi_1. \tag{5.4}$$

Considering now the remaining gravitational equations, if we substitute from (3.7), (4.11), and (5.4) in (5.1), and write for convenience $\rho = \kappa\mu\sqrt{(-g)}$, we obtain after some simplification:

$$\Delta\psi = \frac{1}{4r^2}(L_1F_1 + L_2F_2 + M_1^2 + M_2^2) - \frac{\rho}{2r}, \tag{5.5}$$

$$\psi_1 = -\frac{1}{4r}(L_1F_1 + M_1^2 - L_2F_2 - M_2^2), \tag{5.6}$$

$$\psi_2 = -\frac{1}{4r}(L_1F_2 + L_2F_1 + 2M_1M_2), \tag{5.7}$$

$$\frac{\partial}{\partial x^\alpha}\left[\frac{FM_\alpha - MF_\alpha}{r}\right] = 0, \tag{5.8}$$

$$\frac{\partial}{\partial x^\alpha}\left[\frac{FL_\alpha - LF_\alpha}{r}\right] = -2\rho, \tag{5.9}$$

$$\frac{\partial}{\partial x^\alpha}\left[\frac{ML_\alpha - LM_\alpha}{r}\right] = -2\rho\frac{M}{F}. \tag{5.10}$$

§ 6. Solution of the Gravitational Equations.

We first of all remark that the equations (5.8) to (5.10) are not independent; (5.10) may be deduced from (5.8) and (5.9). To obtain the general solution of the equations we make use of the relation (4.12). Putting

$$F = 1, \tag{6.1}$$

the equations (5.5) to (5.9) become

$$\Delta\psi = \frac{1}{4r^2}(M_1^2 + M_2^2) - \frac{\rho}{2r}, \tag{6.2}$$

$$\psi_1 = -\frac{1}{4r}(M_1^2 - M_2^2), \tag{6.3}$$

$$\psi_2 = -\frac{1}{2r}M_1M_2, \tag{6.4}$$

$$M_{11} + M_{22} - \frac{1}{r}M_1 = 0, \tag{6.5}$$

$$L_{11} + L_{22} - \frac{1}{r}L_1 = -2r\rho. \tag{6.6}$$

We notice that the equation (6.5) expresses the condition of integrability of the equations (6.3) and (6.4). We may therefore attempt to obtain solutions of the equations by choosing a function M satisfying (6.5). The equations (6.3) and (6.4) being integrable then determine ψ. We are then left with the equations (6.2) and (6.6) to determine L and ρ.

The functions L and M, however, are not independent. Owing to the particular choice of the co-ordinate system we have from (5.3)

$$r^2 = FL + M^2, \tag{6.7}$$

and hence by (6.1)

$$L = r^2 - M^2. \tag{6.8}$$

Substituting from (6.8) in (6.6), and using the fact that M satisfies (6.5), the equation (6.6) becomes

$$M_1^2 + M_2^2 = r\rho. \tag{6.9}$$

We may consider this equation to define the density distribution, and then the equations (6.3) to (6.6) are all satisfied. If we now calculate $\Delta\psi$ from (6.3) and (6.4), and substitute the result in (6.2), this equation becomes identical with (6.9), so that it also is satisfied. We have therefore shown that the general solution of the equations depends upon an arbitrary solution of the equation (6.5).

If we write $\phi = x^3$, $ct = x^4$, we have for the fundamental form

$$ds^2 = e^{2\psi}(dr^2 + dz^2) + (r^2 - M^2)d\phi^2 + 2Mcd\phi dt - c^2dt^2, \tag{6.10}$$

where M is any solution of (6.5), ψ is determined from (6.3) and (6.4), and where the density μ is given by the equation

$$\kappa\mu = \frac{1}{r^2}e^{-2\psi}(M_1^2 + M_2^2). \tag{6.11}$$

We cannot deduce solutions for the external field from the present solution by putting $\mu = 0$, for we then see from (6.11) that this implies $M =$ constant, and the resulting solution is trivial, space-time being then galilean. The reason is that the equation (6.1) was deduced from the vanishing of the divergence of the energy tensor on the supposition $\mu \neq 0$, and hence does not necessarily hold when the energy tensor vanishes.

§ 7 The Field of an Infinite Rotating Cylinder.

We now consider the particular solution of the equations which is obtained by supposing M to be a function of r only. The equation (6.5) then reads

$$M_{11} - \frac{1}{r}M_1 = 0,$$

and this yields on integration

$$M = ar^2, \tag{7.1}$$

where a is a constant of integration. Determining the function ψ from (6.3) and (6.4), we find

$$e^{2\psi}=e^{-a^2r^2}.$$

The equation (6.11) for the density now reads

$$\kappa\mu=4a^2e^{-a^2r^2}. \tag{7.2}$$

We obtain therefore the fundamental form

$$ds^2=H(dr^2+dz^2)+Ld\phi^2+2Md\phi dt-Fdt^2, \tag{7.3}$$

where

$$\begin{cases} H=e^{-a^2r^2}, & L=r^2(1-a^2r^2), \\ M=acr^2, & F=c^2. \end{cases} \tag{7.4}$$

The present system of co-ordinates constitutes, as we have seen, a system of reference relative to which the matter composing the cylinder is at rest. We define the angular velocity of the cylinder to be the angular velocity relative to a non-rotating system of reference associated with an observer on the axis of symmetry, using Walker's definition of non-rotating (1935). Walker defines a non-rotating system of reference for an observer to be such that in it the acceleration of a free isolated particle in the neighbourhood of the observer is independent of its velocity. This clearly corresponds to what is meant by a non-rotating system of reference in Newtonian dynamics. We may call such a system a dynamical rest frame for the observer. Walker has shown that for such a system the unit tangent vectors in the direction of the space axes must be defined by Fermi-transport along the world-line of the observer, which, since the world-line in the present instance is a geodesic, reduces to ordinary parallel transport.

Let us denote the unit tangent vector to the r-lines by ξ^i $(i=1, 2, 3, 4)$. Then if ξ^i is transported parallelly along the world-line of the observer the equations

$$\frac{\partial\xi^i}{\partial x^a}\frac{dx^a}{ds}+\begin{Bmatrix} i \\ ab \end{Bmatrix}\xi^a\frac{dx^b}{ds}=0 \quad (i, a, b=1, 2, 3, 4) \tag{7.5}$$

must be satisfied at the origin. We have

$$\begin{cases} \xi^i=(H^{-\frac{1}{2}}, 0, 0, 0), \\ \dfrac{dx^a}{ds}=(0, 0, 0, F^{-\frac{1}{2}}). \end{cases} \tag{7.6}$$

Substituting from (7.6) in (7.5), and using (3.5), we obtain the two equations

$$D^{-2}(FM_1-MF_1)H^{-\frac{1}{2}}F^{-\frac{1}{2}}=0, \tag{7.7}$$

$$D^{-2}(MM_1+LF_1)H^{-\frac{1}{2}}F^{-\frac{1}{2}}=0. \tag{7.8}$$

We now apply the transformation (4.7), where we write Ω' instead of Ω, to the form (7.3), and calculate the expressions on the left-hand side of (7.7) and (7.8) for this new form, obtaining the equations

$$(7.9) \qquad r(c - a\Omega' r^2) = 0,$$

$$(7.10) \qquad \frac{\Omega' + ac}{r} - 2a^2 r\Omega' + a^3 r^3 c^{-1}\Omega'^2 = 0.$$

These equations are satisfied at the origin only if

$$(7.11) \qquad \Omega' = -ac.$$

It may be shown similarly that the unit tangent vector to the z-lines is propagated parallelly along the world-line of the observer, independent of the value of Ω'. We therefore obtain the expression for the fundamental form in a non-rotating system of co-ordinates, by applying the transformation (4.7), where we substitute for Ω the value given by (7.11). When this is done we obtain the form (7.3), where

$$(7.12) \qquad \begin{cases} H = e^{-a^2 r^2}, & L = r^2(1 - a^2 r^2), \\ M = a^3 c r^4, & F = c^2(1 + a^2 r^2 + a^4 r^4). \end{cases}$$

§ 8. Interpretation of the Solution.

We see from (7.11) that the angular velocity ω of the cylinder is given by

$$(8.1) \qquad \omega = ac.$$

If we denote the density of the cylinder on the axis of symmetry by μ_0, we find from (7.2), if we substitute for κ its value in terms of Newton's gravitational constant γ, that

$$(8.2) \qquad a^2 c^2 = 2\pi\gamma\mu_0,$$

and hence we have for the angular velocity

$$(8.3) \qquad \omega = \sqrt{2\pi\gamma\mu_0}.$$

If we suppose the cylinder to be of the density of water on the axis of symmetry, so that $\mu_0 = 1$, the period of rotation of the cylinder is approximately 2 hours 42 minutes. If we denote by R the value of r on the boundary of the cylinder, then we see from (7.12) that we must have

$$aR < 1,$$

otherwise the coefficient of $d\phi^2$ in the fundamental form is negative in the interior of the cylinder. For a cylinder of given density therefore there is an upper limit for the radius. Since $\omega = ac$, the inequality may also be written

$$\omega R < c,$$

so that the upper limit of R is the same as the upper limit of the radius of a rotating cylinder in the special theory of relativity. The quantity R, however, is not the radius of the cylinder but connected with it by the equation

$$R' = \int_0^R e^{-\frac{1}{2}a^2r^2}dr,$$

where R' denotes the radius. If $\mu_0 = 1$ we find that the maximum radius is approximately $3 \cdot 5 \times 10^8$ K.M.

We now consider with what angular velocity a particle is to describe a ϕ-line if its world-line is to be a geodesic in space-time. We write the equations of the geodesics in the Lagrangian form

$$\frac{d}{ds}\frac{\partial T}{\partial x'^i} - \frac{\partial T}{\partial x^i} = 0, \qquad (i = 1, 2, 3, 4)$$

where

$$T = H(r'^2 + z'^2) + L\phi'^2 + 2M\phi' t' - Ft'^2,$$

dashes denoting differentiation with respect to the arc s. We attempt to satisfy these equations by putting r = constant, z = constant. The only pertinent equation is then seen to be

$$\frac{d}{ds}\frac{\partial T}{\partial r'} - \frac{\partial T}{\partial r} = 0,$$

which gives a quadratic for $d\phi/dt$, namely

$$L_1 d\phi^2 + 2M_1 d\phi dt - F_1 dt^2 = 0. \tag{8.4}$$

Substituting from (7.12) in (8.4), and solving the quadratic, we obtain the roots

$$\omega_1 = ac, \tag{8.5}$$

$$\omega_2 = -\frac{1 + 2a^2r^2}{1 - 2a^2r^2}ac. \tag{8.6}$$

The first root gives the angular velocity of the cylinder, verifying that the world-lines of the particles composing the cylinder are geodesics. If we suppose a thin tube hollowed out in the cylinder along a ϕ-line, the second root gives the angular velocity with which a particle must be endowed in order to traverse the tube in a sense contrary to the sense of rotation of the cylinder.

The unit tangent vectors to the world-lines of the particles composing the cylinder must always be time-like. To investigate whether this is the case, we consider the null-directions in the (ϕ, t)-surfaces. These are given by

$$L d\phi^2 + 2M d\phi dt - F dt^2 = 0. \tag{8.7}$$

Solving this quadratic we obtain the values of $d\phi/dt$ corresponding to the null-directions. They are found to be

$$\Omega_1 = \frac{1 - a^3r^3}{1 - a^2r^2}\frac{ac}{ar}, \tag{8.8}$$

$$\Omega_2 = -\frac{1 + a^3r^3}{1 - a^2r^2}\frac{ac}{ar}. \tag{8.9}$$

We see from (8.8) that, for all values of r, we have $\Omega_1 > ac$, and hence the tangent vectors to the world-lines of the particles are always time-like. Comparing (8.6) and (8.9), we see that when $ar = \frac{1}{2}$, we have $\Omega_2 = \omega_2$. Hence when $ar \longrightarrow \frac{1}{2}$, the velocity with which a particle must be projected in order to describe a ϕ-line, in a sense opposite to the sense of rotation of the cylinder, tends to the velocity of light. At points where r exceeds this value it will be impossible for a particle to describe a ϕ-line in this sense. If we consider the case of a cylinder whose density and radius are such that $aR = \frac{1}{2}$, then on the boundary a light signal sent out in the direction of a ϕ-line, and in a sense opposite to the sense of rotation of the cylinder, will travel along the ϕ-line. An observer therefore on the surface of the cylinder will be able to look right round the cylinder. Assuming $\mu_0 = 1$, we find that if the observer is at rest on the surface of the cylinder, a ray of light sent out by him returns in approximately 40 minutes.

Returning to equation (8.6), we see that when $ar = \frac{1}{\sqrt{2}}$, the angular velocity ω_2 becomes infinite. The ϕ-line is then a space-like geodesic. The length l of a ϕ-line is given by

$$l = \int_0^{2\pi} r(1 - a^2r^2)^{\frac{1}{2}} d\phi = 2\pi r(1 - a^2r^2)^{\frac{1}{2}},$$

and we see that l is a maximum when $ar = \frac{1}{\sqrt{2}}$. As r increases beyond this value the length of successive ϕ-lines diminishes. When $ar \longrightarrow 1$ the length of the corresponding ϕ-line $\longrightarrow 0$. If the cylinder is such that $aR = 1$, then all geodesics issuing from the origin in the planes $z = \text{constant}$, meet again on the boundary, which then reduces to a line, the antipodal line of the axis of symmetry. We will return to this point when we consider the external solution.

Before proceeding to the question of the external field and boundary conditions, we may consider the Newtonian analogue of the present solution. If in Newtonian potential theory an infinite liquid cylinder of uniform density is endowed with a constant angular velocity Ω, there

exists a definite value for Ω, which will reduce the pressure everywhere to zero. If μ is the density of the cylinder and γ the gravitational constant, we find for Ω the value

$$\Omega = \sqrt{2\pi\mu\gamma}.$$

The present paper is concerned with the gravitational field of such a cylinder according to the general theory of relativity. We see that we obtain the same value for the angular velocity, but in the present case the density is not constant, it decreases with increasing distance from the axis. If the radius of the cylinder is small, however, the density does not vary very much from its value on the axis of symmetry.

§ 9. Lewis's Solution for the External Field.

In order to complete the solution we must now consider the external field and the boundary conditions. We make use of Lewis's solution for the external field of a rotating cylinder (Lewis, 1932). The form in which Lewis gives this solution is not a convenient one from the point of view of determining the constants occurring in it by means of the boundary conditions. We will therefore here obtain this solution by a different method, and in a form more convenient for our purposes.

We return to the equations (5.5) to (5.10), where we put $\rho = 0$. With Lewis we remark that the conditions of integrability of (5.6) and (5.7), and the condition of compatibility of these equations with (5.5), are contained in the system (5.8) to (5.10). These last three equations are not independent, any one being a consequence of the remaining two. We consider (5.9) and (5.10), and make the substitution

$$(9.1) \quad . \quad . \quad . \quad . \quad . \qquad u = \frac{F}{L}, \qquad v = \frac{M}{L}.$$

Using the relation (6.7), the equations (5.9) and (5.10) become

$$(9.2) \quad . \quad . \quad . \qquad \frac{\partial}{\partial x^\alpha}\left[\frac{r u_\alpha}{u + v^2}\right] = 0, \qquad \frac{\partial}{\partial x^\alpha}\left[\frac{r v_\alpha}{u + v^2}\right] = 0.$$

We attempt to obtain solutions of these equations by putting

$$(9.3) \quad . \quad . \quad . \qquad u_\alpha = \Theta_\alpha(u + v^2), \qquad v_\alpha = \Phi_\alpha(u + v^2).$$

Substituting from (9.3) in (9.2), we see that the functions Θ and Φ must satisfy the equation

$$(9.4) \quad . \quad . \quad . \quad . \quad . \qquad \Theta_{11} + \Theta_{22} + \frac{1}{r}\Theta_1 = 0.$$

Choosing two arbitrary solutions of this equation, we now attempt to obtain u and v from the system of equations (9.3). This system of first

order partial differential equations is complete, that is to say, the conditions of integrability are identically satisfied if

$$\Theta_1\Phi_2 - \Theta_2\Phi_1 = 0,$$

as may easily be verified. This relation involves

$$\Phi = f(\Theta),$$

and since Θ and Φ are both solutions of (9.4), it is easily seen that we have

$$\Phi = A\Theta + B,$$

where A and B are constants. It is then evident from (9.3) that this implies

$$v = Au + B. \tag{9.5}$$

The two sets of equations in (9.3) are then equivalent, and they reduce to an ordinary differential equation which may be written

$$\frac{du}{A^2u^2 + (2AB+1)u + B^2} = d\Theta. \tag{9.6}$$

In the integration of (9.6) three cases arise, according as $4AB+1$ is $>$, $=$ or < 0. In the case $4AB+1 < 0$, we obtain, introducing different constants,

$$-u = \alpha^2 + \beta^2 + 2\alpha\beta \coth 2\Theta,$$
$$-v = \alpha \coth 2\Theta + \beta,$$

and these give

$$\left\{\begin{aligned} r^{-1}L &= \frac{1}{\alpha}\sinh 2\Theta, \\ r^{-1}M &= -\cosh 2\Theta - \frac{\beta}{\alpha}\sinh 2\Theta, \\ r^{-1}F &= -2\beta\cosh 2\Theta - \frac{\alpha^2+\beta^2}{\alpha}\sinh 2\Theta. \end{aligned}\right. \tag{9.7}$$

Substituting from (9.7) in (5.6) and (5.7) we obtain for ψ the equations

$$\left\{\begin{aligned} \psi_1 &= -\frac{1}{4r} + r(\Theta_1^2 - \Theta_2^2), \\ \psi_2 &= 2r\Theta_1\Theta_2. \end{aligned}\right. \tag{9.8}$$

In the case $4AB+1 < 0$, we have similarly

$$\left\{\begin{aligned} r^{-1}L &= \frac{1}{\alpha}\cos 2\Theta, \\ r^{-1}M &= \sin 2\Theta - \frac{\beta}{\alpha}\cos 2\Theta, \\ r^{-1}F &= 2\beta\sin 2\Theta + \frac{\alpha^2-\beta^2}{\alpha}\cos 2\Theta. \end{aligned}\right. \tag{9.9}$$

The equations for ψ are now

$$(9.10) \quad \begin{cases} \psi_1 = -\dfrac{1}{4r} - r(\Theta_1^2 - \Theta_2^2), \\ \psi_2 = -2r\Theta_1\Theta_2. \end{cases}$$

The case $4AB+1=0$ does not, in the present instance, require separate treatment.

§ 10. Boundary Conditions. The Case $aR < \frac{1}{2}$.

The external field of the cylinder is now obtained by choosing a solution of (9.4) which is a function of r alone. We find

$$(10.1) \quad \Theta = n \log\left(\frac{r}{r_0}\right),$$

We consider first of all the solution given by (9.7). Obtaining ψ from (9.8), we find for H

$$(10.2) \quad H = k\left(\frac{r}{R}\right)^{2n^2-\frac{1}{2}}.$$

We suppose that on the boundary of the cylinder we have $r=R$. We write $2\Theta = \theta_0 + \theta$, where

$$(10.3) \quad \begin{cases} \theta_0 = 2n \log\left(\dfrac{R}{r_0}\right), \\ \theta = 2n \log\left(\dfrac{r}{R}\right), \end{cases}$$

so that $\theta=0$ on the boundary. Substituting in (9.7) we equate the values of the coefficients on the boundary with the internal boundary values given by (7.4). If we write for convenience $ct=x^4$, we obtain the equations

$$(10.4) \quad \begin{cases} \dfrac{R}{\alpha}\sinh\theta_0 = R^2(1-a^2R^2), \\ -\cosh\theta_0 - \dfrac{\alpha}{\beta}\sinh\theta_0 = aR, \\ -2\beta\cosh\theta_0 - \dfrac{\alpha^2+\beta^2}{\alpha}\sinh\theta_0 = R^{-1}. \end{cases}$$

We now calculate the derivatives of the coefficients of the form and equate their values on the boundary, obtaining the equations

$$(10.5) \quad \begin{cases} \dfrac{1}{\alpha}(\sinh\theta_0 + 2n\cosh\theta_0) = 2R(1-2a^2R^2), \\ -(\cosh\theta_0 + 2n\sinh\theta_0) - \dfrac{\alpha}{\beta}(\sinh\theta_0 + 2n\cosh\theta_0) = 2aR, \\ -2\beta(\cosh\theta_0 + 2n\sinh\theta_0) - \dfrac{\alpha^2+\beta^2}{\alpha}(\sinh\theta_0 + 2n\cosh\theta_0) = 0. \end{cases}$$

The continuity of H and its derivative across the boundary gives two more equations

(10.6) . . . . . . $$k = e^{-a^2R^2},$$

(10.7) . . . . . $$2n = \sqrt{(1-4a^2R^2)}.$$

The constants k and n are defined by these equations. To satisfy the six remaining equations we have therefore only the three constants α, β, θ_0 at our disposal. We shall show that the equations are consistent. Eliminating θ_0 from the equations (10.4), and again from the equations (10.5), we obtain the two equations

$$2aR^2\beta + R^2(1-a^2R^2)(\beta^2-\alpha^2) = 1,$$
$$2aR^2\beta + R^2(1-2a^2R^2)(\beta^2-\alpha^2) = 0.$$

Solving these two equations for α and β, we find

(10.8) . . . . $$\begin{cases} \beta = -\dfrac{1-2a^2R^2}{2a^3R^4}, \\ \alpha = \dfrac{\sqrt{(1-4a^2R^2)}}{2a^3R^4}. \end{cases}$$

Substituting these values of α and β in (10.4), we find

(10.9) . . . . $$\begin{cases} \sinh\theta_0 = \dfrac{\sqrt{(1-4a^2R^2)}}{2a^3R^3}(1-a^2R^2), \\ \cosh\theta_0 = \dfrac{1-3a^2R^2}{2a^3R^3}. \end{cases}$$

All the constants have now been determined in terms of the physical dimensions of the system. If the values of the constants that we have found are substituted in the remaining equations of (10.4) and (10.5), it will be found that these are also satisfied. Substituting from (10.8) and (10.9) in (9.7), we obtain the explicit expression for the coefficients of the fundamental form of the external region. It will be convenient to introduce a new constant ϵ, defined by

(10.10) . . . . . $$\tanh\epsilon = \sqrt{(1-4a^2R^2)}.$$

We then obtain for the external field the form (7.3), where

(10.11) . . . . . $$H = e^{-a^2R^2}\left(\frac{r}{R}\right)^{-2a^2R^2},$$

(10.12) . . . . . $$L = Rr\frac{\sinh(3\epsilon+\theta)}{2\sinh 2\epsilon\cosh\epsilon},$$

(10.13) . . . . . $$M = rc\frac{\sinh(\epsilon+\theta)}{\sinh 2\epsilon},$$

$$F=\frac{rc^2}{R}\frac{\sinh(\epsilon-\theta)}{\sinh\epsilon}, \tag{10.14}$$

and where θ is defined by

$$\theta=\sqrt{(1-4a^2R^2)}\log\frac{r}{R}. \tag{10.15}$$

The present system of co-ordinates is, as we have seen, to be interpreted as a rotating system. In order to obtain the field in a system of co-ordinates which is a dynamical rest frame for the observer on the axis of symmetry, we must apply the transformation (4.7) where $\Omega=-ac$. When this is done it will be found that for sufficiently great values of r, the coefficient of dt^2 changes its sign. This does not mean that the fundamental form changes its signature, for by (6.7) the determinant of the quadratic (8.7) is positive for all values of r, and hence, as long as L is positive a transformation of the type (4.7) can always be found which will transform the quadratic into the difference of two squares. We see, however, that the unit tangent vector to the world-line of a particle with fixed space co-ordinates in Walker's dynamical rest frame is space-like at sufficiently great distances from the axis of symmetry. The separation of space-time into space and time by means of Walker's rest frame therefore holds only in the neighbourhood of the observer. We can, however, find a system of co-ordinates in which the coefficient F remains positive throughout space. We apply the transformation (4.7), where we put

$$\Omega=-\frac{c}{R}\frac{\sinh\epsilon+\cosh\epsilon}{\sinh 3\epsilon+\cosh 3\epsilon}2\cosh\epsilon. \tag{10.16}$$

We then obtain the form (7.3), where H is given by (10.11) and

$$\left\{\begin{aligned} L&=R^2\left[\frac{\sinh 3\epsilon+\cosh 3\epsilon}{4\sinh 2\epsilon\cosh\epsilon}\left(\frac{r}{R}\right)^{1+2n}+\frac{\sinh 3\epsilon-\cosh 3\epsilon}{4\sinh 2\epsilon\cosh\epsilon}\left(\frac{r}{R}\right)^{1-2n}\right],\\ M&=-\frac{Rc}{\sinh 3\epsilon+\cosh 3\epsilon}\left(\frac{r}{R}\right)^{1-2n},\\ F&=c^2\frac{4\sinh 2\epsilon}{\sinh 3\epsilon+\cosh 3\epsilon}\left(\frac{r}{R}\right)^{1-2n}, \end{aligned}\right. \tag{10.17}$$

and where n is defined by (10.7). We easily verify that in the present system of co-ordinates the cosine of the angle between the ϕ-lines and the t-lines tends to zero as r tends to infinity. Furthermore the angular velocities with which a particle must describe a ϕ-line in order that its world-line may be a geodesic in space-time, tend to become equal and opposite and both tend to zero as r tends to infinity. Hence the present system of co-ordinates may be described as one which is not rotating with respect to the fixed stars. We call the present system of co-ordinates an astronomical rest frame for the observer on the axis of symmetry.

The angular velocity ω' with which the cylinder is rotating in the present system of co-ordinates is obtained from (10.16). Expressing ω' in terms of R by using (10.10) we obtain

$$\omega' = \frac{ac}{2a^4R^4}[1 - 2a^2R^2 - \sqrt{(1-4a^2R^2)}],$$

$$(10.18) \quad . \quad . \quad . \qquad = ac(1 + 2a^2R^2 + 10a^4R^4 + \ldots).$$

Comparing (10.18) and (8.1), we see that Walker's dynamical rest frame rotates with an angular velocity ω'' relative to the astronomical rest frame, where ω'' is given by

$$(10.19) \quad . \quad . \quad . \qquad \omega'' = 2a^2R^2(1 + 5a^2R^2 + \ldots)ac.$$

If aR is small compared with unity, so that the radius of the cylinder is small compared with its maximum permissible value, then ω'' will be very small compared with ω'. If we suppose the cylinder to be of the density of water on the axis of symmetry, and its radius to be that of the earth, then Walker's system of reference will complete a revolution relative to the fixed stars in $7{\cdot}7 \times 10^5$ years.

§ 11. Boundary Conditions. The Case $aR \geqslant \frac{1}{2}$.

The equations (10.8) and (10.9) show that the solution of the preceding section for the external field is valid only if $aR < \frac{1}{2}$, otherwise the constants occurring in the solution are imaginary. The external field in the case $aR = \frac{1}{2}$ is simply obtained from the equations (10.11) to (10.14) by a limiting process. We find

$$(11.1) \quad . \quad . \quad . \quad . \quad \begin{cases} H = e^{-\frac{1}{2}}\left(\dfrac{r}{R}\right)^{-\frac{1}{2}}, \\ L = \frac{1}{4}Rr\left(3 + \log\dfrac{r}{R}\right), \\ M = \frac{1}{2}rc\left(1 + \log\dfrac{r}{R}\right), \\ F = \dfrac{rc^2}{R}\left(1 - \log\dfrac{r}{R}\right). \end{cases}$$

To obtain the external field in the case $aR > \frac{1}{2}$, we return to the solution (9.9). Proceeding exactly as before, we find

$$(11.2) \quad . \quad . \quad . \quad . \quad \begin{cases} H = e^{-a^2R^2}\left(\dfrac{r}{R}\right)^{-2a^2R^2}, \\ L = Rr\dfrac{\sin(3\epsilon + \theta)}{2\sin 2\epsilon \cos\epsilon}, \\ M = rc\dfrac{\sin(\epsilon + \theta)}{\sin 2\epsilon}, \\ F = \dfrac{rc^2}{R}\dfrac{\sin(\epsilon - \theta)}{\sin\epsilon}, \end{cases}$$

where

$$\theta = \sqrt{(4a^2R^2 - 1)} \log \frac{r}{R}, \qquad (11.3)$$

$$\tan \epsilon = \sqrt{(4a^2R^2 - 1)}. \qquad (11.4)$$

The present solution only holds when $aR > \frac{1}{2}$.

We see from (11.2) that L, the coefficient of $d\phi^2$ in the fundamental form, is zero when

$$3\epsilon + \theta = \pi. \qquad (11.5)$$

Denoting the value of r at this point by r', we find

$$r' = R \exp \left[\frac{\pi - 3 \tan^{-1}\sqrt{(4a^2R^2 - 1)}}{\sqrt{(4a^2R^2 - 1)}} \right]. \qquad (11.6)$$

When $r > r'$ the coefficient of $d\phi^2$ becomes negative and the fundamental form changes its signature. It is easy to show, however, that all geodesics in the surfaces z=constant, issuing from the origin, meet again at the point $r=r'$. It follows that these surfaces are closed, the point $r=r'$ being the antipodal point of the origin. We see from (11.6) that when $aR \longrightarrow \frac{1}{2}$, then $r' \longrightarrow \infty$, and hence the external region becomes infinite as the radius of the cylinder approaches its critical value. When $aR \longrightarrow 1$ it is seen that $r' \longrightarrow R$. Hence as the radius of the cylinder increases, the external region diminishes and finally vanishes when the radius reaches its maximum value. The cylinder then fills space completely. We see from (8.5) and (8.8) that both ω_1 and Ω_1 remain finite when $aR \longrightarrow 1$. It follows that the world-lines of the particles on the antipodal line are null geodesics. Hence as $aR \longrightarrow 1$, the velocity of the particles on the boundary tends to the velocity of light. The cylinder can therefore never fill space completely, there must always remain a small filament of empty space surrounding the cylinder.

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X.—The Revised Complete System of a Quadratic Complex. By Professor H. W. Turnbull, F.R.S.

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Introduction.

This is a direct continuation of an earlier communication by the author, on "The Revised Prepared System of the Quadratic Complex," which appeared in vol. lvi (1936) of the Society's *Proceedings*, pp. 38–49. It will be convenient to utilize the same notation and references to earlier works I, II and III as before, with IV to denote the 1936 paper. A complete system of concomitants for a quadratic complex is here worked out; omitting polar forms, it consists of 210 forms. The sign $\equiv$ in a relation $A \equiv B$ is here used to mean "may be replaced by": thus A is equal to a non-zero numerical multiple of B plus reducible terms.

§ 1. The complete system is obtained by combining principal and secondary factors (IV, pp. 45, 47) in such a way as to involve each symbol

$$a, r, 2, 3, 4, 5, A, R, H, K, L, \alpha, \alpha_4,$$

an even number of times, and then rejecting every case which proves to be reducible. At most one principal factor may occur (IV, Theorem 2, p. 42): but two further simplifications can now be made:—

(i) Forms involving the variable π are reducible.

(ii) Forms involving H are reducible.

In proof of (i) consider the identity

$$(D_4\pi) = (D_4\xi\eta\zeta\omega) = 2(D_4R)(Rp) \cdot \mu, \qquad (1)$$

where $\mu = r_\zeta r_\omega$ (see V in "References to Literature," formula (11), p. 222). This holds for any set of symbols D_4, of currency four, where R denotes two equivalent symbols r. It follows that all such factors F_i as involve π may be rejected: for each of them is expressible in terms of simpler factors involving p, and primary factors (D_4R). Such an elimination of π introduces forms of higher degree in r, but this is immaterial since the whole set of possible combinations of the remaining factors, now involving only the variables ξ, p or P, is to be considered. This proves (i). Furthermore, when the whole system is expressed in quaternary symbols

the factor μ may occur to a negative power (V, p. 219). Consequently the above relation (i) may be used, if necessary, to reinstate the variable π. This expedient sometimes allows the R_3-rule (IV, p. 39) to operate, and reduces the form, as in the following instance.

In proof of (ii) first consider the case where (Hp) occurs. Let $H=\alpha R_5$, where $R_5=RR_3$, $\alpha=aA$. Then, omitting obviously reducible terms,

$$\begin{aligned}(Hp)\{H\ .\ .\ .\} &\equiv (Rp)(R_3\alpha)\{\alpha R_5\ .\ .\ .\}\\ &\equiv (Rp)(R_3\alpha)(aR_5)\{A\ .\ .\ .\}\\ &\equiv \mu^{-1}(R_3\alpha)(aR_3\pi)\{A\ .\ .\},\quad . \quad . \quad .\ \text{by (1)},\\ &\equiv \mu^{-1}(a\alpha\pi)\{A\ .\ .\ .\}=0,\end{aligned}$$

on using the R_3-rule. Hence (Hp) is reducible. Reference to the lists of factors shows that H can now only occur convolved either with a or A. Also we may write

$$(Ha) \equiv (\alpha)(aR_5), \qquad (HA)' \equiv (\alpha)(AR_5\epsilon) = (\alpha)(2).$$

These substitutions at once reduce each form involving H to simpler cases.

We therefore delete all factors from the Prepared Systems which involve either π or H: and, since the presence of the symbol α_4 in a principal factor requires its duplicate in a secondary factor which can only be $(\alpha_4\pi)$, we reject all cases where α_4 occurs.

(iii) *Every factor* $(2sp)$ *is reducible*, where s is a symbol equal or equivalent to r and belonging to any simple or compound symbol r, 2, 3, 4, 5, K, L: hence the only remaining p factor involving the symbol 2 is $(2ap)$. Thus

$$(2sp) \equiv 0, \qquad (2ap) \not\equiv 0. \quad . \quad . \quad . \quad . \quad (2)$$

Proof.—Write the symbol 2 in full as AR_5, with $R_5=R_3rt$, $A=ab$. A typical form containing $(2sp)$ is now

$$\begin{aligned}&(AR_3r)(stp)(aR_5)\{b,\ s,\ .\ .\ .\}\\ \equiv\ &(AR_3r)(stp)(aR_3st)\{b,\ r,\ .\ .\ .\},\end{aligned}$$

which reduces by applying the π and the R_3-rule as in the case of H above.

This reduction deletes $(r2)$, (23), (24), (25) from the list of secondary factors, and every case where $X=2p$ in the list of principal factors.

(iv) *A form* $(35,\ 35,\ 5',\ 5')$, containing four symbols 5, two of which are convolved with the symbol 3, *is reducible.* This follows by using the identity $(35,\ 35) \equiv (44',\ 44')$ and then the reduction $(45,\ 45)$.

§ 2. Next the following six types of principal factor are reducible:—

$$(a2LR), \quad (a234R), \quad (a23Kr), \quad (a2L5r), \quad \phi_1\text{(vii), (viii)}. \quad . \quad (3)$$

The first four of these reduce at once if the duplicate symbol a occurs convolved with L, R or K. In fact

$$(a2LR)(Ra) \equiv (a2LR)(R2) + f(a2, a2),$$

where $(R2)$ in the first term reduces by the R_3-rule, and the second term indicates that $a2$ occur convolved twice and reduce by IV (27). Hence R occurs only as

$$(a2LR)(RA)' \quad \text{or} \quad (a2LR)(Rp),$$

of which the first form reduces by transferring the symbol A to the other factor. This leaves the case (Rp). But, by (1),

$$(a2LR)(Rp) \equiv \tfrac{1}{2}\mu^{-1}(a2L\pi) = \tfrac{1}{2}\mu^{-1}(arr'r''\pi) \ldots,$$

on breaking up the symbols 2 and L. The cofactor of μ^{-1} in this last expression is a concomitant to which the R_3-rule at once applies. This replaces $(arr'r''\pi)$ by $(aD_3\pi)$, which in turn can be replaced by $\mu(aD_3R)(Rp)$. Thus three symbols r are removed by the R_3-rule and two replaced by changing from π back to p: in all the degree in r is lowered by unity, so that the original form is expressed in terms of forms already considered, and is therefore rejected.

Similarly for $(a234R)$. On replacing (K, K) by $(r5, r5)$ (IV, p. 46 (28)) and $(L5, L)$ by $(r45, r4)$ the next two forms are reducible likewise. The forms ϕ (vii) (viii) require more detailed examination, which is omitted here.

§ 3. By applying these and preceding methods of reduction, the list of possible irreducible forms has been truncated to the following 210 forms. They are given in abridged notation, from which the necessary variables are frequently omitted. To facilitate the reading of the list a summary of the notation is now given.

Symbols a, $A=ab$, $\alpha=abc$, and A_4, A_5 refer to the first quadric, r, $R=rs$, and R_3, R_4, R_5 to the second: symbols i, j, k are alike,

$$i = A_i R_{7-i}, \qquad i = 2, 3, 4, 5,$$

$$(a) = a_\xi, \qquad (i) = i_\xi, \qquad (ai) = (aip), \qquad (ij) = (ijp), \qquad (ijk) = (ijkP).$$

In full

$$(35) = (A_3R_4 \,.\, A_5R \,.\, p).$$

Also

$$L = A_4R_4, \qquad K = A_5R_3,$$

(aL), (aK), (Ai), (Ar), (aR), (α), (aij) are secondary factors, each containing the variable P.

In full

$$(aL) = (aLP) = (a \,.\, A_4R_4 \,.\, P).$$

The accented factors $(AL)'$, $(AK)'$, $(AR)'$ contain the variable ξ with P. Thus

$$(AL)' = (AlP)(l'\xi) - (Al'P)(l\xi).$$

For clarity the chain notation is often used, $(\alpha_\beta \gamma_\delta \, . \, .)$ meaning $(\alpha\beta)(\beta\gamma)(\gamma\delta)(\delta \, . \, .) \, . \, .$: for example, under K_2(vi),

$$(\xi^3 A^5 a^\xi) = (3)(3A)(A5)(5a)(a) = 3_\xi(3AP)(A5P)(5ap)a_\xi.$$

The variables are ξ, $p = \xi\eta$ and $P = \xi\eta\zeta$. Certain reference marks are used, corresponding to the earlier list in paper II.

§ 4. List of Concomitants.

The K_0 system (8 forms).

$(A_i R_{6-i})^2$, $\quad i = 0, 1, 2, 3, 4, 5, 6$: $\quad r_\xi r_\omega = \lvert\delta\rvert$ . . .	8

The K_1 system (8 forms).

i_ξ^2, $\quad i = a, 2, 3, 4, 5$; $\quad (Ap)^2$, $\quad (\alpha P)^2$, $\quad (Lp)^2$. . .	8

The K_2 system (48 forms).

(i) $(\xi^a i^\xi)$, $\quad i = 2, 3, 4, 5$; $\quad (\xi^i j^\xi)$, $\quad ij = 34, 35, 45$.	7
(ii) $(p^A i^\xi)$, $\quad i = 3, 4, 5$, ...	3
(iii) $(p^A i^a \xi)$, $\quad$,, ; $\quad (p^A i^j \xi)$, $\quad ij = 34, 35, 45, 53$,	7
(iv) $(p^A 3^4 5^\xi)$, $\quad$ (v) $(\xi^3 A^5 \xi)$, $\quad$ (vi) $(\xi^3 A^5 a^\xi)$, . . .	3
(xvi) $(p^L a^\xi)$, $\quad$ (xxii) $(L)(LA)'(A)$, . . .	2
(xvii) $(p^L a^i \xi)$, $\quad i = 3, 4, 5$, $\quad$ (xxiii) $(L)(LA)'(A_i\xi)$, . .	6
(xviii) $(p^L a^i A^p)$, $\quad$,, $\quad (a)(aL)(LA)'(A)$, . .	4
$(a)(aL)(LA)'(A_i\xi)$, $\quad i = 3, 4, 5$, . . .	3
$(a3)^2$, $\quad (a4)^2$, $\quad (a5)^2$, $\quad (35)^2$, . . .	4
$(ij)(jk)(ki)$, $\quad i, j, k = a, 3, 4, 5$, ...	4
$(A4)^2$, $\quad (A5)^2$, $\quad (Ai)(ij)(jA)$, $\quad i, j = 3, 4, 5$. . .	5
	48

The K_3 system (26 forms).

(l) $(a35)^2$, ...	1
(m) $(aij)a_\xi i_\xi j_\xi$, $\quad ij = 23, 24, 25, 34, 35, 45$, . .	6
(n) $(aij)(i_A j)a_\xi$, $\quad ij = 34, 35, 45$, ...	3
(o) $(aij)(ai)j_\xi$, $\quad ij = 34, 35, 45, 53$, ...	4
(p) $(aij)(ai)(aj)a_\xi$, $\quad ij = 34, 35, 45$, ...	3
$(a35)(a3)(35)3_\xi$, ...	1
(ϵ) $(aij)(ai)(jk)k_\xi$, $\quad ijk = 345, 453$, ...	2
(ζ) $(a34)(a3)(4_A 5_\xi)$, ...	1
IV $(a34)(a35)(a45)a_\xi$, ...	1
(a_1) $(a34)(a45)(35)$, ...	1
(b) $(aij)(i_A j)(a_k\xi)$, $\quad ijk = 345, 453, 534$. . . .	3
	26

The ϕ_i system (120 forms).

Fifteen forms involving $(ALR)=\psi_1$:

$\psi_1(L)(R)(A)$,	$\psi_1(AL)'(R)$,	
,, ,, ,, $(A_4\xi)$, or $(A_4 a_\xi)$,	,, ,, $(RA)'(A)$,	
,, ,, $(aR)(A)(a)$,	,, ,, ,, $(A_4\xi)$,	
,, ,, ,, $(A_4 a)$,	,, ,, $(aR)(A)(\alpha)$,	
,, ,, ,, $(A_4\xi_a)$,	,, (aL) ,, $(A_4\xi)$ or (A),	
,, ,, ,, $(A_\xi i_a)$, $i=3, 4$,	,, ,, ,, $(A_4 a_\xi)$. . .	15

Eight forms involving $(\alpha 4R)=\psi_2$:

$\psi_2(\alpha)(R)4_\xi$,	$\psi_2(\alpha)(Ra)(a_\xi 4)$,	
,, ,, ,, $(4a)a_\xi$,	,, ,, ,, $(A4)(A)a_\xi$,	
,, ,, ,, $(4A)(A)$,	,, ,, $(RA)'(A)4_\xi$,	
,, ,, $(Ra)(a4)$,	,, ,, ,, $(A4)$. . . .	8

Twenty-one forms involving $(\alpha Kr)=\psi_3$:

$\psi_3(aK)(\alpha)(air)i_\xi$, $i=3, 4$,	$\psi_3(\alpha)(K)(r)$,	
,, ,, ,, ,, $i_a\xi$, ,, ,	,, ,, ,, $(r_i\xi)$, $i=3, 4$,	
,, ,, ,, (ar),	,, ,, ,, $(r_A p)$,	
,, ,, ,, $(a)(r)$,		
,, ,, ,, $(a)(A)(Ar)$,	,, ,, ,, $(r_a i_\xi)$, ,, ;	
,, ,, ,, $(a_4 A_r)$,	,, ,, ,, $(r_a\xi)$,	
,, ,, ,, $(a_\xi 4_A r)$,	,, ,, ,, $(r_4 A_p)$,	
,, $(AK)'(\alpha)(r)(A)$ or (rA),	,, ,, ,, $(r_A 4_\xi)$,	
,, ,, ,, $(r_\xi 4_A)$, ...		21

Four forms involving $(A34R)$:

$(A34R)(A)(R)(3)(4)$,
,, ,, ,, $(3a)(4)(a)$,
,, ,, $(Ra)(3)(4)(a)$,
,, ,, ,, $(3a)(4)$. 4

Thirteen forms involving $(A3Kr)=\psi_4$:

$\psi_4(Ka)(Ar)(3_\xi a)$ or $(3a)$,	$\psi_4(K)(Ar)3_\xi$,	
,, $(KA)'(Ar)(A)(3)$,	,, ,, ,, $(3_a\xi)$,	
,, ,, $(3)(r)$,	,, ,, $(A)(3r)$,	
,, $(Ka)(A)(r)(3)(a)$,	,, ,, ,, $(3_a r)$,	
,, ,, ,, $(3_a r)(a)$,	,, ,, ,, $(3_a\xi_r)$,	
,, ,, ,, $(a3)(r)$,	,, ,, ,, $(3)(r)$. . .	13

Twenty-two forms involving $(AL5r)=\psi_5$:

$\psi_5(La)(A5)(a3r)3_\xi$,	$\psi_5(L)(A5)(r_Ap)$ or (r),
,, ,, ,, ,, $(3_a\xi)$,	,, ,, ,, $(r_a\xi)$ or $(r_3\xi)$,
,, ,, ,, $(rA)(A)(a)$,	,, ,, ,, (r_a3_ξ),
,, ,, ,, (ra) or $(r)(a)$,	,, ,, $(A)(5)(r)$,
,, ,, ,, (a_3r) or $(a_3\xi_r)$,	,, ,, ,, (5_ar) or (5_3r),
,, ,, $(A)(a)(5)(r)$,	,, ,, ,, $(5_\xi a_r)$,
,, ,, ,, $(a5)(r)$,	,, $(LA)'(5)(r)$ or (5_Ar),
,, ,, ,, $(ar)(a5)(a)$,	,, ,, $(5)(A)(Ar)$. . . 22

Twelve forms involving $(\alpha45r)=\psi_6$:

$\psi_6(a2r)(\alpha)(2)(4)(5)(a)$,	$\psi_6(\alpha)(4_\theta5)r_\xi$,	$\theta=a,\ A,\ \xi$,
,, ,, ,, ,, $(4_A5)(a)$,	,, ,, ,, $(r_a\xi)$,	
,, $(\alpha)(4)(a)(a5)(r_Ap)$,	,, ,, ,, (r_Ap).	 12

Five forms involving $(A345r)$ and one $(a2345r)$:

$(A345r)(A)(3)(4)(5)(r)$,
,, ,, ,, ,, ,, $(r_a\xi)$,
,, ,, ,, ,, (5_ar), $(a2345r)(a)(2)(3)(4)(5)(r)$,
,, ,, $(3_a4)(5_ar)$,
,, ,, ,, $(5_a\xi_r)$, ... 6

Nine forms involving $(\alpha RLX)=(\chi_1X)$:

$(\chi_1\xi)(\alpha)(R)(L)$ or $(L_a\xi)$,	$(\chi_1\xi)(\alpha)(Ra)(La)$,
,, ,, ,, $(LA)'(A)$,	$(\chi_1ap)(\alpha)(Ra)(La)(a)$,
,, ,, $(RA)'(LA)'$,	,, ,, ,, (L),
,, ,, ,, $(La)(A)(a)$,	,, ,, $(R)(L)(a)$. . . 9

Eight forms involving $(ALrKX)=(\chi_6X)$:

$(\chi_6ap)(Ka)(La)(a)(Ar)$,	$(\chi_6AP)(r)(K)(L)$,
,, (K) ,, (Ar),	,, ,, ,, $(L_a\xi)$,
,, ,, ,, $(A)(r)$,	,, ,, (K_aL),
,, ,, $(L)\ (Ar)(a)$,	,, ,, $(KA)'(LA)'$. . . 8

Two forms of type ϕ_2:

$(paLarKap)(\alpha)(r)(L)(K)$,
,, $(\alpha)(r)(La)(K)(a)$. 2

Total 210

§ 5. Inspection shows the degree and order of each of the above forms. The degrees in the coefficients a_{ij} of the first quadric run from zero to 17 inclusive, and the number of forms belonging to these 18 classes in ascending degree are

$$2, 2, 3, 4, 5, 6, 10, 11, 17, 21, 25, 26, 25, 16, 15, 17, 4, 1.$$

The single form of highest degree 17 is, in fact,

$$(\alpha 45r)(a2r)(\alpha)(4A)(A5)(2)(a),$$

a mixed concomitant of order 2 in ξ and 4 in P. There are, as is known from general theory, seven invariants (in the K_0 set), and seven covariants, namely, six quadratics a_ξ^2, i_ξ^2, r_ξ^2 and the sextic Jacobian

$$(a2345r)a_\xi 2_\xi 3_\xi 4_\xi 5_\xi r_\xi,$$

which has six linear irrational factors, giving the Klein set of six fundamental linear complexes.

There are fifteen forms involving the variable p alone, ten quadratics, one cubic, three quartics and one quintic. There are four forms

$$(\alpha P)^2, \qquad (A5P)^2, \qquad (A4P)^2, \qquad (a35P)^2,$$

which involve the variable P only. The first of these is the symbolic form (in senary symbols) of the singular Kummer Quartic Surface associated with the quadratic complex a_ξ^2. The other three appear in the equations for the Kummer surfaces of covariant quadratic complexes, but, taken singly, have a more complicated interpretation.

The remaining forms are mixed concomitants.

§ 6. Each form involving the variable $P=\xi\eta\zeta$ may be polarized by means of the operator $\Sigma Q\frac{\partial}{\partial P}$, $Q=\xi\eta\omega$. When this is done, and all reducible cases are rejected, the system may be translated back into quaternary symbols and variables. Each ξ then denotes a straight line θ, each p or $\xi\eta$ denotes a point x combined with a plane u, each P or $\xi\eta\zeta$ denotes x twice and each Q denotes u twice. Thus a form of orders m, m', m'', m''' in ξ, p, P, Q respectively denotes a quaternary form of orders n, n', n'' in point, line, and plane co-ordinates respectively, where

$$n=m'+2m'', \qquad n'=m, \qquad n''=m'+2m'''.$$

Note: The Greek letter α, which might be confused with the ordinary a, does not occur in the K_2 and K_3 lists above.

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XI.—The Time Lag of the Vacuum Photo-cell. By R. A. Houstoun, M.A., D.Sc., Natural Philosophy Department, University of Glasgow. (With Three Figures.)

(MS. received November 28, 1936. Read March 1, 1937.)

(1) It is generally assumed that the vacuum photo-cell has no appreciable time lag. The electrons start out from the surface as soon as the light falls on it. The most important investigation on the subject is that of Marx and Lichtenecker (1913). By means of a mirror rotated 170 times per second an image of a narrow slit was swept over the cathode of a photo-cell at a distance of 8·3 metres from the mirror; the cathode was a potassium one sensitised with hydrogen. The electrometer reading was noted, and then the mirror was rotated at a much slower speed and the electrometer read again. No appreciable difference was recorded. In the words of the authors an alteration of the time of illumination from $4 \cdot 38 \times 10^{-3}$ to $1 \cdot 46 \times 10^{-7}$ sec., the intensity of radiation being 0·56 erg/sec. cm.^{2}, had no effect on the result to an accuracy of one or two per cent. Theoretical deductions have been drawn from this statement which have been much quoted. It seemed to me that the statement itself was worth verifying, and this paper is the record of an attempt to verify it and push the limit of accuracy further.

In their paper the authors give two sets of readings, each containing about 30 observations, both made on the same photo-cell. The source was a slit 1·3 mm. wide illuminated by a mercury arc which in the case of one of the sets was used with a yellow filter. The 0·56 erg/sec. cm.^{2} was the intensity at the brightest part of the image on the cathode, when the mirror was at rest; it refers to the set with the yellow filter and so is for monochromatic green light. During the experiment every point on the cathode experienced a very short time of illumination followed by a period of darkness; the magnitude of these changed, but their relative value remained unaltered.

I tried three different methods for rotating the mirror. In the determinations of the velocity of light the mirror has always been rotated by a turbine. In his classic determination Foucault attained speeds of 600 to 800 turns per second with a mirror of 1·4 cm. diameter. Michelson used lower speeds than this; in his first determination the blower was driven by a 3 horse-power steam engine. The fastest speeds hitherto

used for physical investigation appear to have been used by Zimmerman (Zimmerman, 1933); he rotated a hexagonal mirror of 1·5 mm. face at 1900 revolutions per second and a rectangular mirror of 3 mm. face at 750 revolutions per second, in both cases by means of a turbine. At these high speeds it is necessary that the rotating system should be perfectly balanced, and the driving forces must be applied in a perfectly symmetrical manner. It is not possible to multiply the speed by toothed wheels or cords and pulleys or any method that puts a sideways force on the axle.

I had a rectangular mirror made in hard steel, the four faces of which measured 20 mm. by 5 mm., the long side being vertical. This rotated on a vertical axis. Oil was fed to the bearings at a pressure which could be adjusted so as to make the oil itself just carry the weight of the rotating system, and curved teeth were fixed to the axle below the mirrors to receive the jets of air; these were produced by an Edwards Rotary Blower, Type IV. With this arrangement a speed of 400 turns per second was attained. The horse-power of the blower was only one quarter, but owing to the small moment of inertia of the system a higher speed was expected. The poor result was attributed to a slight lack of balance in the rotating system. But it was not possible to correct this without re-designing the whole arrangement, so I abandoned it, and turned my attention to the air-driven top of Henriot and Huguenard (Henriot and Huguenard, 1925, 1927).

This is explained in fig. 1. A is a cylindrical brass vessel which has a conical depression on the top. Air enters this vessel under pressure and escapes by the two holes C. B is a solid cone of brass. According to Bernoulli's theorem there is a diminution of pressure outside the holes owing to the velocity of the jets, and the cone is sucked down. But the air must escape round its edge. Thus a state of equilibrium would be reached if the holes were normal to the surface of the containing vessel. In actual practice they are set obliquely, and the surface of the cone is grooved in the direction of its generators. Consequently, the cone begins to spin round, and as it is literally riding on a whirlwind there is no friction, and very high speeds are reached. With a cone, the base of which was $1\frac{1}{4}$ in. in diameter, and the blower already mentioned I obtained 1200 turns per second; with a more powerful blower in the Engineering Department 2000 turns per second was easily reached. When the cone once starts spinning, it goes hours without attention. In order to measure the speed, the top of the cone was painted half black and half white and

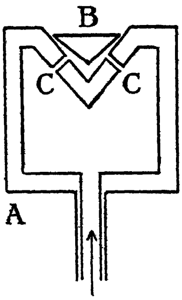

FIG. 1.

exposed to a strong illumination, the diffusely reflected light from one side being caught by a Photronic cell. The photo-electric current went from a maximum to a minimum with each rotation of the cone. It was multiplied up by a two-stage amplifier and the note heard in a telephone.

This method appeared very promising at first and much time was spent experimenting with different models. But the method suffers from two defects; there is always a slight precession and nutation, and when a mirror is attached to the upper surface of the cone, the air-drag causes a great reduction in the velocity. So I abandoned this method in turn, and fell back on a high-speed electric motor which worked at $\frac{1}{8}$ h.p., and which could be made to go at 320 turns per second, when risks were taken with the insulation.

Two mirrors were used: one of magnalium 2 in. broad, $\frac{3}{8}$ in high, and $\frac{3}{16}$ in. thick, and one of glass aluminiumised on one face, 2 in. broad, $\frac{3}{8}$ in. high, and $\frac{1}{4}$ in. thick. Their definition was excellent. They were very carefully centred on the axis of the motor, which was vertical. After being used some weeks the magnalium mirror fractured under its own centrifugal force, which at the ends of the mirror amounted to as much as 10,000 times the weight. The glass mirror was sometimes combined with a stationary one, as shown in the plan in fig. 2 where there are four reflections on the moving mirror. In this case it may be shown by easy geometry that when the mirror rotates 320 times a second, the reflected ray rotates $320 \times 8 = 2560$ times per second. As far as I am aware, this method of multiplying the speed is described here for the first time; I have used it with as many as eight reflections on the moving mirror. It requires that the two mirrors should be accurately parallel, for, if there is an error of 1° in a vertical direction, the final ray is about 16° off the horizontal. For safety the mirror and motor were contained in a box of oak 1 in. thick, and the light entered and left by a slot measuring $10 \times 1\frac{1}{2}$ cm.² The box was steadied by placing a 28-lb. weight on its top.

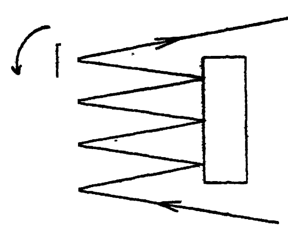

FIG. 2.

As source of light I used the filament of a vacuum glow lamp. The diameter of the filament was $\frac{1}{15}$ or $\frac{1}{10}$ mm., but some definition was lost in the glass wall of the photo-cell, and the breadth of the image on the cathode was about $\frac{3}{10}$ mm. It was found that a brighter and steadier source could be obtained in this way than with a slit illuminated by an arc or a pointolite. Part of the lamp could be screened off and only one leg of the filament used, but in some cases the whole filament was taken.

It is of course necessary that the voltage on the lamp should remain

constant. I sought to attain this by three different methods. First, a 120-volt Robertson lamp was run in series with Osram barretter, No. 304, off the 250-volt lighting circuit, a voltmeter being put in parallel with the lamp. It was hoped that the barretter would take up the fluctuations and leave the voltage on the lamp constant, but expectations were not realised; the barretter diminished the fluctuations, but did not remove them. Then a 4-volt lamp was used off accumulators; the accumulators were steady enough, but the resistance of such lamps seems liable to change, and I had not sufficient accumulators at my disposal for one of the larger lamps. Finally, I used a 220-volt lamp run at 240 volts in series with a variable resistance and in parallel with a voltmeter which read to one-tenth volt; the voltmeter was watched during the experiment through a magnifying lens and any fluctuation at once removed with the rheostat. With practice I became expert at this, and am confident that in this way the voltage was kept constant to one part in a thousand.

The image was only $\frac{3}{10}$ mm. broad and it travelled round in a circle of 1 metre radius. The question arose as to how much of the deflection was due to the image itself and how much to stray light during the period of darkness; owing to this period being relatively so much longer the effect of stray light would mount up. The question was answered by fixing a black paper cap over the mirror with a thin rubber band; the deflection then went down to a fraction varying between $\frac{1}{11}$ and $\frac{1}{35}$ according to the conditions of the experiment. The fraction of the deflection due to "stray" in the actual experiment would be less than this, because the black paper in front of the mirror created additional stray. The stray was due to light reflected from the fittings of the mirror and the front of the box containing it, and in a much lesser degree to the door and windows of the room which were not absolutely light tight; none of it was due to the electrometer lamp.

Observations were made on two photo-electric cells, a K V 6 and K M V 6, both by the General Electric Company. The over-all length of these cells is about 15·5 cm. They are mounted on a four-pin base, but only the anode is connected to a valve pin. The cathode is connected to a terminal on the upper end of the tube. There is an internal guard ring with a terminal which is earthed. The cathode measures $4 \times 2\frac{1}{4}$ cm.^{2}, the longer side being vertical.

The lay-out of the experiment is shown in plan in fig. 3. The source of light S is contained in a box; the light from an aperture in the front of this box fell on a lens of 70 cm. focal length and then on the rotating mirror M. P is the photo-cell and E the Dolezalek electrometer with which it was used. They were contained together with a spring earthing

key inside a tin box 37 cm. high on a base of 25×27 cm.^{2} There were two holes in the sides of the box, closed with glass windows: one 5×3 cm.^{2} for the electrometer, and the other $2\frac{1}{4} \times 4$ cm.^{2} for the photo-cell. The earthing key was operated by pulling a string outside the box. The distance SL from the lamp to the lens was 145 cm., ML the distance from the mirror to the lens was 40 cm., and MP the distance from the mirror to the photo-cell 100 cm. The capacity of the cathode, quadrants,

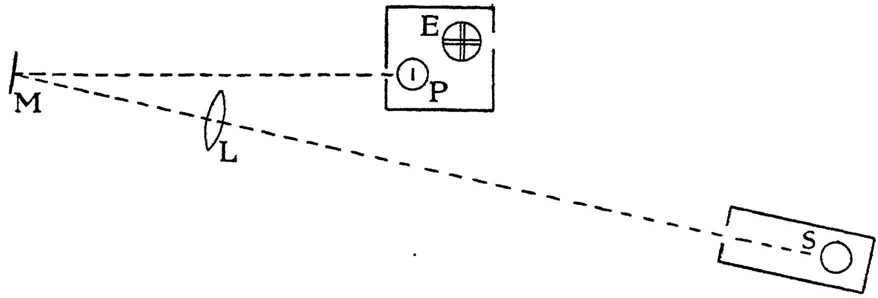

FIG. 3.

and connecting wires was about 58 cm. or $6{\cdot}42 \times 10^{-5}\mu F$, and a deflection of 10 cm. on the scale corresponded to 0·219 volt. The quadrants were mounted on amber pillars and their insulation was excellent, there being no appreciable leak for hours, even in damp weather.

The tin box was not quite airtight, and at first the cathode of the photo-cell leaked to earth in damp weather; but when a beaker containing sulphuric acid was placed inside the box, all trouble on this point was removed.

In making observations I noted the time taken to cover a given 15 cm. on the scale when the mirror was going at 15 turns per second; then the time for 320 turns per second. Then the time for 15 turns per second was taken again, and so on. The table below reproduces a good set of readings:

K M V 6. Source whole filament of 220-volt lamp. Magnalium mirror. Exposures $1{\cdot}59 \times 10^{-6}$ sec. and $7{\cdot}4 \times 10^{-8}$ sec.

Slow.	Fast.	Slow.	Fast.
130·6 sec.	130·7 sec.	127·6 sec.	126·7 sec.
131·5 „	129·4 „	127·1 „	127·3 „
130·5 „	130·1 „	127·0 „	127·5 „
129·0 „	129·8 „	128·1 „	127·0 „
129·4 „	128·6 „	126·0 „	125·6 „
128·3 „	129·6 „	126·8 „	126·2 „
126·8 „	128·2 „	126·9 „	126·6 „

Mean increase in time when mirror is running at high speed $-\cdot 15 \pm \cdot 17$ sec. or $-\cdot 12 \pm \cdot 13$ per cent.

The following table gives the results of the three most reliable experiments on K V 6:—

K V 6. Source one leg of 120-volt filament. Magnalium mirror. Exposures $1\cdot 59 \times 10^{-6}$ sec. and $7\cdot 4 \times 10^{-8}$ sec. Time of charging about 90 sec. 11 determinations in set. Increase in time when mirror is running at high speed $\cdot 30 \pm \cdot 50$ per cent.

K V 6. Source one leg of 120-volt filament. Magnalium mirror. Exposures $1\cdot 59 \times 10^{-6}$ sec. and $7\cdot 4 \times 10^{-8}$ sec. Time of charging about 110 sec. 14 determinations in set. Increase in time when mirror is running at high speed $\cdot 70 \pm \cdot 31$ per cent.

K V 6. Source whole filament of 4-volt lamp. Magnalium mirror. Exposures $1\cdot 59 \times 10^{-6}$ sec. and $7\cdot 4 \times 10^{-8}$ sec. Time of charging about 176 sec. 8 determinations in set. Increase in time when mirror is running at high speed $\cdot 85 \pm \cdot 59$ per cent.

The most serious cause of error is a vertical oscillation of the image at the high speed due to vibration of the motor. This amounted to one millimetre either way and sometimes caused a spurious effect owing to the oscillation changing the proportion of the light which was obscured by the wire-net anode. The only way to avoid this error was to choose a position on the cathode for the image, where the reading was not affected by a slight vertical displacement of the image—a somewhat tedious process. This error may be partly due to the photo-cell not being equally sensitive all over.

The mean of the results stated above for K V 6, ·30, ·70, ·85 per cent. is an increase of 0·62 per cent. This was borne out by other experiments and is, I am convinced, a real result. The values obtained for K M V 6 occurred in both directions, and, as far as my measurements go, this cell shows no change whatever. In fact the persistent difference in the behaviour of the two cells contributed very considerably to the confidence I have in the result.

If we make the assumptions, that when the light falls on the cathode, there is a time lag or period of induction before the electrons start to come off, and that the emission ceases as soon as the light ceases; this time lag according to my result is

$$0\cdot 0062 \times 7\cdot 4 \times 10^{-8} = 5 \times 10^{-10} \text{ sec.}$$

for the K V 6 cell. According to Marx and Lichtenecker it is less than 1·5 or 3×10^{-9} sec. for a cell of the same nature. The only other attempt known to me to determine the time lag for a vacuum cell is that of Lawrence and Beams (Lawrence and Beams, 1927). They used an arrangement which caused a spark and immediately afterwards cut off the anode potential, and found that they could not cut off the potential fast enough

to prevent the electrons getting across. They state that for a potassium-coated cathode the lag must be less than 3×10^{-9} sec. So my result is in agreement with previous workers in the field.

The paper by Zimmerman already referred to deals with the effect on a photographic plate of sweeping the image of a narrow slit across it by an arrangement similar to mine. He worked with exposures as short as $5{\cdot}5 \times 10^{-8}$ sec., and found that for the same quantity of light the blackening was constant down to about 5×10^{-7} sec., but after this it fell off in a marked manner. This is in agreement with my result. A similar result will probably be obtained with the human eye.

(2) It is interesting to consider the numerical values of the energy changes involved in the experiment.

The photo-electric current varied from experiment to experiment. In the second experiment on K V 6 recorded above it was

$$\frac{1{\cdot}5 \times {\cdot}219 \times 6{\cdot}42 \times 10^{-11}}{110} = 1{\cdot}92 \times 10^{-13} \text{ amp.}$$

This is the average value for the whole cathode; for the area actually under illumination the current is

$$1{\cdot}92 \times 10^{-13} \times \frac{2\pi \times 100}{2\frac{1}{4}} \times \frac{1}{3 \times \frac{3}{100}} = 5{\cdot}96 \times 10^{-10} \text{ amp./cm.}^2.$$

If we take the average value of the quantum for the energy absorbed in the photo-electric process as $4{\cdot}4 \times 10^{-12}$ erg, the energy of the electrons emitted is

$$\frac{5{\cdot}96 \times 10^{-10} \times 4{\cdot}4 \times 10^{-12}}{1{\cdot}59 \times 10^{-19}} = 1{\cdot}65 \times 10^{-2} \text{ erg/cm.}^2 \text{ sec.}$$

When we consider the energy of the light falling on the cathode, we are on less certain ground. In order to determine the efficiency of K V 6 an 8-c.p. standard lamp of approximately the same colour as the lamps used in the experiment was set up at a distance of 368 cm. from the photo-cell, and a metal plate with a hole of 2·48 mm. radius placed in front of the latter. It was found that the spot of light travelled 15 cm. on the scale in 19·8 sec. The lumens received was $1{\cdot}142 \times 10^{-5}$, and the current was $1{\cdot}065 \times 10^{-12}$ amp., giving an efficiency of

$$\frac{1{\cdot}065 \times 10^{-6}}{1{\cdot}142 \times 10^{-5}} = 9{\cdot}33 \times 10^{-2}\ \mu\text{A per lumen.}$$

Thus the light received during the period of illumination was

$$\frac{5{\cdot}96 \times 10^{-4}}{9{\cdot}33 \times 10^{-2}} = 6{\cdot}39 \times 10^{-3} \text{ lumen/cm.}^2.$$

If we assume that the lamp is rated at $3\frac{1}{2}$ watts per candle, the intensity of the radiation falling on the image is

$$\frac{6{\cdot}39 \times 10^{-3} \times 7 \times 10^{7}}{2 \times 4\pi} = 1{\cdot}78 \times 10^{4} \text{ erg/cm.}^2 \text{ sec.}$$

Only about 3 per cent. of this is visible light, and about one-fifth per cent. falls within the region for which the cathode has selective emission. So the proportion of the radiation which has a chance of contributing towards the photo-electric emission is, very roughly, about 35 erg/cm.^{2} sec. This is about 2000 times the energy of the electron stream that comes off; the remainder must be lost by reflection and as heat in the cathode.

The energy of ah emitted electron is about $4{\cdot}4 \times 10^{-12}$ erg., and the time of induction is 5×10^{-10} sec. Let us assume that the electron acquires its energy from the incident light during the time of induction. Then A, the catchment area, is given by

$$35A \times 5 \times 10^{-10} = 4{\cdot}4 \times 10^{-12}, \qquad \text{or} \qquad A = 2{\cdot}5 \times 10^{-4} \text{ cm.}^2.$$

There are three different views which may be taken with reference to the absorption of light by the cathode:

(i) The energy of the light-wave is spaced uniformly over the wave front, and each electron emitted draws its energy from a comparatively wide area.

(ii) The energy of the light-wave is concentrated in units, one of which is incident on the atom which contains the electron.

(iii) We must not try to form a picture of the process.

The third view is the popular one at present. The calculation made above is based on (i), and the magnitude of the result was formerly held to constitute an argument in favour of the second view, but this argument has been weakened by the advent of wave mechanics and the discovery that the electron is sometimes not a point charge; but owing to the assumptions implicit in the calculation not much value can be attached to it. All that the experiment shows is that there is a small deviation from the law of proportionality between quantity of light and photo-electric current, when the duration of each flash, into which the total quantity of light is subdivided, becomes excessively small. Unfortunately, the difficulties caused by the vertical oscillation of the image make it impossible to investigate this deviation in further detail.

I am indebted to my colleague, Dr G. E. Allan, for assistance in determining the speed of the mirrors in many preliminary experiments; his trained ear enabled him to state the speed from the note at once and with great accuracy. I am also indebted to the chief mechanic of the Depart-

ment, Mr Wm. Reid, for the trouble he took with the same preliminary experiments and the balancing and adjusting of the mirrors; and I am indebted to the Carnegie Trust for the Universities of Scotland for defraying the special expenses of the investigation.

Note added February 20, 1937. A short letter was printed in *Nature*, vol. cxxxviii, 1936, p. 1011, drawing attention to the results of the above investigation. It called forth a letter from Dr Norman R. Campbell in *Nature*, vol. cxxxix, 1937, p. 330, in which he states that the K M V 6 cell *is* highly evacuated, but the K V 6 always contains an appreciable quantity of hydrogen evolved from the sensitised potassium; it should not be treated as a true vacuum cell. No experiments made in the Research Laboratories of the General Electric Company, Ltd., Wembley, throw any light on the question as to whether the difference I obtained is actually due to this difference in gas content or not, but, according to Dr Campbell, this possibility should not be excluded from consideration.

I agree. Marx and Lichtenecker's work was, of course, done on a cell similar to the K V 6.

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XII.—Studies in Practical Mathematics. I. The Evaluation, with Applications, of a Certain Triple Product Matrix.

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1. Theoretical Basis of the Method.

The solution of simultaneous linear algebraic equations, the evaluation of the adjugate or the reciprocal of a given square matrix, and the evaluation of the bilinear or quadratic form reciprocal to a given form, are all special cases of a certain general operation, namely the evaluation of a matrix product $H'A^{-1}K$, where A is square and non-singular, that is, the determinant $|A|$ is not zero. (Matrix multiplication is like determinant multiplication, but exclusively row-into-column. The matrix H' is obtained from H by transposition, that is, by changing rows into columns.) The matrices H' and K may be rectangular. If A is singular, the reciprocal A^{-1} does not exist; and in such a case the product $H'(\text{adj}\,A)K$ may be required. Arithmetically, the only difference in the computation of $H'A^{-1}K$ and $H(\text{adj}\,A)K$ is that in the latter case a final division of all elements by $|A|$ is not performed.

We have denoted the premultiplying matrix by H' instead of H purely for notational consistency, for we intend to regard its rows as row vectors (rectangular matrices of one row only) h', and the columns of K as column vectors k. Thus in the product matrix $H'A^{-1}K$ a typical row is a vector $h'A^{-1}K$, a typical column is a vector $H'A^{-1}k$, and a typical element is the scalar $h'A^{-1}k$, a bilinear form.

Our present purpose is the practical one of deriving and illustrating a simple and uniform working process for computing this product of matrices. The basis of computation is Cauchy's expansion of a bordered determinant according to elements of the bordering row *and* column and their common co-factors. This allows us to express the reciprocal bilinear form $h'A^{-1}k$ in the well-known way as a bordered determinant; for example:

$$h'A^{-1}k = \left|\begin{array}{c:c} A & k \\ \hdashline -h' & 0 \end{array}\right| |A|^{-1} \equiv \left|\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & k_1 \\ a_{21} & a_{22} & a_{23} & k_2 \\ a_{31} & a_{32} & a_{33} & k_3 \\ \hdashline -h_1 & -h_2 & -h_3 & 0 \end{array}\right| \div |A|. \qquad (|A| \neq 0.)$$

The adjoint or adjugate bilinear form differs from this only in the respect that the division by $|A|$ is not required.

It follows that to evaluate $H'A^{-1}K$ is to evaluate an array of bilinear forms, namely the array of bordered determinants obtained by bordering A with each row of H' associated in turn with each column of K. Hence a convenient scheme for computing all of these forms together by a uniform process will be the following: set up A, border it below with $-H'$, on the right with K, and subject the whole Γ-shaped array to the successive "pivotal" condensations customary in evaluating determinants by Chiò's method (see Whittaker and Robinson, 1929) or any of its several variants. The initial pivot must be an element of A, and pivots at later stages must originate exclusively from elements of A.

$$\begin{array}{c|c} A & K \\ \hline -H' & 0 \end{array}$$

The scheme is therefore as indicated above; the 0 represents a block of zeros. Pivotal condensation is applied until the stage when all entries to the left of the vertical dividing line are cleared off. What then remains on the right of the line is the matrix $H'(\operatorname{adj} A)K$; and a final division of each element of this by $|A|$, which will have been obtained as a by-product of the condensation, yields the desired $H'A^{-1}K$. As will soon appear, it is easy to arrange pivotal divisions which will save the necessity for this last step.

2. Remarks on Technical Procedure.

Pivotal condensation (which is no more than the cross-multiplying process* of elimination with respect to a fixed element customary in reducing simultaneous equations, reinforced by certain suitable divisions) will be carried out in our illustrative example with the *leading* element in each stage as pivot. Each leading element, as soon as it is computed, will be entered marginally at the left; and the elements in the row containing the pivot will be divided by this leader, so that what is written down is a row in which the pivot for the next condensation is unity. The algorithm is in fact the one used by Gauss for solving the normal equations of least squares. The divisions in the pivotal row are few, and each can be performed without clearing the register of the machine. An alternative way is to enter the pivotal row without division, and then to do the division by multiplying each element in turn by the reciprocal of the leader, this

* In any cross-multiplication it is expedient, no matter how the tetrad of elements may be situated, to let the pivotal term retain its sign and to impose the minus on the other product term.

multiplier being left on the setting levers all the time. These are mere details, depending on the resources available to the computer, and his own habits of computation. The elements in non-pivotal rows at any stage are found by ordinary cross-multiplication *without* division.

The following scheme shows with literal symbols how the cross-multiplications proceed stage by stage, how the results are set out, and what the actual elements are at any stage, in terms of the minors of the array which is being condensed. Such minors are represented by their diagonal elements, e.g. $a_1b_2 - a_2b_1$ as $| a_1b_2 |$, pivots are underlined, and pivots as they were before division are set in the margin. The first row of the array to be condensed is $[a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]$, where a_1 is assumed to be not zero.

$$
\begin{array}{lllllll}
 & a_1 & 1 & a_2/a_1 & a_3/a_1 & a_4/a_1 & a_5/a_1 \\
\text{I.} & & b_1 & b_2 & b_3 & b_4 & b_5 \\
 & & c_1 & c_2 & c_3 & c_4 & c_5 \\
 & & d_1 & d_2 & d_3 & d_4 & d_5 \\
\hline
 & |a_1b_2|/a_1 & & 1 & |a_1b_3|/|a_1b_2| & |a_1b_4|/|a_1b_2| & |a_1b_5|/|a_1b_2| \\
\text{II.} & & & |a_1c_2|/a_1 & |a_1c_3|/a_1 & |a_1c_4|/a_1 & |a_1c_5|/a_1 \\
 & & & |a_1d_2|/a_1 & |a_1d_3|/a_1 & |a_1d_4|/a_1 & |a_1d_5|/a_1 \\
\hline
 & |a_1b_2c_3|/|a_1b_2| & & & 1 & |a_1b_2c_4|/|a_1b_2c_3| & |a_1b_2c_5|/|a_1b_2c_3| \\
\text{III.} & & & & |a_1b_2d_3|/|a_1b_2| & |a_1b_2d_4|/|a_1b_2| & |a_1b_2d_5|/|a_1b_2| \\
\hline
\text{IV.} & |a_1b_2c_3d_4|/|a_1b_2c_3| & & & & 1 & |a_1b_2c_3d_5|/|a_1b_2c_3d_4| \\
\hline
\end{array}
$$

The computer, working systematically along row after row, will easily find the routine which is most congenial to him. A piece of card cut to an L-shape, or with an L-shaped window in it, sliding on a straight edge, or some similar device, may be found helpful to the eye in locating without confusion the tetrad of elements in a cross-multiplication; and of course two members of this tetrad, namely the unit pivot and an element vertically below it, remain fixed as one works along any row. A final column containing the "check sum" of elements in each row can, and should, be included in the scheme of computation.

There comes a stage when the pivotal row contains only one element, the unit pivot itself, to the left of the vertical dividing line. At the following stage there is nothing at all on the left; what appears on the right is the product $H'A^{-1}K$; and if the determinant $| A |$ is also required, it is the continued product of the pivots as they were before division, that is, of the marginal entries. If $| A |$ is zero and A of rank $n-1$, the last pivot will be found to be zero; in such a case no division is carried out on the row containing it, the next and final stage proceeds as before, and on multiplication of elements by the continued product of the non-zero pivots, $H(\text{adj } A) K$ is obtained, instead of $H'A^{-1}K$.

The use of leading elements as pivots is of course not compulsory.

It is merely the most arithmetically convenient, and would be adopted naturally in that large class of practical cases in which A is symmetric and positive definite. On the other hand, if the leading element were zero or small and subject to relatively large tabular error, the choice of some other pivot would be unavoidable, by preference one in the leading diagonal of the array at that stage; this may sometimes be arranged by special devices (see the end of § 3 (iv)), but is not always possible. The question of giving the correct sign to a computed array at any stage may therefore arise; probably the simplest way of taking account of this is to take a skeleton grid or lattice of horizontal and vertical lines, representing the rows and columns of the original array, and on it to mark off the position of the pivots as lattice points, remembering that to use a pivot is to delete a row as well as a column from the frame. If at the end, or at any stage, these points are in rows 1, 2, 3, . . . and in columns α, β, γ, . . ., the sign to be prefixed to the continued product of the marginally entered pivotal elements up to that stage is the sign of the permutation α, β, γ, . . . of natural order, and this can be found by any of the rules given in works on the theory of determinants.

3. Special Applications and Examples.

We give an example to illustrate the suggested routine.

Example 1.

$$H'=\begin{bmatrix}1\cdot132 & 2\cdot713 & -3\cdot241\\ 2\cdot329 & -0\cdot5609 & 4\cdot307\end{bmatrix},\quad A=\begin{bmatrix}2\cdot631 & 3\cdot374 & -1\cdot283\\ 4\cdot152 & 7\cdot126 & 3\cdot287\\ 2\cdot218 & -3\cdot569 & 3\cdot035\end{bmatrix},\quad K=\begin{bmatrix}3\cdot614 & 3\cdot827\\ 2\cdot658 & -2\cdot981\\ 1\cdot567 & -1\cdot326\end{bmatrix}.$$

The working appears on the next page, each row as it is completed being summed along and checked by the elements in the "check sum" column, which undergo pivotal cross-multiplication like the rest.

There are various special cases which have practical applications.

(i) In the case $H'=I$, where I is the unit matrix (which has unit elements in the diagonal, zeros everywhere else), the evaluation gives $A^{-1}K$; and in the case $K=I$ it gives $H'A^{-1}$.

It is not necessary to give numerical illustrations of these. The matrices are such as would arise in solving equations of the type $Ax=By$, for example:

$$\begin{aligned}a_{11}x_1+a_{12}x_2+a_{13}x_3 &= b_{11}y_1+b_{12}y_2,\\ a_{21}x_1+a_{22}x_2+a_{23}x_3 &= b_{21}y_1+b_{22}y_2,\\ a_{31}x_1+a_{32}x_2+a_{33}x_3 &= b_{31}y_1+b_{32}y_2.\end{aligned}$$

(ii) In the case where H' is a single row vector h', and K is a single column vector k, we obtain the reciprocal bilinear form $h'A^{-1}k$. When A is a symmetric matrix, i.e. $A'=A$ and $h=k$, we have the reciprocal

						Σ.
	[2·631	3·374	−1·283	3·614	3·827	12·163]
2·631	*1·0000*	1·2824	0·48765	1·3736	1·4546	4·6229
	4·152	7·126	3·287	2·658	−2·981	14·242
	2·218	−3·569	3·035	1·567	−1·326	1·925
	−1·132	−2·713	3·241	·	·	−0·604
	−2·329	0·5609	−4·307	·	·	−6·0751
1·8015		*1·0000*	2·9485	−1·6904	−5·0072	−2·7491
		−6·4134	4·1166	−1·4796	−4·5523	−8·3287
		−1·2613	2·6890	1·5549	1·6466	4·6292
		3·5476	−5·4427	3·1991	3·3878	4·6918
23·027			*1·0000*	−0·53506	−1·5923	−1·1274
			6·4079	−0·57720	−4·6690	1·1617
			−15·903	9·1960	21·151	14·444
				2·8514	5·5343	8·3857
				0·6869	−4·1713	−3·4844

$$\text{The determinant } |A| = 2{\cdot}631 \times 1{\cdot}8015 \times 23{\cdot}027 = 109{\cdot}14.$$

$$H'A^{-1}K = \begin{bmatrix} 2{\cdot}851 & 5{\cdot}534 \\ 0{\cdot}687 & -4{\cdot}171 \end{bmatrix}.$$

quadratic form $h'A^{-1}h$. The special use of such forms is in mathematical statistics, especially in the theory of correlated variables; for example, the "variance of error" of a set of variables x, expressed in standard measure and having a matrix R of total correlations, is equal to the positive definite form $x'R^{-1}x$.

Example 2.

$$h' = [1{\cdot}132 \quad 2{\cdot}713 \quad -3{\cdot}241], \qquad A = \begin{bmatrix} 2{\cdot}631 & 3{\cdot}374 & -1{\cdot}283 \\ 4{\cdot}152 & 7{\cdot}126 & 3{\cdot}287 \\ 2{\cdot}218 & -3{\cdot}569 & 3{\cdot}035 \end{bmatrix}, \qquad h = \begin{bmatrix} 3{\cdot}614 \\ 2{\cdot}658 \\ 1{\cdot}567 \end{bmatrix}.$$

The matrix chosen for illustration is that of Example 1, and the vectors are the first row of H' and the first column of K in that example. The details of working can therefore be followed from that example, if we disregard or delete the other bordering rows and columns and those derived from them at later stages. Thus the final value desired is the leading element (in this case it would be the only element) on the right at the last stage, namely 2·851.

(iii) If we put $H' = I$, and also $K = I$, we obtain the adjugate and the reciprocal of A; the latter only if A is non-singular.

Example 3.—To find the reciprocal of A as given in Example 1.

For economy of space we confine ourselves to an illustration of the third order only. Actually one would not in practice use this method for so low an order, the nine co-factors being easily calculable otherwise, and the determinant $|A|$ from four of them. The advantages of the present method appear only when the matrices are of the fourth or higher orders.

							Σ.
	[2·631	3·374	−1·283	1	.	.	5·722]
2·631	*1·0000*	1·2824	−0·48765	0·38008	.	.	2·1748
	4·152	7·126	3·287	.	1	.	15·565
	2·218	−3·569	3·035	.	.	1	2·684
	−1	.	.	.	.	.	−1·000
	.	−1	.	.	.	.	−1·000
	.	.	−1	.	.	.	−1·000
1·8015		*1·0000*	2·9485	−0·87599	0·55509	.	3·6276
		−6·4134	4·1166	−0·84302	.	1·0000	−2·1398
		1·2824	−0·48765	0·38008	.	.	1·1748
		−1·0000	.	.	.	.	−1·0000
		.	−1·0000	.	.	.	−1·0000
23·027			*1·0000*	−0·28059	0·15460	0·043427	0·91744
			−4·2688	1·5034	−0·71185	.	−3·4772
			2·9485	−0·87599	0·55509	.	2·6276
			−1·0000	.	.	.	−1·0000
			$A^{-1}=$	0·30562	−0·05189	0·18538	0·43911
				−0·04867	0·09925	−0·12804	−0·07746
				−0·28059	0·15460	0·043427	−0·08256

[Check by multiplying A by A^{-1}, row by column.]

$$\text{The determinant } |A| = 2·631 \times 1·8015 \times 23·027$$
$$= 109·14.$$

The reciprocal of A is thus obtained, and, pending examination of the possible error, we may take the elements to be correct to four digits. Careful analysis of the steps discloses that, apart from common factors due to pivotal division, the elements at intermediate stages are the same as those which would occur in the course of a method (Smith, 1927) of building up the adjugate of a matrix by successive bordering. In that method, however, special treatments are required for "outside elements" and "corner elements," a necessity which does not arise in the present method.

(iv) Finally, the solution of ordinary simultaneous equations is also a special case; for in matrix notation these may be written, $Ax=k$, or in transposed fashion, $x'A'=k'$, and the solution, if A is non-singular, is given by $x=A^{-1}k=IA^{-1}k$.

It is easier on the computing sheet to condense by stages downward than across. Hence, for this reason of convenience and no other, it is better to use the *transposed* form of the equations to be solved and accordingly to set up the scheme

$$\begin{array}{c|c} A' & -I \\ \hdashline k' & 0 \end{array}$$

in readiness for pivotal condensation, which then proceeds as before. When everything to the left of the vertical dividing line is cleared off, what remains on the right is a single row; its elements are the values of all the unknowns. Thus, perhaps at a slight sacrifice of speed as compared with some other extant methods, there is the advantage of uniformity; the usual resubstitution after elimination, the "back solution," is not required. Where the matrix of the equations is symmetrical, diagonal pivots will be used and the property of symmetry will halve the labour.

Complete symmetry in the condensations of a symmetrical matrix would be preserved if the pivotal row *and column* were divided by the square root of the pivot as first found, the pivot being thus reduced to unity by two divisions by its square root; but this is troublesome, and symmetry in all but the pivotal row and column will be preserved if, as usual, the pivotal *row alone* is divided by the pivot itself, and not by its square root, the corresponding *undivided* elements being entered in the pivotal column.

A *non-symmetric* set of simultaneous equations, $Ax=k$, may often with advantage be replaced, for purposes of solution, *by the corresponding normal equations* $A'Ax=A'k$, which are easily formed from the original set, and which possess the advantages of symmetry and positive definiteness. For example, diagonal or leading pivots could always be taken in such a case.

Example 4.

$$\begin{aligned} 2{\cdot}631x+3{\cdot}374y-1{\cdot}283z &= -\ 1{\cdot}032, \\ 4{\cdot}152x+7{\cdot}126y+3{\cdot}287z &= 18{\cdot}350, \\ 2{\cdot}218x-3{\cdot}569y+3{\cdot}035z &= 2{\cdot}783. \end{aligned}$$

We set up the detached coefficients in columns instead of rows, as on the next page, border on the right with negative units, and proceed.

The values of x, y, and z are as shown. A final check, which should always be applied, is that of substituting the values in the original equations, as has been done in the present example. The agreement is not perfect, but is satisfactory, subject, in the general case, to important reservations which it is now appropriate to mention.

				x.	*y.*	*z.*	Σ.
	[2·631	4·152	2·218	−1	·	·	8·001]
2·631	*1·0000*	1·5781	0·84302	−0·38008	·	·	3·0410
	3·374	7·126	−3·569	·	−1	·	5·931
	−1·283	3·287	3·035	·	·	−1	4·039
	−1·032	18·350	2·783	·	·	·	20·101
1·8015		*1·0000*	−3·5600	0·71185	−0·55509	·	−2·4032
		5·3117	4·1166	−0·48765	·	−1·0000	7·9407
		19·979	3·6530	−0·39224	·	·	23·240
23·027			*1·0000*	−0·18538	0·12804	−0·043427	0·8992
			74·778	−14·614	11·090	·	71·254
			Solutions	−0·752 *x*	1·515 *y*	3·247 *z*	4·010
			Checks by substitution	−1·033	18·346	2·780	

$$\text{Determinant } |A| = 2{\cdot}631 \times 1{\cdot}8015 \times 23{\cdot}027 = 109{\cdot}14.$$

4. The Question of Accuracy and Tabular Error.

What degree of accuracy can be attributed to the computed values of the unknowns, or of the elements in the product matrix $H'A^{-1}K$? Supposing, for example, that our initial elements are known to be trustworthy to four significant digits but are dubious beyond that, what principles must guide us in assessing how many digits to retain at later stages? This is a most difficult matter, not to be disposed of in a paragraph. It is the author's own experience that even when final digits in the intermediate stages of any computation are doubtful or actually wrong it is better not to discard them lightly, but to carry the computation through with them and to defer to the end of the whole work the consideration of what error may have been incurred. This does not mean, of course, that we are to burden the computation with several dubious digits in each item; two, perhaps one, will be enough. In the present case our final estimate of the accuracy of the results will be based on what is known (*e.g.* Etherington, 1932) concerning the maximum range and the probable range of error in determinants and in those special quotients of determinants which occur in the solution of linear equations; and we cannot afford to leave out of account the errors that may have been superinduced by the special order of computation, what pivots have been used, and so on. The value of the final check of resubstitution in the

original equations is not to be denied; but it is not a crucial test, for a set of equations having a determinant relatively small compared with its first minors (so that the elements of the reciprocal matrix are not fractions of the size of unity or so, but are quite large numbers) possesses instability, in the sense that the equations can be satisfied, to the order of accuracy of the given coefficients, by sets of solutions (*e.g.* Muir, 1930, p. 128) which may be very different from each other. The extreme case of this is the complete indeterminacy which arises when the determinant of the system is zero. In a case of suspected instability it would be better to compute, for inspection, the reciprocal matrix itself, and to operate with it * upon the vector of right-hand members of the equations. Thus, operating with A^{-1} as evaluated in Example 3 above, the elements being satisfactorily small, we might solve Example 4 as follows:—

Example 5.

$$A^{-1}k = \begin{bmatrix} 0\cdot30562 & -0\cdot05189 & 0\cdot18538 \\ -0\cdot04867 & 0\cdot09925 & -0\cdot12804 \\ -0\cdot28059 & 0\cdot15460 & 0\cdot043427 \end{bmatrix} \begin{bmatrix} -1\cdot032 \\ 18\cdot350 \\ 2\cdot783 \end{bmatrix} = \begin{bmatrix} -0\cdot7517 \\ 1\cdot5151 \\ 3\cdot2473 \end{bmatrix}.$$

We conclude that the solutions, as found in Example 4, may be safely adopted.

5. Final Remarks.

In an earlier paper (Aitken, 1933) the author, with a special application in view, used a method of solving simultaneous equations which differs from § 3 (iv), Example 4 above, only in the stages at which pivotal divisions are performed. This former method, in spite of the theoretical advantage of showing at every stage actual minors of the array and not quotients of minors (as in § 2 above), appears to be slightly more laborious. The methods of the present paper also supersede, in the opinion of the author, those of his earlier papers (Aitken, 1931, 1932) on related topics.

* This course would be taken in any case if there were several sets of equations with the same matrix, but differing in the right-hand members. See, for example, R. A. Fisher, *Statistical Methods for Research Workers*, 6th ed., 1936, p. 158.

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XIII.—Ions and Isotopes. By **Professor James Kendall**, F.R.S., Department of Chemistry, University of Edinburgh. (With One Figure.) (Address delivered to the Royal Society of Edinburgh on February 1, 1937.)

(MS. received March 10, 1937.)

Introduction.

The fact that the atomic weight of an element is not invariable, but that the same element may exist in two or more forms which are chemically identical even though their atomic weights differ, was first noted by Soddy (1910, p. 285) in connection with radioactive elements. A little later Thomson (1913) showed, by the method of positive ray analysis, that neon, with an average atomic weight of 20·2, was a mixture of two kinds of atoms with masses 20 and 22 respectively. The brilliant extension of this method of attack by Aston has established that the majority of the common elements are similarly inhomogeneous. Chlorine (35·457) is a mixture of Cl^{35} and Cl^{37}; lead (207·22) exists in as many as eight forms, with atomic masses ranging from 203 to 210. These chemically-identical species of the same element with diverse masses are known as *isotopes*.

The separation of isotopes has been strenuously attempted by many chemists during the last twenty years. There is only one instance where nature has, in a sense, performed this separation for us—lead derived from uranium disintegrations consists mainly of the isotope with mass 206, while lead derived from thorium disintegrations contains chiefly the isotope with mass 208. In all other cases the proportions of the various isotopes of any particular element appear to be practically constant, whatever the source. Partial separation has proved possible, although exceedingly difficult, in several investigations; complete separation has been achieved, very recently, with one element—hydrogen.

This communication summarizes a series of researches in this field which has been carried out by the author and his co-workers during the past eighteen years. Two lines of attack on the problem have been followed, both connected with properties in which it seemed likely that isotopes might exhibit slightly different behaviour, namely *ionic mobility* and *ionic discharge potential*.

PART I.—IONIC MOBILITY.

The development of the ionic migration method for effecting difficult chemical separations has been described in a number of articles appearing in the *Proceedings of the National Academy of Sciences* (Kendall and others, 1923–1925) and in the *Journal of the American Chemical Society* (Kendall and others, 1926).

The work was started, as noted above, as an attempt to devise a practicable method for the separation of isotopes. Other investigators had succeeded in obtaining, at best, only a very slight degree of separation of isotopic elements into their various atomic species after heroic expenditure of time and labour by other methods, and it appeared that there did exist the possibility here of obtaining a quick and decisive result. The situation with regard to ionic mobility may be explained very briefly. A long-standing controversy has been waged on this property; one school insisting that ionic mobility is fundamentally dependent upon ionic volume, another being equally confident that it is fundamentally dependent upon ionic mass. The results available in the literature for homologous series of organic anions and cations have been utilized by both parties to prove their respective points; but, since we have no definite knowledge as to what amount of solvent accompanies any ion in its journey towards an electrode and since it is the *total mass* or *total volume* of the ion and of its accompanying solvent envelope which must be taken into account, such data obviously offer us no means for definitely determining the problem.

The discovery that isotopes possess equal atomic volumes first put us in a position to impose a crucial test, for isotopic ions necessarily *differ in mass*. If mass is influential, therefore, it should be possible to obtain a separation of isotopes by taking advantage of the fact that the lighter ion will migrate more rapidly than the heavier. This idea of an "isotopic race," however, cannot be carried out experimentally as simply as it might seem at first sight. Ions do not compete under the influence of the electric current in the manner of a track meet, unless we extend our experiences to imagine a continuous relay race. We cannot start all of our ions at one point and obtain a separation by noting when those of a certain species have passed a given goal, for there must be maintained a steady supply all the way from one electrode to another in order for the current to pass, and it will not help us much if a faster ion hurries ahead of its slower neighbour, since it will merely find itself in the company of other slower ions which happened to start a little in advance. By a modification of the experimental procedure, nevertheless, our "isotopic

race" may be converted into a "parade" which can be suitably regulated.

The apparatus used is shown in the accompanying diagram, and its applicability may be illustrated by a condensed description of the technique employed in the case of chlorine.

An agar-agar gel A containing sodium chloride is inserted as a short middle section in a long horizontal tube of pyrex glass, one and a half inches in internal diameter. On one side of the chloride gel is added a gel B containing sodium hydroxide; on the other side a gel C containing sodium acetate. The ends of the tube are connected with right-angled pyrex bends of the same diameter, and the gels continue well up into these bends, as in the diagram. Above the hydroxide gel, after it has set, is poured concentrated sodium hydroxide solution D, and above the acetate

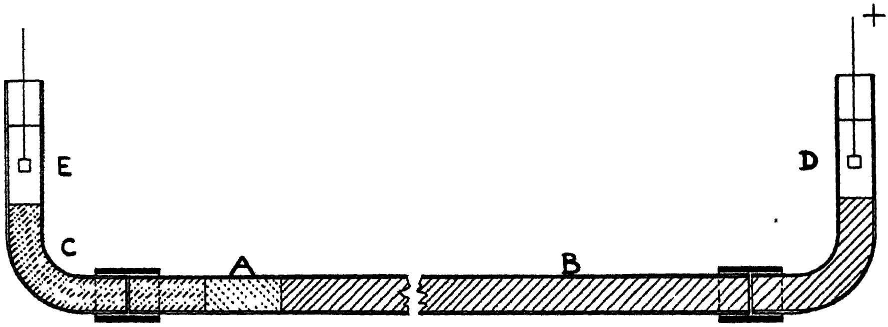

FIG. 1.

gel a solution E of sodium acetate in concentrated acetic acid. Platinum electrodes are placed in these two solutions, and a current is passed through the tube, the electrode in D being made the anode and the electrode in E the cathode. The heating effect of the current upon the gel may be minimized by submerging the tube in a long trough filled with running tap-water.

At the beginning of the run, the boundary surfaces between the various sections of the gel are perfectly sharp. When the current is turned on, the boundaries move toward the anode. Inasmuch as there is a faster-moving ion in front of the chloride ion, and a slower-moving ion behind it, admixture of the salts is absolutely prohibited and the boundaries remain distinct throughout the whole experiment. In order to maintain the width of the chloride section approximately constant, it is well to arrange the concentrations of the various salts in their respective gels in accordance with the transference numbers of their anions. Even if this is not done exactly, however, the boundary concentrations soon automatically adjust themselves to the required ratios. Care must be taken

that the solution around the cathode always contains sufficient excess of acetic acid to neutralize the sodium hydroxide that is there formed.

The rate at which the boundaries move depends upon the potential drop between the electrodes, the length of the tube, and the concentrations of the solutions. In actual practice, the horizontal tube is made up of several three-foot sections connected by rubber bands, and the current is regulated (100 to 500 volts) so that the boundaries advance about 12 to 18 inches a day. When the front chloride boundary has almost reached the end of the tube, the apparatus is taken apart. The two rear sections are discarded, two new sections filled with hydroxide gel are inserted in front of the chloride, and new bends are fitted on as before. The chloride ions are now forced to migrate into these two new sections, and the whole procedure is repeated until they have progressed through about 100 feet of the gel. The chloride gel is then removed from the tube and immediately cut up into strips about 1 cm. in width.

If, now, the isotopic chloride ions with atomic masses 35 and 37 respectively possess significantly different mobilities, the front segments of the gel will contain only the faster-moving and the rear segments only the slower-moving isotope. Preliminary experiments in which a mixture of two sodium salts with anions of known mobilities was substituted for the middle sodium chloride section fully demonstrated this point. Thus when a mixture of sodium iodide and sodium thiocyanate was employed and the boundary was moved only a few feet, the front segments were found to contain only sodium iodide and the rear segments only sodium thiocyanate. The observed lag of the slower ion was almost exactly equal to that calculated from the difference in mobilities. In this particular case the difference is approximately 16 per cent. In subsequent experiments this difference was narrowed by choosing other suitable pairs of ions: *e.g.* barium and calcium, barium and strontium, and iodide and chloride. The difference in mobilities are here reduced to 8 per cent., 5 per cent., and 1 per cent. respectively, and the results obtained showed that the method could be depended upon to provide a very considerable degree of separation even at the lowest limit tested.

It was highly disappointing, therefore, to be forced to admit from the results of all our experiments with isotopic ions that no significant separation could ever be detected. The only conclusion that can logically be drawn is that the mobilities of isotopic ions are well within 1 per cent. of equality, and that those workers are substantially justified who contend that volume is the decisive factor in determining ionic mobilities. More recent theoretical advances, it must be added, support this view, although the matter is still not absolutely settled (Jette, 1927).

Experimental difficulties encountered in connection with this work on isotopes cannot be discussed in full detail here, but mention may be made, in passing, of two points. One was the impossibility of determining whether any separation was proceeding within a section except by interrupting its progress irretrievably through segmentation and analysis. In other words, there was no way of telling in advance if a run of 100 feet was any more advantageous than a run of 10 feet, or whether a run of 500 feet should be preferred. It was proposed by one assistant to solve this difficulty by incorporating with the material under examination a small amount of a substance containing a coloured ion with a mobility intermediate between those of the two isotopes. The end of an experiment would then be automatically announced by the appearance of a thin coloured strip in the centre of the section, the gel in front of this strip containing only the faster isotope and the gel in the rear of it only the slower. Unfortunately, no coloured ion with the properties required has yet been discovered. A second difficulty consisted in fixing the exact position of the boundaries during a run. Slight changes in the colour of the agar-agar gel induced by the different salts present, or slight differences in the refractive indices of the various sections, were sometimes sufficiently good indications, but the much larger variations in their electrical resistances offered, in general, more dependable assistance. The approximate position of a boundary could frequently be determined merely by lifting the apparatus from the water-trough for a few minutes, running the hand along the tube and noting the place at which a temperature gradient became evident. Finally, more exact information was obtained by fusing two short platinum wires through the glass at suitable points on the tubes and establishing the passage of a boundary past these points by noting when an abrupt change occurred in the electrical resistance of the gel between them.

Although all the work on isotopes, as has been stated, led to no successful separations, yet the positive results obtained in the test experiments on known mixtures did inspire the hope that the ionic migration method might, after all, prove of practical service in the separation of more familiar materials which are ordinarily obtained by the chemist in a pure state only with extreme difficulty. The most important instance of this type is furnished by the family of the rare earth metals. The group of elements known as the rare earths comprises the elements of atomic number 57 to 71 inclusive, and yttrium with atomic number 39. These elements are distinguished by such extraordinarily alikeness in chemical and most physical properties, due to the identical arrangements of the two outer shells of electrons in their atoms, that

they actually approach isotopes in similarity. It is necessary, in practice, to take advantage of slight differences in solubility observed for corresponding salts and to resolve a given mixture into its components by a long series of fractional crystallizations or precipitations, ranging in number from several hundred to many thousand according to the elements present and the degree of purity desired. These operations are so laborious and time-consuming that chemically pure samples of individual rare earth compounds are practically unknown, save on the shelves of a few skilled workers in the field of atomic weights. The "rareness" of the rare earths is due not so much to the lack of abundance of their ores in nature as to the lack of a simple method for their separation.

Little is known of the relative mobilities of the rare earth cations in aqueous solution, but ionic volume and hydration variations within the group may be expected to cause differences in ionic mobilities which will, in most cases, exceed one per cent., so that a ready analysis of a given mixture into its pure components should be, in general, feasible. The experiments described below furnish, in fact, two instances of the successful separation of typical binary rare earth mixtures by the ionic migration method.

The first separation attempted was upon a mixture of yttrium and erbium, kindly furnished by Professor James, of the University of New Hampshire. Potassium was used as a preceding faster ion and trivalent chromium as a following slower ion. Not only could the position of the rear boundary be more conveniently followed in this particular case because of the colour of the chromic solution, but it was also found possible to observe the actual progress of the separation within the rare earth section in a very simple way by means of a small direct-vision hand spectroscope. The majority of the rare earths give solutions with characteristic absorption spectra, and, by noting the increase or decrease in intensity of the most prominent lines in various parts of a section, the experimenter could immediately detect any change in composition in the whole length of the section without interrupting the run. Here yttrium rapidly accumulated in the front half of the section and erbium was relegated to the rear. After a run of fourteen days, during which the boundaries moved two meters, an almost perfect separation had been effected.

The next mixture tested consisted of neodymium and praseodymium, two elements which are so nearly alike that their first differentiation by Otto von Welsbach still forms one of the triumphs of technique in this difficult field. Neodymium salts exhibit a purple solution in water, while praseodymium salts give a green solution. A beautiful crystal

of neodymium nitrate and an equally fine specimen of praseodymium sulphate were secured from the Chandler chemical museum of Columbia University and the careful work of several years was deranged in a few minutes by mixing these two salts to obtain a solution with an intermediate neutral tint. With this solution as a central section and with the same arrangement as in the preceding experiment, it required only a few days' migration to disclose the fact that the front portion of the section was becoming distinctly green and the rear portion purplish. Observation by means of the spectroscope simultaneously showed that the characteristic absorption lines of praseodymium were becoming more pronounced in the front section and fading out in the rear, with the opposite behaviour for the neodymium. At the end of ten days, substantial progress towards a final separation had been accomplished.

Experiments with other mixtures of rare earths showed that, while complete separation could not be secured in every case, yet in most mixtures the mobilities of the ions were sufficiently divergent to compel a very rapid concentration of one particular component in the front or in the rear section. The method may, therefore, be considered as a general new method for obtaining pure samples of the rare earth elements with the expenditure of much less time and trouble than is required by the classical method of fractional crystallization (see, however, Selwood and Hopkins, 1929).

Important industrial uses for the rare earth elements will undoubtedly be discovered as soon as more convenient means for their isolation are developed. Aluminium remained a chemical curiosity until this same end was achieved, and while the later members of the aluminium family may not duplicate the successful career of their brilliant brother, yet it would indeed be strange if nature had omitted to endow them, alone of all the elements, with no properties of service to mankind.

The success of the experiments with rare earths suggested that the method might be applicable to the separation of radium from barium. The concentration of radium from the barium residues of carnotite ores at present involves a very tedious series of fractional crystallizations, and, since it had already been found that barium could be separated from the other elements of the alkaline earth family by the method here under discussion, it appeared very probable that a similar separation from radium, the last member of this same family, could also be accomplished. Samples of barium residues containing known amounts of radium and of mesothorium (an isotope of radium) were obtained and, after a few days' migration, the sections were segmented and their radioactivity examined. It was found that the radioactive components of each mixture

tested accumulated very rapidly in the front part of the section. The ease of the separation in this case is so striking as to suggest that the ionic migration method may come into technical use for the concentration of radium in barium residues.

An attempt to separate hafnium and zirconium did not lead to such conclusive results. Hafnium exists to the extent of several per cent. in all zirconium ores, and the similarity in properties is so pronounced that the actual discovery of hafnium was not definitely established until 1923. A sample of hafnium-rich zirconium oxide was kindly furnished us by Professor Hevesy. The elements were not amenable to separation by the ionic migration method in the form of positive ions, owing to hydrolysis, and only after considerable search was a suitable negative complex ion discovered in the form of a complex oxalate. After long migration, analysis showed a very slight accumulation of hafnium in the rear of the section, the mobilities of the complex anions being evidently so close together as to render a complete separation impracticable.

The results in an entirely different field have been of better promise. Many of the alkaloids particularly useful for medicinal purposes are derived from natural sources as mixtures of several individual members which can be separated by the ordinary methods of organic chemistry only with great difficulty. The alkaloids, however, are weak bases which form soluble hydrochlorides, and the mobilities of the cations of these salts are not identical. It should, consequently, be possible to obtain a pure sample of an especially valuable alkaloid from the mixture of similar materials with which it naturally occurs by use of the ionic migration method, and several preliminary experiments indicate that the method is indeed applicable in a majority of cases.

Finally, the possibility may be mentioned of the applicability of the method to the separation of organic isomers of various types. More complex bio-chemical problems, such as the concentration or isolation of specific proteins or even of vitamins from natural sources, are probably also open to attack by the ionic migration method, but the experimental technique in such cases has not yet been worked out in detail.

PART II.—IONIC DISCHARGE POTENTIAL.

Isotopic ions differ only in their minute nuclei, and these nuclei are surrounded by a relatively-distant cloud of external electrons, increasing in number as we proceed up the periodic system. When an ion is discharged at one of the electrodes during the process of electrolysis, all that happens is that one or more electrons is added to, or taken from, this

external cloud. Although, therefore, the free energy changes involved in the discharge of two isotopic ions are not necessarily identical, it is unlikely that the discharge potentials will be markedly different in the two cases, when expressed in volts (Lindemann, 1918). If we could pass an electric current through a solution containing two isotopic ions, adjusting the voltage so that the decomposition potential of one species was *just exceeded* while the decomposition potential of the other species was *just not attained*, then a perfect separation should be feasible. Owing, however, to the impossibility of adjusting the voltage with sufficient delicacy, and to the fact that reverse interaction of the discharged material with the solution, unless its immediate withdrawal from the system was effected, would probably tend to re-establish equilibrium conditions fairly rapidly, it appeared that the utmost that would be attainable in practice would be an "electrolytic fractionation."

Encouragement that a partial separation might be achieved by this method was derived from the knowledge that it is possible to vary the composition of a tin-lead alloy, obtained by electrolysis of a mixed solution of tin and lead salts, very significantly by varying the voltage employed. The discharge potentials of tin and lead ions differ by approximately 0·01 volt, so that it seemed that there was a likelihood of being able to establish some slight degree of separation of isotopic ions by fractional electrolysis, even if their discharge potentials were still more nearly identical.

The choice of an element to test the method presented great difficulty. Other things being equal a light element would obviously be preferable, since the external electrons are fewer in number and shield the nucleus less effectively. The experimental obstacles in the case of lithium, the lightest known isotopic element at the time when this work was started (1921) are, however, exceptionally grave, owing to the activity of the free metal and to the elaborate technique necessary for its exact atomic weight determination. Other light elements presented similar objections, and the selection finally fell upon mercury. It is true that mercury is very near to the wrong end of the periodic system, its isotopes ranging in atomic mass from 197 to 204, but the free metal is inert and the smallest change in its average atomic weight can be immediately and readily detected by means of density determinations. It was consequently decided to attempt the fractional electrolysis of mercury first, and to continue with lithium later, if the results obtained were sufficiently promising.

The results obtained have been presented in detail in a Ph.D. thesis from Columbia University (Haring, 1924). Electrolysis of an acid solution of mercurous nitrate, using a low voltage and low current density,

gave mercury with a density only 0·999981 that of ordinary mercury. This value was the average of fourteen different electrolyses. Parallel experiments in which a high voltage and high current density were employed gave mercury with density unchanged.

The reduction in density, it will be seen, was only 19 parts per million, which represented three times the average deviation of the individual results. The greatest difficulty encountered in the whole investigation was the exact setting of the mercury level in the pycnometers and, although the error involved thereby was minimized by repeating each setting five times and averaging the results of the five weighings, the probable error involved still amounted to 7·5 parts per million. This fact, together with the disappointingly small change (0·004 unit) in the average atomic weight of the electrolyzed mercury, discouraged us from proceeding further at the time.

I have always been convinced, however, of the reality of the indicated fractionation and, in the expectation that more conclusive results would be obtained with a lighter element than mercury, Mr James M'Laren of Syracuse University began a systematic study of the electrolytic method of isotopic fractionation under the direction of myself and Dr E. B. Ludlam at the University of Edinburgh in 1930. A considerable amount of work had been done on lithium, but no final results attained, when the discovery of the isotopic character of hydrogen turned our attention early in 1932 to that element, which obviously offered still greater chances of success. Spectrographic evidence suggesting that the proportion of the heavier hydrogen isotope in the first fraction of the electrolysis of water was less than in ordinary hydrogen had been obtained when parallel investigations in America (Washburn and others, 1932–33; Lewis, 1933) were brought to a successful conclusion. The isolation of pure "heavy hydrogen" and the preparation of pure "heavy water" from the final fractions of the repeated electrolysis of ordinary water must be regarded as the most significant achievement in chemistry in recent years. It is disappointing to have been anticipated, but there is some consolation available in being the first to employ the method by which the separation was made.

Summary.

It was for a long time a matter of dispute whether the mobility of an ion (the speed with which it migrates through the solution towards the electrode under the influence of the electric current) is dependent upon its *volume* or upon its *mass*. No decision was possible until the existence

of isotopes (ions of the same element, with identical volumes but different masses) was established. An apparatus was devised in which a short section of a solution containing a mixture of isotopic ions was run for a considerable distance along with a series of tubes, with the idea that if mass were the significant factor affecting mobility the faster species would, at the end of the run, all be in the front of the section and the slower species all at the rear, and a quantitative separation of natural isotopes, hitherto unattainable, would thus be achieved.

The results were entirely negative, and it therefore appears that volume is the significant factor in ionic mobility. By means of similar experiments with synthetic mixtures containing two distinct ions of slightly different mobilities, however, it was found that many important separations, most difficult to make by standard methods such as fractional crystallization, could be very readily effected. Mixtures of the rare earth elements, for example, from which the two components were obtainable in a pure state only after many hundreds of fractional crystallizations, were separable within a few days. It was shown that the method could also be successfully used in the separation of radium from barium residues. It might also be applicable in resolving mixtures of naturally occurring alkaloids.

Another method by which the separation of isotopes was attempted was that of fractional electrolysis, and in 1923 a slight fractionation of mercury was achieved thereby. In the expectation that more conclusive results would be obtained with a lighter element than mercury, a systematic study of this method was begun in Edinburgh in 1930. A large amount of work had already been done on lithium, when the discovery of the isotopic character of hydrogen diverted attention to that element. Spectrographic evidence that the proportion of "heavy hydrogen" in the first fractions of the electrolysis of water was less than in ordinary hydrogen had been obtained, when investigators in America announced the isolation of pure "heavy hydrogen" from the final fractions of the electrolysis.

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XIV.—The Genetical and Mechanical Properties of Sex Chromosomes. III. Man. By P. C. Koller, D.Sc., Ph.D., Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D., D.Sc. (With Two Plates and Twelve Figures.)

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CONTENTS.

	PAGE
Introduction	194
Material and Technique	196
Chromosome Behaviour during Mitosis	196
Behaviour of the X and Y Chromosomes	198
(i) Mitosis	198
(ii) Meiosis	200
Structural Hybridity in Man	205
Structure of the X and Y Chromosomes	207
Summary	212
References to Literature	213
Description of Plates	214

Introduction.

In the male chromosome complement of man, Painter (1923) found that one chromosome pair is heteromorphic and forms an asymmetrical or unequal bivalent during meiotic metaphase, and consequently two kinds of gametes are produced. One member of this heteromorphic pair is about three times longer than the other and is designated the X, the other is described as the Y chromosome. Though the sex chromosomes of man were carefully studied and identified with certainty throughout the nuclear cycle of mitosis and meiosis by several investigators (*cf.* Shiwago and Andres, 1932), no further attempt was made to correlate the peculiar pairing properties of the heteromorphic chromosomes with their qualitative and structural differentiation.

A more detailed analysis, however, became of primary importance in view of the recently discovered and daily accumulating data on human genetics. Characters in man, such as hæmophilia and colour-blindness, follow the law of sex-linked inheritance; the genes of these characters

are therefore necessarily borne on the sex chromosome, and more exactly in the region which is differential and is concerned with sex determination. The great advance of cytogenetics during the last few years enables us to undertake a further analysis of the behaviour and structure of the heteromorphic X and Y chromosomes during the entire nuclear cycle in order to define the degree and manner of their differentiation.

It is generally believed that normal segregation of chromosomes during the reduction division is conditioned by pairing at meiotic metaphase. The abundant data of genetics (Dobzhansky, 1932) and cytology (Darlington, 1937) have brought forward evidence that the post-pachytene association of homologous chromosomes is a result of genetical crossing-over, *i.e.* of an exchange between partner chromatids of the homologues, followed by chiasma formation.

Unequal chromosome pairs, such as the X and Y of man and several other mammals, are necessarily differentiated into a pairing and non-pairing or differential segment, and the post-diplotene association between them is maintained by chiasmata in the pairing segment, as was found in the mouse (Crew and Koller, 1932), rat (Koller and Darlington, 1934), ferret and mole (Koller, 1936*a*). The important factor which determines the meiotic behaviour of these chromosomes is the position of the pairing and non-pairing segments in relation to the centromere (spindle attachment). In the mouse, rat and ferret, chiasmata were found in the pairing segment of the X and Y on either side of the centromere, and the presence of post-reduction of these chromosomes is cytological proof of a suggestion previously made on genetical grounds in *Bombyx* (Goldschmidt and Katsuki, 1931). Those genes which are localised in the pairing segment exhibit crossing-over with sex if the chiasmata lie between the locus of the gene and the region in which the X and Y differ. In view of this observation it was suggested by Darlington, Haldane and Koller (1934) that the unequal X and Y of man most probably show a similar behaviour, and that genetical data may prove the existence of partial sex linkage of particular genes. In 1935 Philip found partial sex linkage in *Drosophila melanogaster*. Recently Haldane (1936*a*) published a provisional map of sex chromosomes in man, and identified six genes which appear to be incompletely sex linked which he assumes to be localised in the pairing segment.

The aim of the present study is to analyse the behaviour of the X and Y chromosomes during the entire nuclear cycle in order to determine their internal structure, *i.e.* the position and length of the pairing and differential segments in relation to the centromere. This may provide cytological proof of partial sex linkage in man.

Material and Technique.

Testicular material was fixed in strong Flemming, Hermann, Allen-B, Bouin-15 and Champy-Minouchi solution. The sections were cut at 10–12–15–20 μ thickness and stained with iron hæmatoxylin, gentian violet, toluidin blue and Feulgen's basic fuchsin.

The drawings were made with the aid of a Zeiss camera lucida, using a 1·4 apochr. oil-immersion objective, and $\times 20$ or $\times 30$ comp. eyepiece. The approximate magnification is given in the legends.

The author takes this opportunity to thank Dr H. W. Beams (State University of Iowa, U.S.A.) for part of the material used during this study. He wishes also to acknowledge the helpful criticism of Dr C. D. Darlington (John Innes Horticultural Institution, London).

Chromosome Behaviour during Mitosis.

The first complete account of human chromosomes was given by Painter (1923, 1924), who found the diploid number to be 48 in both sexes. He observed that in the male there is one pair of dissimilar chromosomes, while in the female all pairs are composed of two similar chromosomes. Evans and Swezy (1929) in their detailed monograph on human chromosomes verified Painter's data.

In spermatogonial cells in which the chromosomes were scattered, I counted 48 chromosomes (text-fig. 1, *a*). They vary in length and shape. The longest is about 6–7 μ and the smallest 1·5 μ. Three kinds of chromosomes could be identified, namely, V-shaped with equal arms, V-shaped with unequal, and rod-shaped.

The sizes of chromosomes vary in different cells, partly due to size differences of the cells themselves. Several instances were found, however, in which this difference was brought about by contraction; the chromosomes in the spermatogonia appear as thick, egg- or ball-shaped bodies. They are either scattered in the cell (text-fig. 1, *d*) or closely grouped together (text-fig. 1, *f*) resembling bivalents formed by small chromosomes during meiotic metaphase, *e.g.* in *Dahlia*, *Sigara*, etc. Such a condition in the human spermatogonia was first noticed by Painter (1923), who described the chromosomes as "balled" chromosomes. Cells with such chromosomes usually occupy a large area. It is probable that these cells may represent spermatogonia showing the premeiotic division. The strongly contracted chromosomes indicate a time lag, the prophase commenced earlier and hence it is longer, and the chromosomes contract more. The "precocity" of prophase, however, is not so far advanced that it should commence before the division of chromosomes; such extreme precocity is characteristic of the following meiotic division (Darlington, 1931).

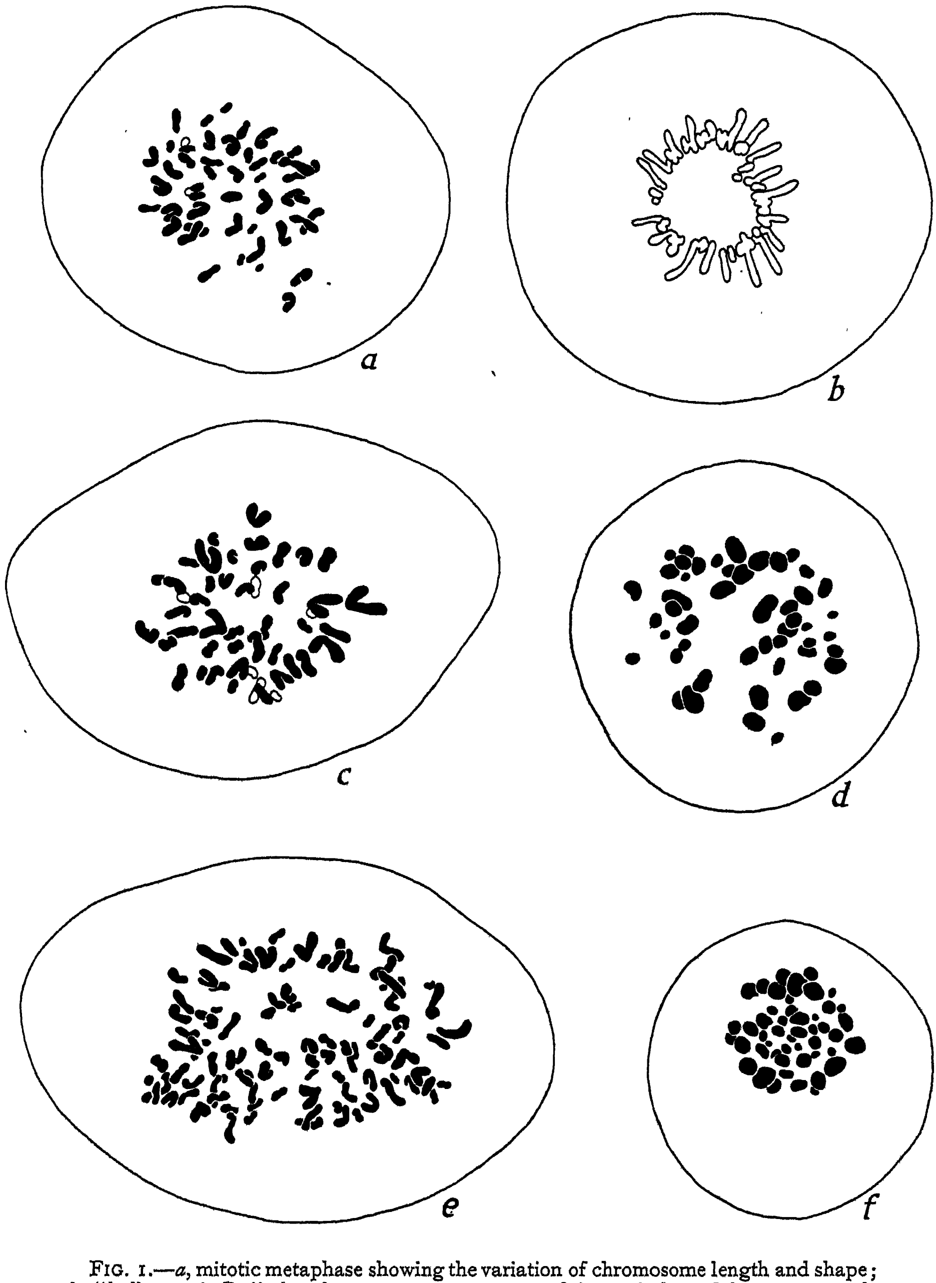


FIG. 1.—*a*, mitotic metaphase showing the variation of chromosome length and shape; *b*, "hollow spindle," the chromosomes are arranged in a circle and less contracted; *c*, giant spermatogonial cell with diploid chromosome number; *d*, "balled" or strongly contracted chromosomes; *e*, tetraploid cell at metaphase; *f*, "balled" chromosomes at metaphase closely grouped together. ×3000.

The chromosomes in several spermatogonia were arranged at the centre of the cell during mitotic metaphase and formed a circle. Because the centromeres are localised at the periphery of the spindle, the inner region of the spindle is therefore free from chromosomes (Pl. I, D; text-fig. 1, *b*). This chromosome arrangement is similar to the metaphase pattern which Belar (1928) described in *Salamander* and *Triton*, and designated as the "hollow spindle." It was found that such a chromosome arrangement is invariably associated with a smaller degree of contraction; the chromosomes are thinner and longer than they are in other spermatogonia with chromosomes evenly distributed on the equatorial plate. Similar chromosome behaviour was noted by Kemp (1930) in tissue culture of human embryonic heart. The same correlation between contraction and distribution has been found by Darlington (1936*a*) in pollen grains of *Fritillaria*.

Giant spermatogonia with the diploid chromosome number are occasionally present in the seminiferous tubules. They show the characteristic stages of mitosis (text-fig. 1, *c*). Some polyploid spermatogonia were also found, and in these the chromosomes were numerous; in one cell 90 were counted (text-fig. 1, *e*). Painter (1923) reports that polyploid cells undergo meiosis, but in the material at my disposal this has not been observed.

The frequency of the above-described anomalies is given in Table I.

TABLE I.—CHROMOSOME BEHAVIOUR OF SPERMATOGONIAL CELLS DURING METAPHASE.

No. of Slides.	Normal Mitosis.	Hollow Spindle.	Giant Cells.	Balled Chromosomes.	Polyploid Cells.	Total.
A 1	210	12	10	74	5	
B 12	120	3	..	33	11	
F 6	142	7	17	56	4	
Total	472	22	27	163	20	704
Per cent.	67·1	3·1	3·8	23·1	2·8	

Several other abnormalities were also encountered during the present study, such as tripolar spindles, binucleate spermatogonia, double-headed sperms, chromosome interlocking during meiotic prophase and spermatocytes with several univalents.

BEHAVIOUR OF THE X AND Y CHROMOSOMES.

(i) *Mitosis*.

The sex chromosomes in the spermatogonial cells do not exhibit precocity as regards contraction or condensation, and therefore they

cannot be identified at the mitotic prophase. The small member of the sex complex, which is called the Y chromosome, may, however, be recognised by its size during the metaphase, being the smallest chromosome in the complement. According to Oguma (1930) the Y is not a real sex chromosome, but it is a chromatoid body or the remnant of the plasmosome; therefore the heterogametic sex in man is of XO type and has only 47 chromosomes. Recent investigations (King and Beams, 1936), however, clearly demonstrate the existence of the small Y chromosome. It was also recognised during the present study.

Chromosomes lying off the equator were often found at mitotic metaphase. Their number is 1–2, very rarely 3 (Table II).

TABLE II.—THE OCCURRENCE OF MITOTIC METAPHASE SHOWING DISPLACED CHROMOSOMES.

Total No. of Metaphases.	Metaphases without Displaced Chromosomes.	Metaphases with Displaced Chromosomes. 1.	2.	3.
183	153	21	13	2
Per cent.	80·9	11·1	6·9	1·1

In view of the fact that chromosomes which lie off the equatorial plate in other organisms such as rat, mouse, ferret were identified by their behaviour as the sex chromosomes, it is probable that in man the sex chromosomes also exhibit the same property. The larger chromosome, assumed to be the X, if it is displaced, remains nearer to the equator than the small Y (Pl. II, E). The maximum distance between the sex chromosomes and the centrosomes is about half of the distance between the equatorial plate and the centrosomes. A very similar arrangement of the sex chromosomes in rat has been attributed to their lower surface charge (Koller and Darlington, 1934).

The length of the X chromosome varies between 4–5 μ, while that of the Y was calculated to be approximately 1·5 μ. The small variation in the length of X may be due to the fact that the chromosome does not lie in one plane; furthermore, there is a definite size difference between different spermatocytes and consequently between their chromosome complement. A definite range of variation in the length of autosomes of man was reported by Shiwago and Andres (1935).

The shape of the X chromosome is a crescent with a more or less distinct subterminal constriction which divides the chromosome into two segments. The Y chromosome is oval and shows no constriction. Lagging of the Y was occasionally observed during anaphase (Pl. II, I; text-fig. 2). The other chromosomes, including the X, segregate to the poles,

while the Y remains at the equator showing a double structure (text-fig. 2, *b*). In a few mitotic telophases it was found that the membranes of the daughter nuclei were formed before the Y chromosome completed its segregation (text-fig. 2, *c*). The lagging of the small sex chromosome must be causally connected with its position at the preceding metaphase when it lies near to the pole, and it is very probable that the degree of lagging is a measurement of the distance of the Y chromosome from the equator.

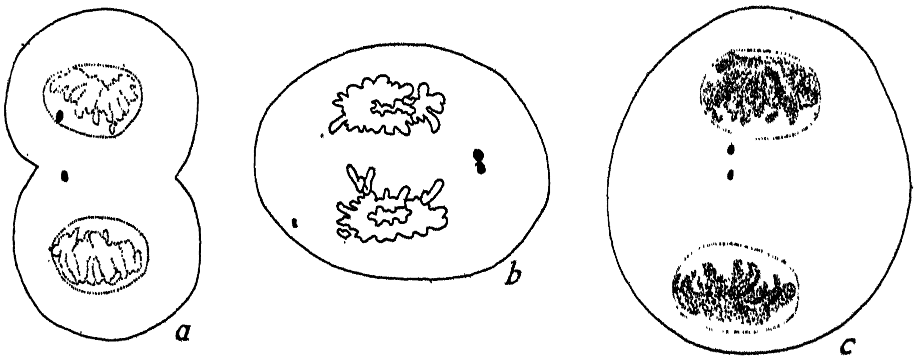


FIG. 2.—*a*, *b* and *c*, lagging of the small Y chromosome at mitotic anaphase and telophase. × 3000.

(ii) *Meiosis.*

During leptotene and zygotene the nuclei of the primary spermatocytes contain two or three chromatin aggregates. They stain deeply with gentian violet and hæmatoxylin, and are scattered amongst the fine, thin nuclear network. One of these aggregates represents the nucleolus, which is a characteristic constituent of the human nucleus. The other bodies could be identified as the precocious sex chromosomes. They are usually associated, and form a bipartite structure which may be seen more distinctly at pachytene (Pl. I, B; text-fig. 3, *a*). The associated or paired sex chromosomes at that stage closely resemble the XY complex found during the early meiotic prophase of *Trichosurus* and *Pseudochirus* (Koller, 1936*b*). The free or unpaired regions of the sex chromosomes are unequal, and the length of the paired segment varies in the different nuclei; sometimes it is about 1/3 of the total length of the X chromosome, though more frequently the X and Y are associated in a very short terminal region only, which is clearly seen at pachytene (text-fig. 3, *b*). Occasionally the sex chromosomes appear to be unconnected though they lie close together.

At pachytene the paired threads of the autosomal chromosomes are frequently arranged with their ends turned towards an attraction pole, displaying a strong polarisation (Pl. I, G). At this stage the XY complex

can easily be identified; its position is at random, it lies amongst the polarised pachytene chromosomes and is not connected with the nucleolus. The unpaired segments of the XY complex lie distinctly apart and show a great degree of condensation, forming two deeply stained terminal "knobs" of varying size (Pl. I, G). The paired region includes the subterminal segments, though in a few instances only terminal connections were seen.

At late diplotene the autosomal bivalents show loops separated by chiasmata; their number is between 1–4 in different bivalents, according to their length (text-fig. 3, *c*). The association of the X and Y is either terminal or subterminal, the latter type being more frequent. At that

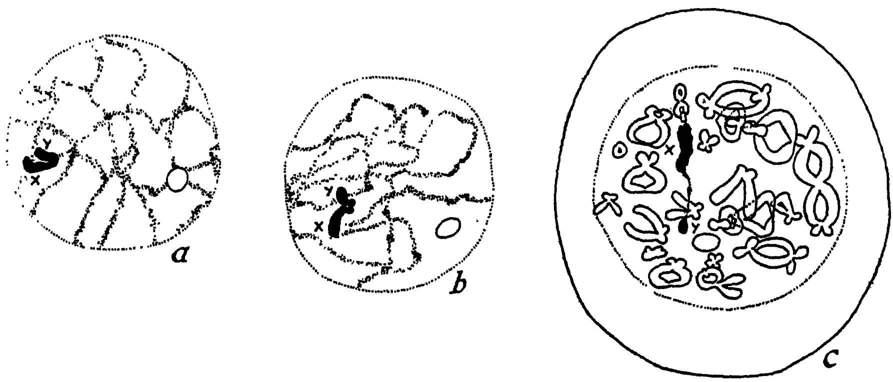


FIG. 3.—*a* and *b*, pachytene stage showing the structure of the precociously condensed XY complex; *c*, diplotene stage, the XY bivalent is associated terminally. ×3000.

stage the X and Y chromosomes exhibit almost the same degree of condensation as the autosomal bivalents. In the following diakinesis several instances were found in which the XY bivalent still contained subterminal chiasmata (Pl. II, F, G; text-fig. 4, *a*, *f*).

At metaphase the autosomal bivalents are closely packed together in the equatorial plate. The XY complex showing the inequality had been identified in about 30–35 per cent. of the spermatocytes at this stage because either both the X and Y (text-figs. 4, *e*; 5, *b*, *e*), or the X chromosome lies off the metaphase plate (text-figs. 4, *b*, *c*, *d*; 5, *a*, *d*, *f*). If the X and Y are pushed off the equatorial plate, they lie at the periphery of the spindle (Pl. I, C; II, B). A similar behaviour of the sex bivalent was found in other mammals, *e.g.* mouse, rat, ferret, mole and marsupials, and is most probably an outcome of the specific differentiation of that particular chromosome pair.

The unequal (asymmetrical or heteromorphic) sex bivalent in man is easily recognisable owing to its shape and position. The members of

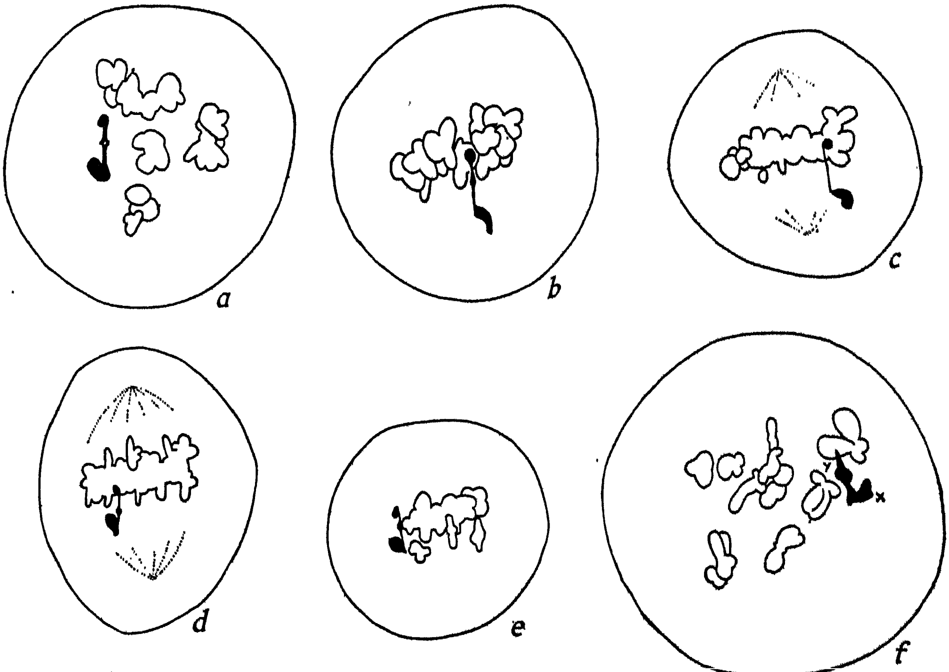


FIG. 4.—The asymmetrical XY bivalent with subterminal chiasma: *a* and *f*, diakinesis; *b*, *c*, *d* and *e*, metaphase. × 2800.

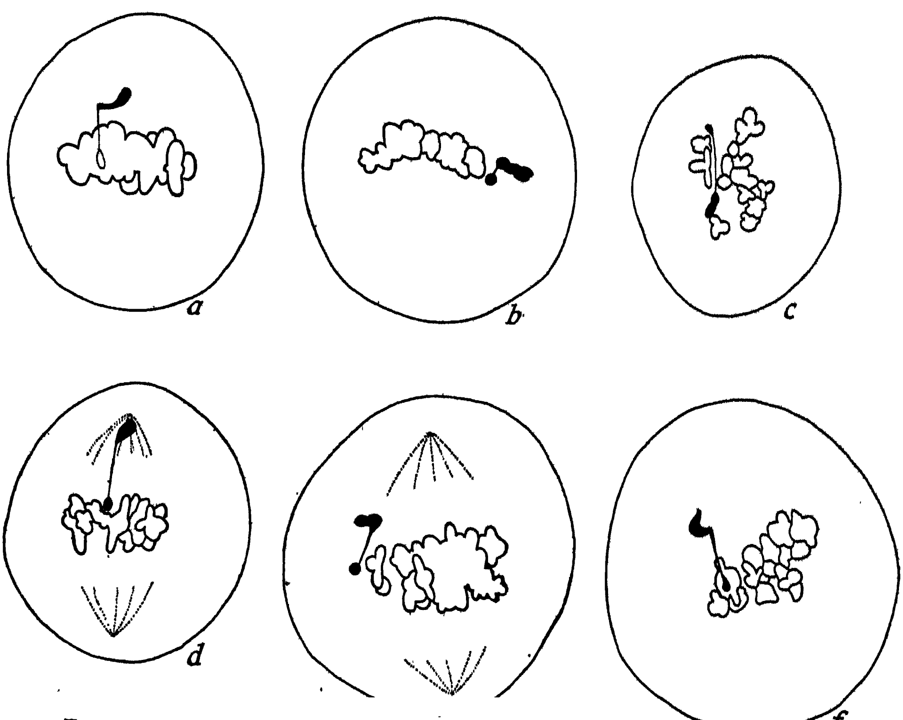


FIG. 5.—The asymmetrical XY bivalent with terminal chiasma: *a*, *b*, *d*, *e* and *f*, metaphase; *c*, diakinesis. × 2800.

the sex bivalent of this type are connected only by a long and thin thread, which contains sometimes one or two intercalary knobs of various sizes and in different positions (Pl. I, F; text-fig. 4, *b*, *c*, *d*, *e*). The X and Y sometimes lie far apart, which indicates either a strong centromere repulsion or a longer metaphase stage as compared with the autosomes (text-fig. 5, *c*, *d*). Such characteristic configuration of the asymmetrical sex bivalent is probably due to the marked precocity which is its inherent property. At

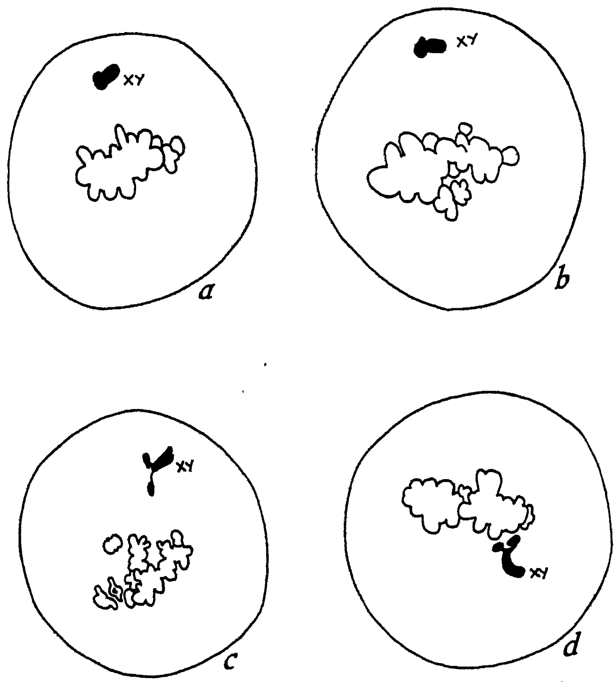


FIG. 6.—*a–d*, symmetrical XY bivalent lying off the equator during meiotic metaphase. × 2800.

the end of diakinesis it was found that the paired sex chromosomes are orientated towards the poles in advance of the autosomal bivalents which lie scattered within the cell. Similar behaviour of the sex bivalent was observed in *Trichosurus* and *Pseudochirus* (Koller, 1936*b*).

The X chromosome of the asymmetrical sex bivalent lies nearer the pole than the Y; the latter frequently occupies a position on the equator amongst the autosomes and therefore cannot be identified (Pl. II, D; text-fig. 5, *a*). It is interesting to note that in man as well as in the rat during mitotic metaphase the small Y lies farther from the equatorial plate than the X, while during meiotic metaphase the X is pushed off more frequently than the Y.

The long limb of the X chromosome is very often bent as a result of

cytoplasmic movements within the cell (text-figs. 4, *b*, *c*; 5, *f*). The condensation of the sex chromosomes in asymmetrical association is sometimes differential; the X, either in its total bulk, or partially, exhibits a smaller degree of condensation (as estimated by difference in staining) than the Y (Pl. II, H).

In addition to the asymmetrical sex bivalent, another characteristic

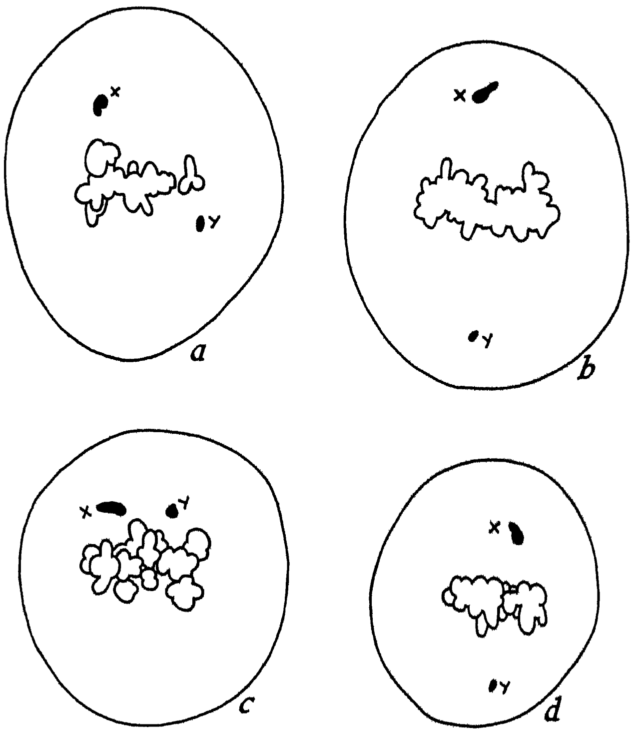


FIG. 7.—*a–d*, unpaired sex chromosomes during meiotic metaphase. ×2800.

structure was observed frequently during meiosis which closely resembles the symmetrical sex bivalent identified in rat, mole and ferret (Pl. I, A, H; II, A, C; text-fig. 6, *a*, *b*, *c*, *d*). It may be seen to consist of two clearly defined parts: a short, thick, deeply stained terminal knob, and a longer single arm. This configuration was encountered in several primary spermatocytes, and its frequency is estimated to be between 10–15 per cent. It is highly improbable that this complex represents an association of a small autosomal bivalent and univalent lying close together; such coincidence in a great number of cases would be very unusual, and would necessarily demand an explanation. It is true that in a few instances an autosomal bivalent or bivalents identified by their definite structure were seen lying off the equator as a result of crowding, but these were found only a short

distance from the metaphase plate, while the configuration described above was located near, or at the pole (Pl. I, H; II, A, C). Furthermore, the presence of this particular symmetrical bivalent is associated with an absence of the asymmetrical XY complex. It is important to note that in all material taken from different individuals the symmetrical complex was observed with about equal frequency; the only variation involved the size, and was probably due to variation in general cell size. Painter (1923) and Evans and Swezy (1929) had illustrated "unpaired" chromosomes in man, the structure of which is closely similar to that described above. These facts strongly suggest that this characteristic structure may represent the symmetrical XY bivalent in man, a type which is brought about by chiasma formation in the pairing segment between the centromere and the differential segment.

In about 3–4 per cent. of the spermatocytes the X and Y are left unpaired during the meiotic prophase, and at metaphase they may be seen lying separately either on the same side of the equatorial plate (text-fig. 7, *c*) or on the opposite sides (text-fig. 7, *a*, *b*, *d*). In some cases, however, the exactly corresponding position of the unpaired X and Y suggests rather a precocious anaphase separation as was illustrated by King and Beams (1936). Lagging of the X and Y chromosomes at the first anaphase is very rare.

Structural Hybridity in Man.

In a primary spermatocyte a double chromatid bridge and two acentric fragments were found at anaphase (Pl. I, E; text-fig. 8, *a*). It is known (Richardson, 1936) that such configuration can arise from double crossing-over of the complementary type within an inverted region (text-fig. 9). In view of the possibility that one individual from whom material was obtained may be a structural heterozygote, extensive analysis of first meiotic anaphase was undertaken in order to determine the presence, the length and the position of the inverted region, if any, in the chromosome complement. The following data show the frequency of those spermatocytes which indicate the presence of a structural difference between a region or regions of two otherwise homologous chromosomes:

8	show	a single chromatid bridge.
4	,,	,, ,, ,, and fragment.
3	,,	double bridges.
2	,,	,, ,, and fragments.
135	,,	neither bridge nor fragments.

All these data, obtained from testicular material of one individual alone, strongly suggest that the individual is heterozygous for an inversion

or inversions. The rarity of spermatocytes showing bridges or fragments (text-fig. 8, *a–f*), as a result of inversion, is due primarily to the smallness of the inversion or inversions and to the fact that the first anaphase in man is a very short stage.

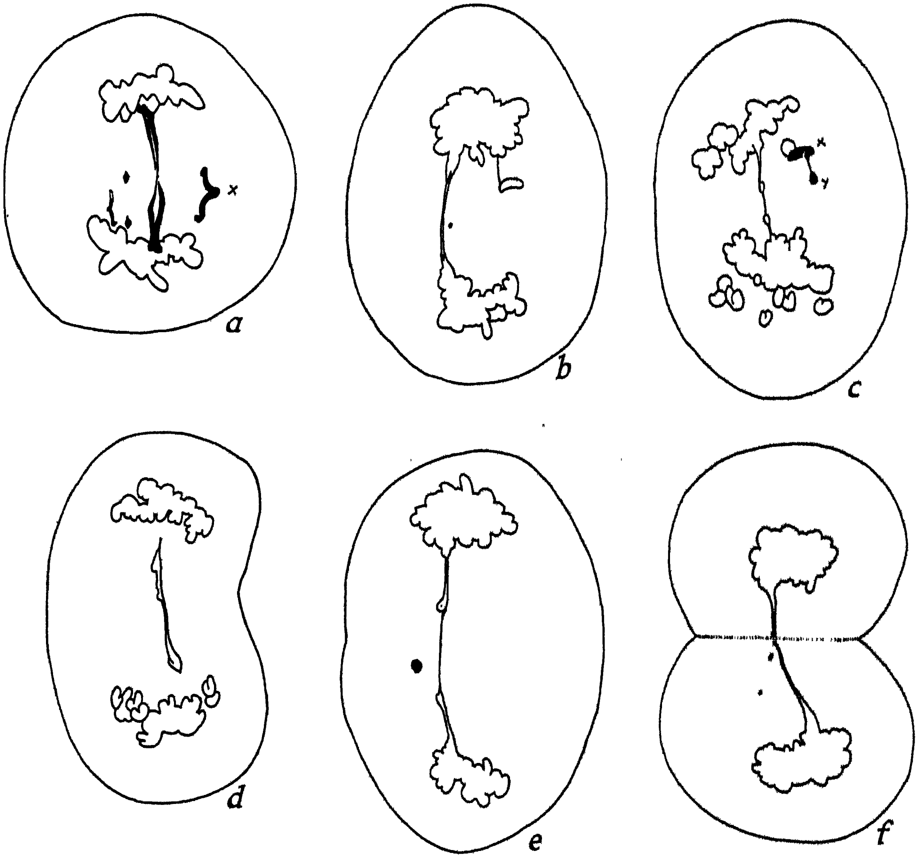


FIG. 8.—First meiotic anaphase, showing the result of crossing-over in inversions: *a*, double chromatid bridge with fragments and the equationally dividing unpaired sex chromosomes; *b*, *e* and *f*, chromatid bridge with fragment; *c* and *d*, chromatid bridge without fragment. × 2800.

It is interesting to note that the individual from whom the material containing inversion was obtained has a pedigree including a racial crossing. One grandfather was a Frenchman and the other grandparents were Scottish. One may assume that structural differences between pairing chromosomes, shown in racial crosses, represent important racial differences. The testicular material from two other men whose pedigrees are unknown and may or may not contain racial crosses did not show chromatid bridges or fragments at the first meiotic anaphase. In view of Darlington's (1936*b*) observation on grasshoppers, it is more probable

that structural hybridity is a prevalent condition in the population, irrespective of crossing between endogamous groups. It would be therefore of primary importance to make a cytological survey of racial crossings in man on a larger scale.

SINGLE AND DOUBLE CROSSING-OVER IN A TERMINAL INVERSION.

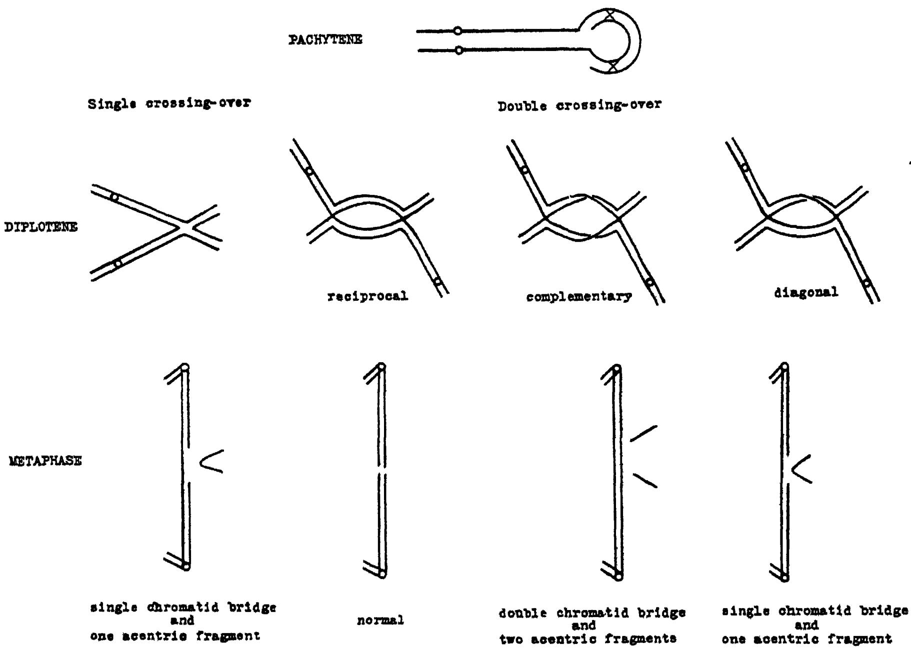


FIG. 9.—Diagram illustrating the result of crossing-over within inverted regions of a chromosome pair.

STRUCTURE OF THE X AND Y CHROMOSOMES.

It is generally accepted that chromosome association during meiosis is conditioned by homology, and consequently the pairing segments of any two chromosomes are assumed to be identical in their internal organization. The members of an unequal chromosome pair are necessarily composed of two regions, a pairing and a non-pairing, or differential, segment. The size of these segments can be determined directly by analysing the structure of the associated unequal chromosome complex at pachytene. It was found that at this stage the pairing segment in man involves about 1/3 of the total length of the X chromosome and nearly the whole of the Y.

The locus of the centromere in the unequal chromosomes can be inferred from the type of the bivalent in the post-diplotene stages which is determined by the position of the pairing segment in relation to the

centromere. If the pairing segment is terminal and the centromere is localised in the non-pairing segment, then the metaphase bivalent is always asymmetrical and the structural inequality is invariably maintained by pre-reduction, *i.e.* by segregation at the first meiotic division. This was found to be the case in various genera of marsupials where the pre-reductional segregation of the sex chromosomes is obligatory (Koller, 1936*b*). On the other hand, an intercalary position of the centromere within the pairing segment brings about facultative pre-reduction, because two kinds of bivalents may be formed, one the asymmetrical XY which leads to pre-reduction, the other the symmetrical XY bivalent resulting in a post-reductional segregation of the structural inequality.

It is assumed that in man not only the asymmetrical, but very probably also the symmetrical type of sex bivalent was found during meiosis. While the configuration of the asymmetrical sex bivalent suggests that the centromere is intercalary and has a subterminal locus, that of the symmetrical XY complex indicates that the centromere is included in the pairing segment, hence chiasmata, representing genetical crossing-over, may be formed on either side of it.

The centromere within the pairing segment can be localised more exactly by calculating the frequency of the symmetrical and asymmetrical types of sex bivalent at the meiotic metaphase (Table III, and text-fig. 10).

TABLE III.—THE FREQUENCY OF SYMMETRICAL AND ASYMMETRICAL SEX BIVALENTS.

Number of Preparations.	Number of Spermatocytes with Symmetrical XY Bivalents.	Number of Spermatocytes with Asymmetrical XY Bivalents.	Total Number of XY.
A 1	3	31	34
A 3	7	63	70
B 12	5	41	46
C 3	4	33	37
F 6	3	29	32
	22	197	219

During the present study more than 500 meiotic metaphases were under observation, and only 219 showed the sex chromosomes so that their structure could be recognised more or less distinctly. These include not only those spermatocytes in which both X and Y were identified, but also those in which the X alone was recognised, lying farther away from the autosomal bivalents while the Y was concealed amongst them (Pl. II, D). Such an arrangement is very frequent. In approximately 50 per cent. of the spermatocytes neither the X nor the Y were seen at all.

It is very probable that a great number of such spermatocytes contain the symmetrical XY, because this type is more easily concealed by the autosomes than is the other. Hence it may be assumed that the frequency of the symmetrical sex bivalent is higher than that given in Table III, and consequently the centromere lies rather nearer to the middle of the

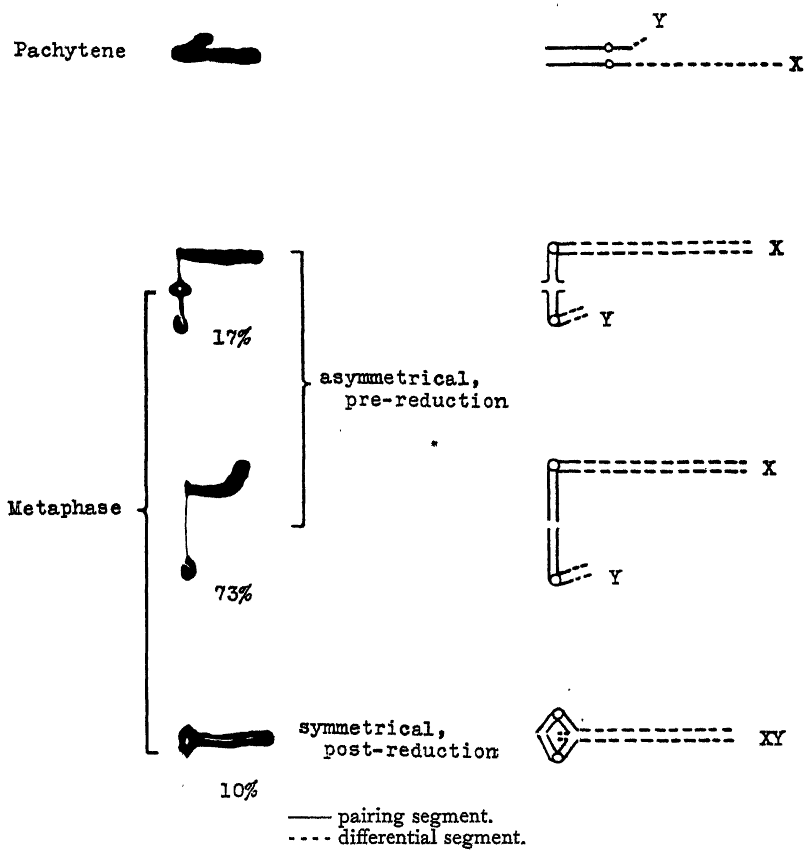


FIG. 10.—Diagram illustrating the various types of the XY bivalent.

pairing segment than to the end which is next to the differential segment. The frequency of the symmetrical XY bivalent was estimated to be about 10 per cent. in the rat, 30 per cent. in the ferret; in man it is no less than 10 per cent., but in view of the above facts it may be 15–20 per cent. or even more.

The centromere divides the pairing segment into two unequal regions, inner and outer, in relation to the differential segment. The inner pairing segment, *i.e.* the region between the centromere and the differential segment, is about 1/10 to 1/3 of the total length of the pairing segment.

The asymmetrical sex bivalent during diakinesis has frequently an interstitial chiasma represented by a subterminal knob (text-fig. 4, *a*, *f*). At metaphase, however, the number of sex bivalents with subterminal chiasmata decreases at the expense of a terminal association (Table IV, and text-fig. 11).

TABLE IV.—THE STRUCTURE OF ASYMMETRICAL BIVALENTS.

Stages.	Number of Sex Bivalents with Subterminal Xma.	Terminal Xma.	Total.
Diakinesis . .	8 (40 per cent.)	12 (60 per cent.)	20
Metaphase . .	37 (18·8 per cent.)	160 (81·2 per cent.)	197

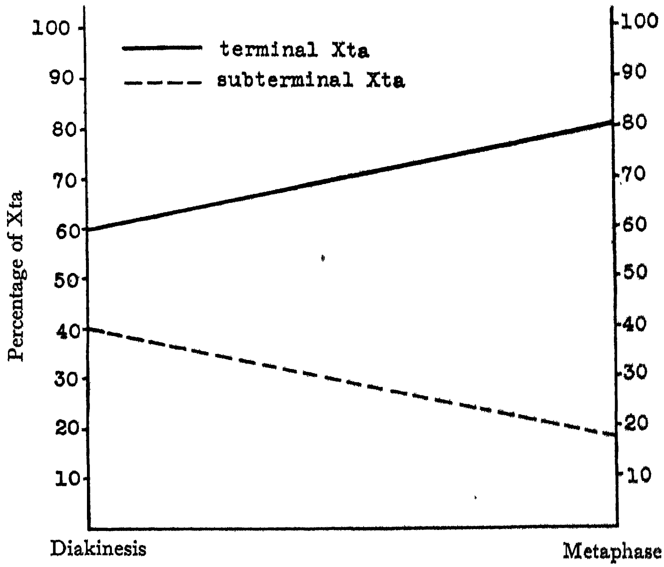
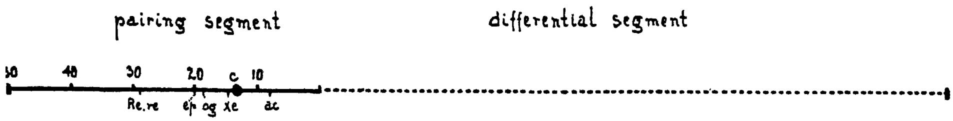


FIG. 11.—Graph showing the frequency of subterminal and terminal chiasmata in the sex bivalent during diakinesis and metaphase.

The decrease of subterminal and the increase of terminal chiasmata from diakinesis to metaphase is due to the movement of chiasmata, called terminalisation (Darlington, 1937).

Chiasmata arise at diplotene interstitially, and in the post-diplotene stage an inequilibrium of the forces of repulsion operating between the pairs of associated chromatids and the centromeres is responsible for the movement of chiasmata towards the distal ends. One may assume from the presence of subterminal chiasmata at metaphase that such chiasmata had been formed initially in the close neighbourhood of the centromere and had been overtaken by metaphase before terminalisation could be completed. Furthermore, it strongly suggests that chiasmata are more or less evenly distributed in the outer pairing segment.

The occurrence of the asymmetrical and symmetrical XY bivalents during meiosis, if their structure is interpreted on Janssens' partial chiasmatype hypothesis, must be considered as a definite cytological proof that crossing-over takes place freely in the region which is identical in the X and Y chromosome, and represents the pairing segment. Those genes which are borne on that particular segment will necessarily exhibit crossing-over with the differential segment determining sex. Haldane (1936 *b*) localised six genes in the pairing segment of the sex chromosomes of man (text-fig. 12), and the cytological evidence as presented in this paper suggests the possibility that the number of genes which exhibit incomplete sex linkage may be even greater.



Incompletely sex-linked characters (genes in the pairing segment):

Re: *Retinitis pigmentosa* (dominant).
re: ,, ,, (recessive).
ep: *epidermolysis bullosa dystrophica* (recessive).
og: Oguchi's disease (recessive).
xe: *xeroderma pigmentosum* (recessive).
ac: *achromatopsia* (recessive).
c: *centromere.*

Completely sex-linked characters (genes in the differential segment):

colour-blindness.
hæmophilia.

FIG. 12.—Genetical map of the pairing segment according to Haldane.

While the genetical length of the pairing segment is given by Haldane to be about 27–30 morgan or cross-over units, the cytological length is calculated to be about 1·5 μ. According to Darlington's interpretation of Janssens' partial chiasmatype hypothesis, one chiasma in a bivalent represents 50 per cent. crossing-over, which means that the genetical length of a chromosome forming one chiasma during meiotic prophase must be at least 50 cross-over units long. In man the X and Y chromosomes are associated by chiasmata in both arms, which indicates that the total length of the pairing segment is more than 50 morgans. Consequently those genes which are located near the distal end of the outer pairing segment of the sex chromosome farthest from the differential segment should show very little sex linkage. In the light of our knowledge concerning the chromosome length and the number of genes in other organisms such as *Drosophila*, it is evident that the genetical data given by Haldane are strongly supported by cytological observations presented in this paper.

SUMMARY.

1. The length of the X chromosome in man is 4–5 μ; that of the Y, 1·5 μ. The X has a subterminal centromere (centric or spindle attachment constriction).

2. The sex chromosome or chromosomes frequently lie off the metaphase plate during mitosis; the Y lies nearer to the pole than the X.

3. At mitotic anaphase lagging of the small chromosome is frequent.

4. The sex chromosomes are precociously condensed during the leptotene and zygotene stage of meiosis.

5. The sex chromosomes are divided into two regions: one is the pairing, the other is the non-pairing or differential segment. The non-pairing segment is about 2/3 of the total length of the X chromosome; it is very small in the Y, if present at all.

6. Two types of sex bivalent may be found at meiotic metaphase lying off the equatorial plate: one is asymmetrical and indicates the pre-reduction, the other symmetrical and indicates the post-reduction of the structural inequality as represented by the X and Y during meiosis.

7. The two types of sex bivalent suggest that the centromere lies in the pairing segment and chiasmata may be formed on both sides of the centromere, hence the pre-reduction of sex chromosomes during meiosis is facultative.

8. The asymmetrical sex bivalent sometimes has a subterminal chiasma during metaphase.

9. The frequency of a symmetrical bivalent is calculated to be 10 per cent. or probably more. The pairing segment has two regions: inner and outer. The inner, *i.e.* the region between the centromere and the differential segment is 1/5th–1/3rd of the length of the outer region.

10. The association of the X and Y by chiasmata means that crossing-over takes place in the pairing segment; consequently those genes which are localised in that region will exhibit incomplete sex linkage.

11. The occurrence of chiasmata in both arms indicates that the total length of the pairing segment is more than 50 morgan or cross-over units, so that the genes farthest from the differential segment should show very little sex linkage.

12. Non-pairing of the X and Y chromosomes was found occasionally. The unpaired sex chromosomes were lying off the equator either on the same side or on the opposite sides of the metaphase plate.

13. Chromatid bridges and fragments were observed in one individual, and it is suggested that he is heterozygous for an inversion or inversions.

14. Structural hybridity in man may be due to racial differentiation in the chromosome complements, but it is more probably a prevalent condition in the population, as was found to be the case in other organisms.

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DESCRIPTION OF PLATES.

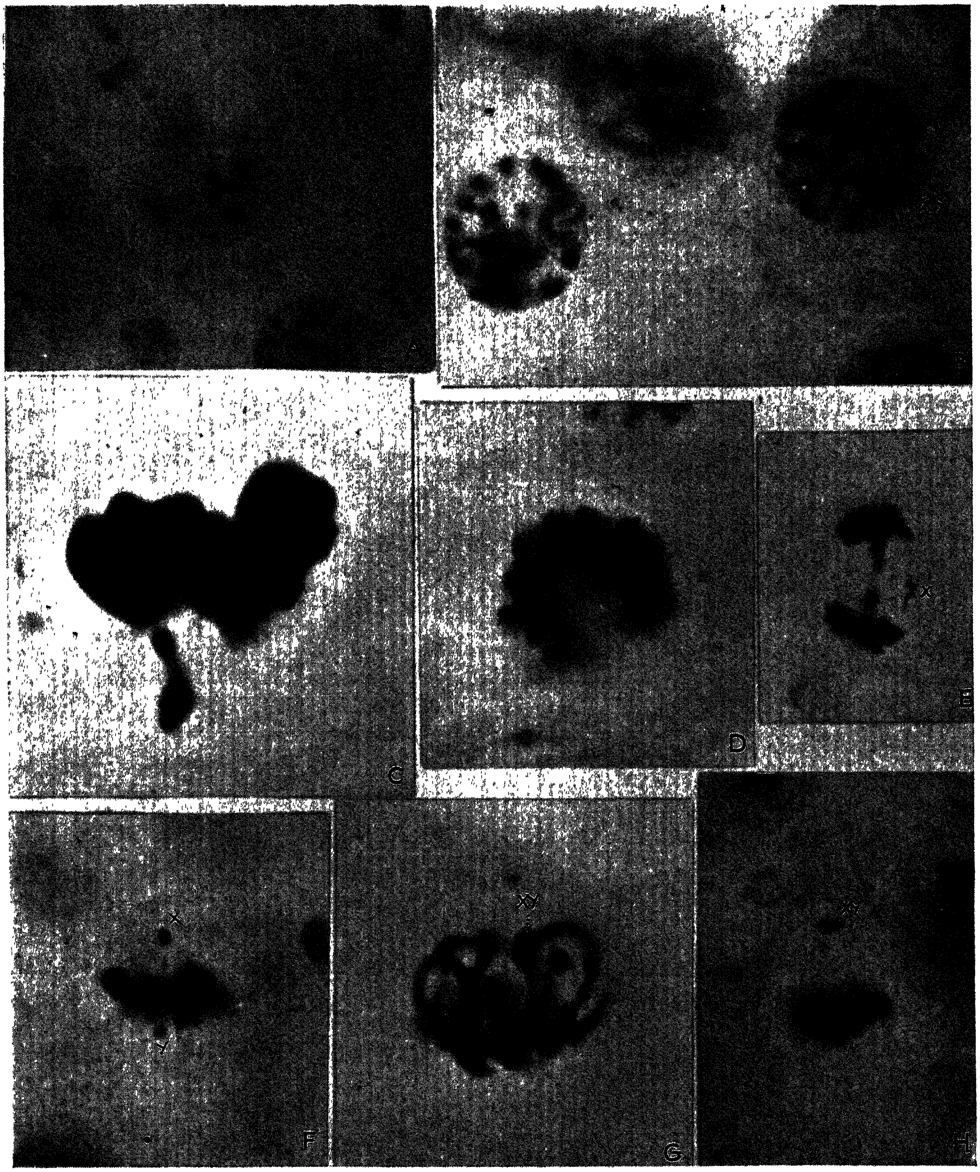
PLATE I.

A. Diakinesis showing the symmetrical XY bivalent. × 3000.
B. Zygotene stage, the paired X and Y precociously condensed. × 2500.
C. Asymmetrical sex bivalent lying off the equator. × 4200.
D. "Hollow" spindle, showing chromosomes arranged in a circle. × 4200.
E. First meiotic anaphase showing double bridge and the equationally dividing X chromosome (*cf.* fig. 8, *a*). × 2800.
F. Distant conjugation of X and Y chromosomes. × 2800.
G. Pachytene polarisation and the associated sex complex. × 4000.
H. Symmetrical XY bivalent lying at the pole. × 2800.

PLATE II.

A and C. Symmetrical sex bivalent. × 2800.
B. Unequal or asymmetrical XY showing terminal chiasma. × 2400.
D. Asymmetrical XY bivalent; the Y is concealed amongst the autosomal bivalents. × 2800.
E. Unorientated sex chromosomes at mitotic metaphase. × 3000.
F and H. Asymmetrical XY bivalent showing subterminal chiasma. × 3000.
G. Diakinesis; the XY is asymmetrical and has a subterminal chiasma. × 2800.
I. Mitotic anaphase showing the lagging Y chromosome. × 2800.

(*Issued separately June* 9, 1937.)



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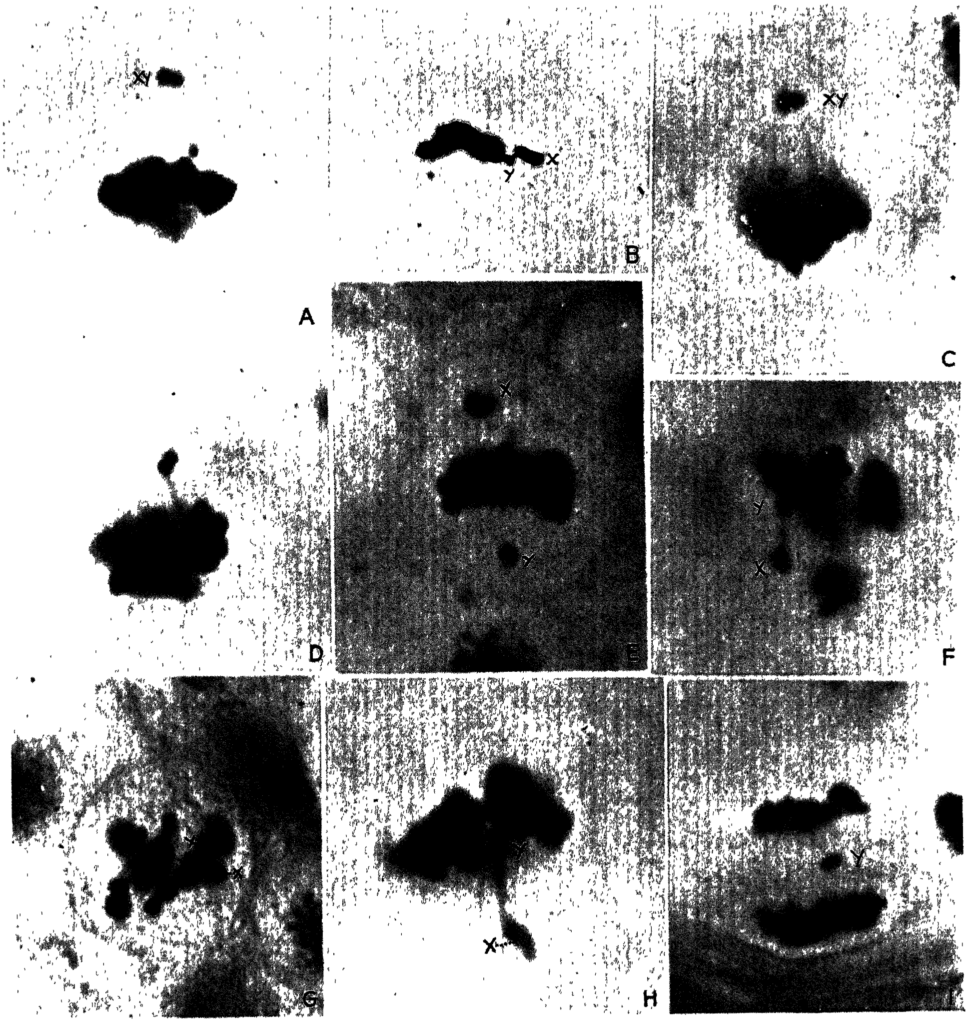
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XV.—Quantitative Evolution. II. Compositæ Dp-ages in Relation to Time. By **Professor James Small,** D.Sc., Department of Botany, Queen's University, Belfast. (With One Figure.)

(MS. received March 3, 1937. Read May 3, 1937.)

THE ages of the various tribes and sub-tribes of Compositæ have been calculated in doubling periods, according to Yule's formulæ; these are known briefly as Dp-ages (Small and Johnston, 1937; Udny Yule, 1924). Yule advises caution in interpreting Dp-ages in terms of geological time, but if his conceptions are to be applied to realities there should be some method of comparing Dp-ages with actual ages. Further, in the case of Compositæ, an almost complete parallel seriation has been found for Dp-ages and order of origin of both tribes and sub-tribes (Small and Johnston, 1937). Since the order of origin has been provisionally correlated with geological time (Small, 1919) it is possible to compare the calculated Dp-ages with suggested points of origin in geological time. This has been done by Small and Johnston (1937, Tables, VI, VII).

For convenience of reference these two tables may be summarised here; see Table I.

TABLE I.

Tribes.	Tribal Dp-ages.	Tribal Average Dp-ages.	Time.		Sub-tribal Average Dp-ages.	
Calenduleæ	2·43			U	1·0	
Arctotideæ	2·355	2·66	Pliocene	M	2·28	2·54–(2·57)
Helenieæ	3·20			L	2·80	
Vernonieæ	3·59			U	3·05	
Eupatorieæ	3·85	4·19	Miocene	M	3·21	3·45–(3·40)
Cynareæ	4·39			L	4·08	
Inuleæ (ltd.)	3·99	5·06	Oligocene			5·37–(5·44)
Mutisieæ	6·14					
Cichorieæ	5·90					
Anthemideæ	7·04	5·46	Eocene			5·39–(5·88)
Astereæ	5·06					
Heliantheæ	3·86					
Gnaphalieæ (ltd.)	4·65	5·96	Basal			6·34
Senecioneæ	7·26					

The values in brackets are calculated for grouped sub-tribes; the others are arithmetical averages of the Dp-ages of the sub-tribes or tribes arising in each period or division of a period.

The seriation of Dp-ages with geological age is obvious. The tribal and sub-tribal values, having different sources, are not the same, but they show a similar magnitude and a similar gradation.

The Time-Scale.—In order to compare these Dp-ages with actual time in years some kind of scale is required. Such a scale can be approximate only, but, so long as the relative lengths of the periods remain similar, any variation in the total length of the scale will not affect the placing of the Dp-ages on the scale.

The scale used throughout this work consists of 10–12 million years for Pliocene with 15–13 m.y. for Miocene, giving 25 m.y. for the two younger periods together; 10–15 m.y. for Oligocene with 25–20 m.y. for Eocene, giving 35 m.y. for the two older periods, also 10 m.y. for the last of the Upper Cretaceous. This is based upon the scale given in Schuchert and Dunbar (1933), where the Cenozoic is given as 60 m.y. divided into an earlier 35 m.y. and a later 25 m.y.

The only detailed scheme available is that by Barrell (1917), where minimal and maximal data for the Tertiary are given as follows: Recent with Pleistocene, 1–1·5 m.y.; Pliocene, 6–7·5 m.y.; Miocene, 12–14 m.y.; Oligocene, 16 m.y.; Eocene, 20–26 m.y.; total Tertiary, 55–65 m.y. The scale adopted before Barrell's list was found agrees very closely with this, except for the longer Pliocene which may cover some of Barrell's Upper Miocene, but such minor differences cannot fundamentally affect the general placing of the Dp-age points along the uniform time-scale; and in any case Urry (1937) gives 13 ± 1 m.y. as the age of at least one Pliocene basalt.

The BAT Curve.—On any ordinary assumptions it was to be expected that age as calculated in doubling periods (Dp-ages) would vary directly with time, so that the Dp-age points would be on a straight line when plotted against time in million years. When the Dp-ages are plotted against the time-scale the result is not a straight line, but a curve, as shown (fig. 1).

Since this is the basic curve for further investigations and consists of what may now be termed "apparent τ values," it has been named the Basic Apparent τ Curve or BAT curve.

The mid-points of the periods are taken at 6, 18, 32, and 49 m.y., with a last point at 70 m.y. as the mid-point of that part of the late Upper Cretaceous which concerns the initial stage of the family. In fig. 1 the points are given for tribal and sub-tribal average Dp-ages, together

with a smooth curve which moves downwards from 6·0 at 70 m.y. to 5·1 at 35 m.y., to 4·2 at 17·5 m.y., 3·3 at 8·75 m.y., 2·4 at 4·375 m.y., and 1·5 at 2·187 m.y. This is the BAT curve upon which lie all five tribal average Dp-age values for the periods. The sub-tribal values vary slightly around this curve, and both the Pliocene and Miocene values, which are low, approach the BAT curve more closely when the sub-tribes of the Helenieæ

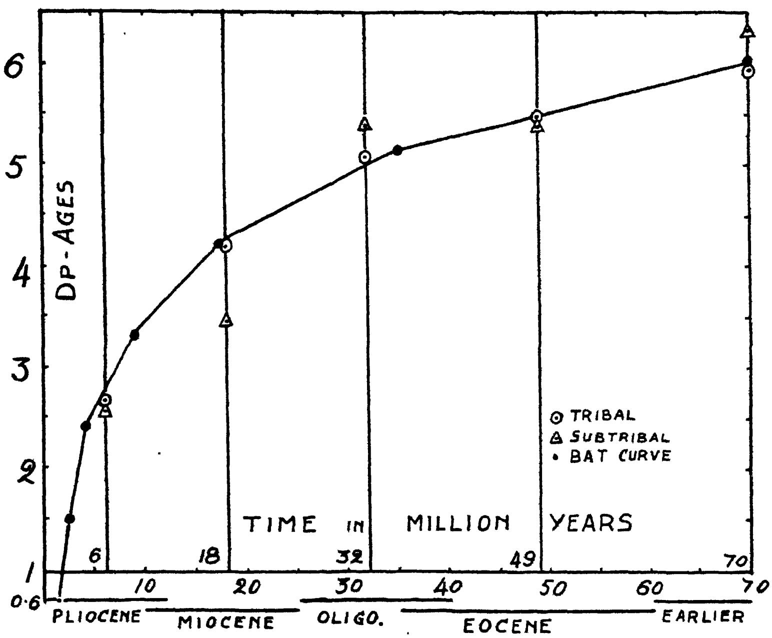

FIG. 1—The BAT Curve.

and Arctotideæ are adjusted between the two periods in accordance with a revision made in a previous contribution (Small and Johnston, 1937).

Following the BAT curve it will be seen that it falls by 0·9 Dp-age unit for each halving of the time. Dp-ages, therefore, do not vary directly with time but increase by 0·9 for each successive doubling of time from 1·094 m.y. backwards. The formulæ for this curve is

$$\text{Dp. } k+nd=t_k \cdot 2^n,$$

where k=a constant Dp-age value, d=a constant difference of Dp-age, t_k=time value in million years at the Dp-age k. For Compositæ, and apparently for average Angiosperms, k=0·6, d=0·9, and t_k=1·09375 m.y. The data from which this is calculated were obtained by following the curve of the tribal period averages along a smooth curve thus—

TABLE II.

DERIVATION OF THE BAT CURVE

OBSERVED TRIBAL DP-AGES AT	DP-AGE SMOOTHED	TIME	M.Y. VALUE OF DP-AGE UNIT
70 M.Y. 5·96	6·0 = 0·6 + 6(·9)	= 70 M.Y.	70/6 = 11·6̇
32 M.Y. 5·06	5·1 = 0·6 + 5(·9)	= 35 M.Y.	35/5·1 = 6·86
18 M.Y. 4·19	4·2 = 0·6 + 4(·9)	= 17·5 M.Y.	17·5/4·2 = 4·16
6 M.Y. 2·66	3·3 = 0·6 + 3(·9)	= 8·75 M.Y. AT 5·4 M.Y.	8·75/3·3 = 2·65 5·4/2·7 = 2·0
	2·4 = 0·6 + 2(·9)	= 4·375 M.Y.	4·375/2·4 = 1·82
	1·5 = 0·6 + 1(·9)	= 2·1875 M.Y.	—
	0·6 = 0·6 + —	= 1·09375 M.Y.	—

No real accuracy is claimed for these decimal fractions of the million-year unit. They are given here only to show the derivation of the formula. The fall of 0·9 is traced in the tribal Dp-ages (Table I) from 5·96 to 5·06 at 32 m.y., and again from 5·06 to 4·19 at 18 m.y.

On comparing the Dp-ages of Compositæ with actual time we find these rather surprising results, namely, (1) that Dp-age does not increase directly with time; and (2) that a given increase in Dp-age requires double the length of time as we pass backwards into the geological past. Clearly there is something here which merits further investigation.

Since the year-values are all relative to the 60 m.y. scale for the Tertiary, the use of other estimates of the lengths of the geological periods would involve only a corresponding change of the actual time-values in strict proportion, but no fundamental difference in the relative values, so that the characteristics of the BAT curve would remain unchanged. Altering the length of the Tertiary from 60 m.y. to 76 m.y. would only alter the various time-values and by contracting the abscissæ or horizontal scale the curve could be retained exactly as in fig. 1. No reasonable re-arrangement of periods would give a straight line and bring the Dp-ages to vary directly with time.

THE YEAR-VALUE OF THE DOUBLING PERIOD.

Udny Yule (1924, pp. 83–84) concludes his memoir with an attempt to estimate the average year-value of the doubling period for the whole

of the flowering plants for all time, but there were no very close data available. He, however, comes to the following conclusion: "In any case the figures are quite definite as to order of magnitude. If the age of the flowering plants is 100 million years or thereabouts, the doubling period for species is probably of the order of some 2 or 3 million years; it is, say, almost certainly over 1 million and less than 6 million." He recognises that these are average figures and "may well differ for different families and genera." This estimate allowed for proportional killing-off by a variety of external causes.

Within the Compositæ we have found the Dp-age values closely comparable with one another for sub-tribes within tribes, and the average Dp-ages for sub-tribes and tribes arising about the same time closely comparable with geological time. The approximate year-value of the doubling period is readily obtained from the Pliocene values: 6/2·66 for the tribes gives 2·22 m.y., while 6/2·54 for the sub-tribes gives 2·36 m.y.; but from the BAT curve it would appear that this year-value increases regularly as we proceed backwards in time, reaching 70/6 or 11·66 at 70 m.y. The value 2·22 at the mid-point 6 m.y. is an average for the period, not an accurate value for the time 6 m.y. The BAT curve is smooth, so that some allowance can be made for this apparent increase in the length of the unit of Dp-age, and it can be calculated that the year-value is 2 m.y. at 5·4 m.y. on the time-scale. This is a convenient unit, and since the BAT curve is very nearly straight in this region (because of the short intervals on the time-scale) 2 m.y. may be taken as a sufficiently close approximation to the true value of the Dp-age unit. There is no reason to suppose that the doubling period has in fact become regularly shorter since the end of the Mesozoic, and further inquiry is suggested by this apparent regular change in its value.

The next question which arises concerns the applicability of the BAT curve and its consequential data to other groups of Angiosperms, and this has been worked out in some detail for the Gramineæ.

Conclusion and Summary.

Average Dp-ages in Compositæ, when placed upon a time-scale in million years, similar to that of Barrell (1917), are found to approximate closely to an exponential curve with the formula $Dp.\ k+nd=t_k \cdot 2^n$; where $k=0{\cdot}6$, $d=0{\cdot}9$, and $t_k=1{\cdot}09375$ m.y. This is named the BAT curve, and from it the approximate year-value of the doubling period is deduced as 2 m.y.

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(*Issued separately June* 9, 1937.)

XVI.—Quantitative Evolution. III. Dp-ages of Gramineæ.

By **Professor James Small,** D.Sc., Department of Botany, Queen's University, Belfast. (With One Figure.)

(MS. received March 3, 1937. Read May 3, 1937.)

APPLYING Udny Yule's formulæ (1924) to the Compositæ, Small (1937) found that the average ages in doubling periods (Dp-ages) of the tribes of Compositæ, when plotted against a time-scale, gave points on an exponential curve called the BAT curve. If this curve is characteristic of average families of Angiosperms it should be possible to place the Dp-ages of tribes within other families on this curve as plotted against geological time, and thus obtain an order of geological origin which is quite independent of actual fossil records and which can be checked against any facts known concerning the evolutionary history of the family.

The variety and interest of the quantitative information which can be deduced from the BAT curve led to the development of a relatively rapid method of determining Dp-ages. The complete curves from the basic tables as given by Udny Yule (1924) and extended by K. I. Small (in Small and Johnston, 1937) were plotted as a series of large scale graphs, upon which the Dp-ages can be found, accurately to the second decimal place, by reference to the $f1$ and M co-ordinates. Using this and a "calculator" it is easy to determine the Dp-ages of normal groups of genera within large families. These can then be compared with time along a BAT curve also plotted on a large scale in order to get reasonably accurate time-values: the time-values are also calculated accurately from the formula for the BAT curve.

The present contribution deals with the Dp-ages of the Gramineæ. The evolutionary history of the grasses has been considered in some detail by Bews (1929), whose *World's Grasses* has been taken as the source of all the numerical data and phylogenetic suggestions.

There are fifteen tribes, which are best taken individually.

1. BAMBUSEÆ. 506 species; 33 genera; 5 monotypes; 4 ditypes. Dp-age 4·653. This is the most primitive group and nearest to the other Monocotyledons (Bews). The Dp-age would indicate about 26 m.y. on the time-scale, but the morphological evidence for the basal position is strong. This is the only real exception in the series of Dp-ages, and is to be explained by the normal death of genera as well as species in a group which has its origin well down in the Upper Cretaceous or even earlier.

The Panicoideæ form a complete series-sequence of Dp-ages with the order of origin as suggested by Bews, and these Dp-ages place the groups in time when the BAT curve is used to get m.y. values.

2. PANICEÆ. 1399 species; 60 genera; 17 monotypes. Dp-age 6·33; m.y. 95. This is described as relatively primitive, remotely connected with the Bambuseæ. The primitive genera of this group are scramblers and climbers in tropical forests.

3. MELINIDEÆ. 61 species; 9 genera; 4 monotypes. Dp-age 5·48; m.y. 50. A small side-line, hardly distinct from the Paniceæ.

4. ARUNDINELLEÆ. 92 species; 4 genera; 0 monotypes. Since this tribe of four genera has no monotypes the Dp-age is not determinable; but Bews states that the tropical and subtropical Arundinelleæ possibly should be included in the Melinideæ (1929, p. 90). Uniting the two groups we get a Dp-age for the combined group of 5·00; m.y. value 34. This may indicate the time of origin for the Arundinelleæ group.

5. ANDROPOGONEÆ. 778 species; 72 genera; 21 monotypes. Dp-age 4·57; m.y. 25. This is given as derived from the Paniceæ and advanced in morphology.

6. MAYDEÆ. 25 species; 7 genera; 3 monotypes. Dp-age 2·76; m.y. 6. This is given as derived from the Andropogoneæ, almost a sub-tribe, and highly developed. The Panicoideæ series is complete and perfectly simple in its time and origin relations to Dp-ages (fig. 1).

The Poöideæ is not so simple and the Dp-age calculations involve various complications.

7. FESTUCEÆ. 1142 species; 100 genera; 37 monotypes. Dp-age 5·792; m.y. 62. This is given as the most primitive group after the bamboos, connecting fairly closely with the herbaceous forms in the Bambuseæ. The Dp-age and m.y. value are less than those for the Paniceæ, so that the latter group would appear to be the older one. As in the Oryzeæ, see below, the primitive structural characters of the Festuceæ may well be due to their affinity with the Bambuseæ, rather than to an ancient origin. The Festuceæ is on the whole a temperate tribe (Bews, 1929, p. 87), whereas the Paniceæ is "mostly tropical and subtropical; few temperate" (Bews, 1929, p. 146), so that by applying Bews's general view of ecological evolution from moist tropical to drier and more temperate grasses (Bews, 1929, p. 262), we find that the ecological and Dp-age data agree in indicating the Festuceæ as less ancient, even if more primitive in structure, than the Paniceæ.

8. AGROSTIDEÆ. 1054 species; 59 genera; 29 monotypes. Dp-age (5·274); 42 m.y. This is said to be connected with the Festuceæ, but

representing probably an earlier separation (Bews, p. 90). Here the monotypes are abnormally numerous, about twice the normal and off the tables. Taking $f1$ at half its value the Dp-age can be determined as above. Since s/g varies mainly inversely with $f1$, and Dp in the usual range of $f1$ varies mainly directly with M, an abnormal $f1$ may be adjusted to a normal value in order to get a provisional Dp value on the scale on a basis comparable with other related groups; the adjustment being kept in mind as possibly indicating either special activity in throwing genera or possible senescence of the group, according to whether the $f1$ is unusually high or unusually low. The actual value of $f1$ has a minor effect on the Dp-age, but small differences of Dp-age above 5·1 have a large m.y. effect.

9. ZOYSIEÆ. 46 species; 17 genera; 10 monotypes. Dp-age 4·81; 30 m.y. This group is removed from the neighbourhood of the Andropogoneæ, which arose later (fig. 1). Bews (1929, p. 69) in this re-arrangement follows Stapf and Hitchcock, and the revised arrangement brings the Dp-age into series, where with Hackel's position the relation would be wrong. The revised position is near the Agrostideæ as a sideline (Bews, p. 91).

10. HORDEÆ. 201 species; 19 genera; 6 monotypes. Dp-age 4·71: m.y. 28. This is closely allied to Festuceæ, and "more advanced" than Chlorideæ. If the Chlorideæ are regarded as a recent off-shoot from the relatively primitive Festuceæ, the more advanced morphology of the Hordeæ is quite in harmony with an earlier origin of that group from the Festuceæ.

11. AVENEÆ. 397 species; 33 genera; 7 monotypes. Dp-age 4·23; m.y. 18. This tribe also is close to Festuceæ, but more advanced (Bews, p. 89), and of earlier origin than Chlorideæ. The $f1$ value is again off the tables, but it is near enough to enable the Dp-age to be determined by the "graph" technique.

12. PHALARIDEÆ. 77 species; 6 genera; 0 monotypes. This small tribe of six genera has no monotypes, but it is derivative from the Aveneæ, and its time of origin is therefore later than 18 m.y., if not quite recent, as shown in fig. 1.

13. CHLORIDEÆ. 269 species; 43 genera; 11 monotypes. Dp-age 3·07; m.y. 8. This tribe is close to Festuceæ and more recent than the Aveneæ. The determination of the Dp-age involves the inclusion in the monotypes of *Trichoneura* given as 1 or 2 (Bews, p. 178). The Festuceæ series, with these doubtful points allowed, is also complete in its sequence with m.y. time and Dp-age.

14. PHAREÆ. 35 species; 7 genera; 1 monotype; 2 ditypes.

Dp-age 2·78; m.y. 7. This small tribe is placed next the Bambuseæ by Stapf (Bews, p. 88). There are only seven genera with one monotype,

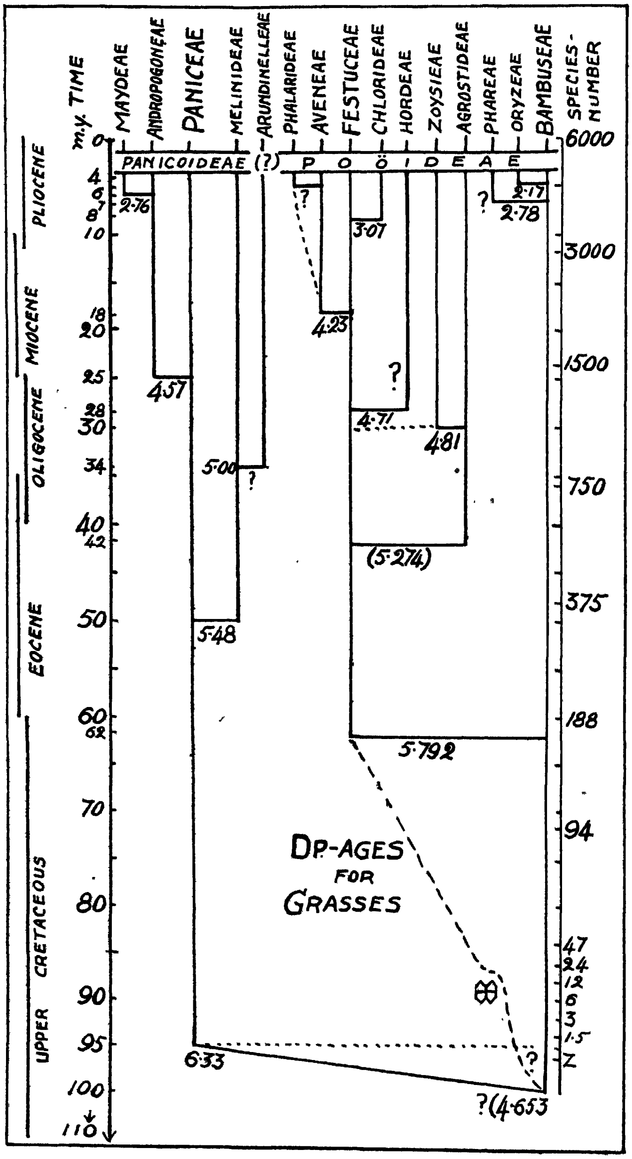

FIG. 1.

and for the determination of the Dp-age one possibly split ditype is calculated as monotypic. This low $f1$ may really be a sign of generic senescence and the group may be old bamboo one, but the morphology indicates it to be a recent offshoot which may belong to the Paniceæ series.

15. ORYZEÆ. 38 species; 12 genera; 5 monotypes. Dp-age 2·17; m.y. 4–5. This tribe is described as somewhat isolated but retaining certain primitive features, such as six stamens. It is figured (Bews, p. 91) as derived directly from the Bambuseæ stock, which would account for the primitive characters in an apparently recent group. This gives a complete series of Dp-ages with time and origin for the direct Bambuseæ series.

If these brief notes are read in conjunction with the evolutionary time-table in fig. 1, it should be clear that in the Dp-age values we have an arithmetical or quantitative measure of evolutionary developments. At the moment the quantitative evaluation can be applied with security only when there has not been extensive generic death as well as the normal species-death (apparently after a life-time of 12 m.y.) for 99 per cent. of angiosperm species. This limited life-time is involved in later developments of quantitative evolution. It is hoped, by extending the work to older groups of angiosperms to obtain a correcting factor for generic death which can be applied to the apparent Dp-ages of old groups.

The palæobotanical records of grasses are scanty, but Schuchert and Dunbar (1933, p. 64) give the Eocene Epoch as showing the rise of grasses, cereals, and fruits, and (*op. cit.*, p. 436) quote the well-known development of grazing mammals in the Miocene as indirect but very clear evidence of extensive grasslands. The evidence of the Dp-ages indicates a meagre, narrow development of the grasses in the Upper Cretaceous, followed by the origin of two new groups of genera in the Eocene, and a more extensive differentiation during Oligocene-Miocene times. Only four relatively small tribes are indicated as recent on the Dp-age time-scale, together with the larger Chlorideæ, which is very closely connected with the Festuceæ and has rather ill-defined limits; it may, indeed, be little more than a selection of Festucean genera.

The pre-Miocene development is of some interest. The primitive genera in the Paniceæ are scramblers or climbers or shade-loving grasses of tropical and subtropical forests, and the temperate pasture grasses are regarded by Bews as derivative forms. The Festuceæ begin with large reed-grasses and pass to temperate pasture, so that their primitive features may be the result of a later origin from the bamboo stock, instead of an earlier beginning than the Paniceæ. The general history of the grasses, as checked by Dp-ages, is summarised in fig. 1.

SPECIES-NUMBER IN GRASSES.

The following brief account of species-number in the grasses has its most suitable place here, as an explanation of the data in fig. 1, even

although it does anticipate certain points which will be demonstrated in later contributions. Since there may have been variation in the Dp-age unit within the Gramineæ the geological places assigned to the tribes are not to be considered hard and fast, but as given they are in accordance with all the facts known up to the present.

The total number of living species in the Gramineæ is approximately 6000. Bews (*op. cit.*, p. 259) gives 5871 as an approximation, but the careful count of his actual data as used here gives 6120. These existent species have apparently arisen during the last 12 m.y., doubling in number each 2 m.y. (Small, 1935). Working backwards, halving the number in six successive 2 m.y. stages, we get about 95 of the existent species arising at 12 m.y. and therefore now senescent, showing advancing sterility of individuals and scarcity of individuals either in one locality or in several widespread localities (*cp.* Small, 1935). These relatively infertile species could doubtless be enumerated, but this is a general phenomenon which requires special treatment, and it is scarcely necessary to occupy space at the moment with such a list.

We must distinguish carefully between the number of living species, which arise along a 2^n exponential curve of progressive origin with a 2 m.y. period, and the total species-number of the family, which is a residual value and increases along a 2^n exponential curve with a 12 m.y. period. Considering species-number as distinct from living species, and working backwards, halving the number in seven successive 12 m.y. stages, we reach 47 species at 84 m.y. (fig. 1). At this point we come to the first 12 m.y. in the history of the grasses, during which the species-number doubled with a 2 m.y. period, unchecked by the death of species for 12 m.y. Moving backwards we halve 47 in round numbers as 24, 12, 6, 3, then 1·5, and reach ·75 or zero at 96 m.y. This is just below the maximum Dp-age point as determined (Paniceæ, 6·33 : 95 m.y.), but above the supposed point of origin for the Bambuseæ.

It should be noted here that the suggested death of all plant species from senescent sterility (Small, 1935) does not invalidate, but rather corroborates, the position of "Age and Area" in that for living species only 1·5625 per cent. are due to die in the next 500,000 years, since only 1·5625 per cent. arose 12 m.y. ago. Such a percentage would not be detected by "Age and Area" methods (Willis, 1922).

Conclusion and Summary.

The BAT curve is used to space Dp-ages calculated for the subdivisions of Gramineæ along a time-scale in million years. It is found

that, with only one distinct exception (the archaic Bambuseæ), this evolutionary sequence in terms of time is in reasonable agreement with the views expressed by Bews (1929) concerning the phylogeny of the Grasses, and also in reasonable agreement with the little that is known of the geological history of the family.

It would appear, therefore, that with the BAT curve as an approximate measure of age in terms of millions of years, we may proceed further with these quantitative methods, applying them to other associated groups of plants and possibly of animals also, since Yule (1924) uses Chrysomelidæ, Cerambycinæ, Snakes, and Lizards. The successful application of a *curved* scale of age (the BAT curve) to the Gramineæ strengthens the verisimilitude so much that it becomes necessary to search for an explanation of the curvature which, being regular, should have a relatively simple relation to the basic direct correlation of age and time. This relation has been determined, but the present contributions are confined to the more obvious facts of the reality of the BAT curve (*cp.* Small, 1937) and its application to a group other than Compositæ.

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XVII.—Tests for Randomness in a Series of Numerical Observations. By **W. O. Kermack**, M.A., D.Sc., and **Lt.-Col. A. G. McKendrick**, M.B., D.Sc. (From the Laboratory of the Royal College of Physicians, Edinburgh.)

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It is often a matter of considerable importance to decide whether the fluctuations exhibited by a series of observations are random in character, or whether some factor operating according to a definite law must be assumed to exist. One general method of testing a statistical hypothesis is to work out results which would be expected theoretically, and then to compare these with the observations. A statistic such as the mean or standard deviation to be observed on a particular hypothesis may be calculated and compared with the actual mean or standard deviation as evaluated from the observed figures. Agreement does not necessarily imply that the hypothesis is a true one, or even that it is consistent with the observations, because it might be found that when another statistic was calculated, the agreement was no longer satisfactory. Obviously the greater the number of tests applied and found satisfactory, the greater the confidence which can be placed in the conclusions.

The object of the present paper is to develop simple criteria which will help us to decide whether a series is random or not. A series which conforms to these criteria is not necessarily a random one, but in the absence of evidence to the contrary it is likely to be so. On the other hand, the failure of a series to conform is evidence of its non-random nature.

Let us consider the following series of numbers, no two of which are equal: 2, 4, 51*, 6, 5, 17, 98*, 57, 43, 80*, 3, 86*, 73, 50. Certain numbers, those marked with an asterisk, are greater than either of their immediate neighbours, they form, as it were, maxima in the series, and we shall call them maximal numbers. Similarly we shall call numbers which are smaller than either of their immediate neighbours minimal numbers. We define a "run down" as the interval between a maximal number and the next succeeding minimal number, and the "size" of a run down as the number of numbers involved, including the maximal and minimal numbers themselves. A "run up" is similarly the interval between a

minimal and the next succeeding maximal number. Clearly the smallest run up or run down is of size 2. The interval between two successive maximal numbers is called a "gap," and the number of numbers involved including the two maximal numbers is taken as the size of the gap. Similarly we have a gap between every two successive minimal numbers. Each gap between two maximal numbers evidently contains one minimal number and *vice versa*. The smallest possible gap is clearly of size 3.

Under defined conditions the distribution of runs or gaps in terms of their size will, in general, be unambiguous, although the actual distribution may be difficult to evaluate. The problem for a series of finite length is somewhat complicated, but in the case which is important in practice, namely, when the series contains an indefinitely large number of numbers, the distribution functions prove to be relatively simple and eminently suitable for arithmetical calculation.

It is first of all necessary to define the problem precisely. Let us imagine that we have in a box a large number of tickets on each of which is written a decimal number between 0 and 1, no two of the numbers being equal. Let us suppose that these are thoroughly mixed, drawn at random one by one, and placed in serial order. We count the numbers of runs of sizes 2, 3, 4 . . ., and divide these by the total number of runs counted. Let the resulting fractions be $q_2, q_3, q_4, \ldots$ Different experiments will give different values of $q_2, q_3, q_4, \ldots$, but if we increase the number of tickets drawn we obtain larger and larger numbers of runs, and the values $q_2, q_3, q_4, \ldots$ will tend in the statistical sense to limits $x_2, x_3, x_4, \ldots$ It is required to calculate x_k the probability that any run selected at random from the series will be of size k. It is obvious from considerations of symmetry that the probability of a run up of size k is equal to the probability of a run down of the same size. Similarly we may define g_k as the probability that any gap between two successive maxima will be of size k. Again from conditions of symmetry g_k is also the probability that any gap between two successive minima is of size k. It is evident that

$$\sum_{2}^{\infty} x_k = 1, \quad \text{and} \quad \sum_{3}^{\infty} g_k = 1. \tag{1}$$

Distribution of Runs of Various Sizes.

If we take n unequal numbers and arrange them in any random order, they may contain 1, 2, 3, . . . $n-1$ different runs. Let p_n be the chance that the n numbers are arranged in either ascending or descending order. It is easily seen that $p_n = \frac{2}{n!}$ for there is one way of arranging the numbers

as a single run up, and one way as a single run down, whilst the total number of permutations is $n!$. Let us now choose any number a_1 in the random series of numbers considered above. Let the n numbers immediately succeeding the arbitrarily chosen number be $a_2, a_3, \ldots a_{n+1}$, and let a_0 be the one which immediately precedes it. Clearly the chance that the n numbers $a_1, a_2, \ldots a_n$ form the whole or part of a single run is p_n.

Let y_n be the chance that the first maximum or minimum following the arbitrarily chosen number a_1 is a_n. If now the numbers $a_1, a_2, \ldots a_n$ form a part or the whole of a single run, then a_{n+1} must either continue the run, or else a_n must be maximal or minimal.

Hence

$$p_n = p_{n+1} + y_n,$$

or

$$\begin{aligned} y_n &= p_n - p_{n+1}, \\ &= -\Delta^1 p_n. \end{aligned}$$

Now either a_1 is maximal or minimal, or else a_0, the number which precedes it, forms part of the run which we have been considering. If then z_n is the chance that a number chosen at random is maximal or minimal, and that the first succeeding maximum or minimum will be at the $n-1$th succeeding number, it follows that $y_n = y_{n+1} + z_n$. For, regarding a_0 as arbitrarily chosen, y_{n+1} is obviously the chance that the first maximal or minimal number after a_0 is a_n.

Hence

$$z_n = y_n - y_{n+1} = -\Delta^1 y_n = \Delta^2 p_n = \Delta^2 \frac{2}{n!}.$$

Now x_n may be regarded as the chance that the first maximal or minimal number succeeding a minimal or maximal number taken at random is the $n-1$th succeeding number. If now c is the chance that any number taken at random is maximal or minimal, x_n is the compound chance that an arbitrarily chosen number will itself be maximal or minimal, and that the first succeeding maximal or minimal number will be the $n-1$th succeeding number.

Clearly $z_n = cx_n$, so that $x_n = \frac{z_n}{c}$.

Of any three numbers the chance that the middle one is the greatest is obviously $\frac{1}{3}$, whence $c = \frac{2}{3}$, so that

$$x_n = \tfrac{3}{2} z_n = 3\Delta^2 \frac{1}{n!} = 3\frac{(n^2+n-1)}{(n+2)!}.$$

Thus

$$\begin{aligned}
x_2 &= \tfrac{5}{8} &&= 0{\cdot}6250\\
x_3 &= \tfrac{11}{40} &&= 0{\cdot}2750\\
x_4 &= \tfrac{19}{240} &&= 0{\cdot}0792\\
x_5 &= \tfrac{29}{1680} &&= 0{\cdot}0173\\
x_6 &= \tfrac{41}{13440} &&= 0{\cdot}0031\\
x_7 &= \tfrac{55}{120960} &&= 0{\cdot}0005\\
x_8 &= \tfrac{71}{1209600} &&= 0{\cdot}00006.
\end{aligned}$$

Clearly

$$\begin{aligned}
\sum_2^\infty x_n &= 3\sum_2^\infty \Delta^2 \frac{1}{n!}\\
&= 3\left(-\Delta^1 \frac{1}{n!}\right)_{n=2}\\
&= 3\left(\frac{1}{2!}-\frac{1}{3!}\right)\\
&= 1,
\end{aligned}$$

so that the x's satisfy condition (1).

Further the average size of a run is

$$\begin{aligned}
\sum_2^\infty nx_n \div \sum_2^\infty x_n &= 3\sum_2^\infty n\Delta^2 \frac{1}{n!}\\
&= 3\left\{2\left(-\Delta^1 \frac{1}{2!}+\Delta^1\frac{1}{3!}\right)+3\left(-\Delta^1\frac{1}{3!}+\Delta^1\frac{1}{4!}\right)+\ \ldots\right\}\\
&= 3\left\{-2\Delta^1\frac{1}{2!}-\Delta^1\frac{1}{3!}-\Delta^1\frac{1}{4!}-\Delta^1\frac{1}{5!}-\ \ldots\right\}\\
&= 3\left\{2\left(\frac{1}{2!}-\frac{1}{3!}\right)+\frac{1}{3!}\right\}\\
&= 2\tfrac{1}{2}.
\end{aligned}$$

A slight modification of the theory is required in the case where the numbers are not all unequal. We may imagine any set of equal numbers to be replaced by another set of numbers which differ only very slightly from each other and from the observed set. When the numbers do not immediately follow each other in the series it is clear that no modification of the runs will be involved as the result of slight changes in their magnitude. When, however, two successive numbers are equal, the slight alterations in magnitude will have approximately equal chances of rendering the first greater or of rendering it less than the second. Therefore it is appropriate to carry out the count on the assumption (*a*) that of the two equal numbers, the first is greater than the second, and

(*b*) that it is less; and to take the mean of the results, we thus take a run such as 1, 2, 5, 5, 9 first as a run of 3 up, 2 down, and 2 up, and then as a run of 5 up, and enter the group as contributing 3 half-runs of types 3 up, 2 down, and 2 up, and one half-run of the type 5 up. When three equal numbers follow in succession a corresponding modification may have to be made.

Distribution of the Sizes of Gaps.

The numbers of points in a "gap," as defined above, is the total number in two successive runs taken at random. This problem, like that of the single runs discussed above, is really a particular case of the more general problem of finding the chance that the number of numbers present in any r successive runs taken at random will be n. We denote this chance by x_{nr}, so that x_{n1} is identically equal to x_n of the previous section, and x_{n2} is identically equal to g_n. The problem of finding the general expression for x_{nr} is treated in a separate paper; here it is sufficient to quote the result for $r=2$.

It can be shown that

$$\begin{aligned} g_n \equiv x_{n2} &= \tfrac{3}{2}\Delta^2 \frac{2^n}{n!} \\ &= \tfrac{3}{2}2^n\left(\frac{1}{n!} - \frac{4}{(n+1)!} + \frac{4}{(n+2)!}\right) \\ &= 3 \cdot 2^{n-1}\frac{(n^2-n-2)}{(n+2)!} \\ &= 3 \cdot \frac{2^{n-1}}{n!}\frac{(n-2)}{(n+2)}. \end{aligned}$$

Thus

$$\begin{aligned} g_3 &= \tfrac{2}{5} &&= 0{\cdot}4 \\ g_4 &= \tfrac{1}{3} &&= 0{\cdot}3333 \\ g_5 &= \tfrac{6}{35} &&= 0{\cdot}1714 \\ g_6 &= \tfrac{1}{15} &&= 0{\cdot}0667 \\ g_7 &= \tfrac{4}{189} &&= 0{\cdot}0212 \\ g_8 &= \tfrac{3}{525} &&= 0{\cdot}0057 \\ g_9 &= \tfrac{2}{1485} &&= 0{\cdot}0013 \\ g_{10} &= \tfrac{4}{18175} &&= 0{\cdot}0002. \end{aligned}$$

This formula is true for all values of n equal to or greater than 3, as naturally no gap can be of size less than 3.

It is readily shown that $\sum_3^\infty g_n = 1$ and that the mean length of a gap

$$\sum_3^\infty n g_n \div \sum_3^\infty g_n = 4.$$

The latter result is evidently consistent with that previously found that every third number on the average is a maximal number.

It may be added that the obvious possible generalization of the above results, namely that $x_{nr}=k\Delta^2\frac{r^n}{n!}$, where k is a constant, does not hold, and that the formulæ become increasingly complex as the value of r increases.

Errors of Mean Size of Run and Mean Size of Gap.

The mean size of run and the mean size of gap are respectively 2·5 and 4 in terms of the system of measurement adopted above. It is of considerable advantage to know the probable errors of these quantities. For this purpose we make use of the properties of the numbers P_{nr}, the number of ways in which n unequal numbers can be arranged so as to give r runs.

It has been shown by André that

$$\sum_{r=1}^{n-1} P_{nr}=n!,$$

and that

$$\sum_{r=1}^{n-1} rP_{nr}=\frac{2n-1}{3}n!,$$

whilst it may be proved that

$$\sum_{r=1}^{n-1} r^2P_{nr}=\frac{40n^2-24n-19}{90}n!.$$

If now we take $p_{nr}=\frac{P_{nr}}{n!}$, p_{nr} regarded as a function of r for a fixed value of n describes the frequency distribution which occurs as the result of a large number of observations on random sets of n numbers. The mean of runs to be observed in such a set is therefore $\frac{2n-1}{3}$, whilst the second moment about the mean is

$$\frac{40n^2-24n-19}{90}-\left(\frac{2n-1}{3}\right)^2,$$

$$=\frac{16n-29}{90}.$$

For large values of n, the mean number of runs is $\frac{2n}{3}$, and the standard deviation of the distribution is $\sqrt{\frac{8n}{45}}$, whilst, as each gap corresponds to

two successive runs, the corresponding figures for gaps are $\frac{n}{3}$ and $\sqrt{\frac{2n}{45}}$ respectively. (The total number of gaps is really twice this number, but as the gaps from minimum to minimum are equal in number to those from maximum to maximum, they do not provide an independent series. Thus in the following paragraphs s_2 is approximately equal to $\frac{n}{3}$, even although both series of gaps ($2s_2$ in all) may have been used in calculating the average gap length. Similar considerations suggest that $P(\chi^2)$ calculated from the $2s_2$ gaps is usually too low.)

If now l_1 is the average length of a run, in a case where there are s_1 runs, then

$$l_1=\frac{n-1}{s_1}+1$$

so that

$$\delta l_1=\frac{n-1}{s_1^2}\,\delta s_1.$$

Thus

$$\sigma_{l^1} \doteqdot \frac{n}{s_1^2}\sigma_{s_1}=\sqrt{\frac{3}{5s_1}}.$$

Hence the average length of a run is $2\frac{1}{2}$ and its standard deviation is $\sqrt{\frac{3}{5s_1}}$, s_1 being the number of runs counted $\left(\doteqdot\frac{2n}{3}\right)$. Similarly if s_2 $\left(\doteqdot\frac{n}{3}\right)$ is the number of gaps counted, and l_2 the length of a gap, then

$$l_2=\frac{n-1}{s_2}+1,$$

and

$$\sigma_{l_2}=\frac{n}{s_2^2}\sigma_{s_2}=\frac{3s_2}{s_2^2}\sqrt{\left(\frac{2}{45}\times 3s_2\right)}$$

$$=\frac{3}{s_2}\sqrt{\frac{2s_2}{15}}$$

$$=\sqrt{\frac{6}{5s_2}}.$$

[It is to be noted that in calculating the average value of a run or of a gap, it is necessary to employ all the runs or gaps in the series.] These errors are smaller than what perhaps might have been expected from a cursory examination of the problem. In n numbers we expect, when n is large, $\frac{2n}{3}$ runs and $\frac{n}{3}$ gaps, and as any group of numbers can occur in the selected group, it might seem that the distribution would be similar to a Poisson one with mean $\frac{2n}{3}\left(\text{or } \frac{n}{3}\right)$. In this case σ would be approximately $\sqrt{\frac{2n}{3}}\left(\text{or } \sqrt{\frac{n}{3}}\right)$. This is approximately twice the value of σ obtained in

the above calculation. The discrepancy is due to the fact that the distribution function is more concentrated about the mean than in the case of a simple Poisson series. This is brought about by the following considerations. The smallest possible run is of size 2, so that the largest number of runs in a series of n points is $n-1$. Thus the distribution function becomes zero at the point $n-1$, and therefore drops more rapidly above the mean than in the case of a Poisson series. It must therefore also rise more steeply up to the mean, and so is evidently more concentrated round the mean.

(a) *Tippett's random numbers.*

Runs of.	Observed.	Calculated.
2	637	625
3	265	275
4	77	79
5	18	17
6	2	3·6
7	1	

$n'=5$, $P > 0·9$.
mean: observed, 2·486.
mean: theoretical, 2·5.
S.D. : theoretical, 0·024.

Gaps of.	Observed.	Calculated.
3	161	160
4	129	133
5	71	69
6	31	27
7	6	8
8	2	3
9	0	

$n'=6$, $P=0·9$.
mean: observed, 3·997.
mean: theoretical, 4·0.
S.D. : theoretical, 0·055.

(b) *Telephone numbers,* reversed and regarded as preceded by a decimal point (*e.g.* 65432 regarded as 0·23456).

Runs of.	Observed.	Calculated.
2	458	456
3	206	201
4	50	58
5	11	13
6	4	2·6
7	1	

$n'=5$, $P=0·5$.
mean: observed, 2·493.
mean: theoretical, 2·5.
S.D. : theoretical, 0·029.

Gaps of.	Observed.	Calculated.
3	139	143
4	124	119
5	64	61
6	20	24
7	9	7·6
8	2	2·6

$n'=6$, $P=0·9$.
mean: observed, 4·005.
mean: theoretical, 4·0.
S.D. : theoretical, 0·058.

(c) *Brownlee's Measles,* London, 1840–1912, quarter years.

Runs of.	Observed.	Calculated.
2	70·5	96·9
3	37·5	42·6
4	37·0	12·3
5	6·0	
6	2·5	3·2
7	1·5	

$n'=4$, $P < 10^{-6}$.
mean: observed, 2·948.
mean: theoretical, 2·5.
S.D. : theoretical, 0·062.

Gaps of.	Observed.	Calculated.
3	49	58·8
4	16	49·0
5	40·5	25·2
6	12·5	9·8
7	14	3
8	9	
9	3	10·8
10	2·5	
11	0·5	

$n'=6$, $P < 10^{-6}$.
mean: observed, 4·871.
mean: theoretical, 4·0.
S.D. : theoretical, 0·127.

(d) *Swedish death-rates*, 1740–1930.

Runs of.	Observed.	Calculated.
2	49	61·3
3	26	27·0
4	15	7·8
5	6 }	2·0
6	2 }	

$n'=4$, P = 0·000006.
mean: observed, 2·837.
mean: theoretical, 2·5
S.D. : theoretical, 0·079.

Gaps of.	Observed.	Calculated.
3	26	38·4
4	21	32
5	22	16·5
6	16	6·4
7	6 }	2·7
8	4 }	
9	1 }	

$n'=5$, P $< 10^{-6}$.
mean: observed, 4·698.
mean: theoretical, 4·0.
S.D. : theoretical, 0·112.

(e) *Edinburgh rainfall*, 1785–1930.

Runs of.	Observed.	Calculated.
2	62	59·4
3	20	26·1
4	12	7·5
5	1 }	1·97
6	0 }	

$n'=4$, P = 0·17.
mean: observed, 2·473.
mean: theoretical, 2·5.
S.D. : theoretical, 0·079.

Gaps of.	Observed.	Calculated.
3	37	37·6
4	31	31·3
5	13	16·1
6	10	6·3
7	3 }	2·7
8	0 }	

$n'=5$, P = 0·6.
mean: observed, 4·053.
mean: theoretical, 4·0.
S.D. : theoretical, 0·156.

(f) *Edinburgh temperatures*, 1764–1930.

Runs of.	Observed.	Calculated.
2	67	65·6
3	22	28·8
4	12	8·3
5	4 }	2·2
6	0 }	

$n'=4$, P = 0·20.
mean: observed, 2·552.
mean: theoretical, 2·5.
S.D. : theoretical, 0·075.

Gaps of.	Observed.	Calculated.
3	44	41·6
4	29	34·7
5	18·5	17·8
6	6·5	6·9
7	3 }	3·0
8	3 }	
9	0 }	

$n'=5$, P = 0·38.
mean: observed, 4·082.
mean: theoretical, 4·0.
S.D. : theoretical, 0·156.

(g) Ectromelia.

Runs of.	Observed.	Calculated.
2	56	61·9
3	24	27·2
4	14	7·8
5	5	2·1

$n'=4$, P = 0·019.
mean: observed, 2·677.
mean: theoretical, 2·5.
S.D. : theoretical, 0·078.

Gaps of.	Observed.	Calculated.
3	35	38·4
4	22	32
5	22	16·5
6	14	6·4
7	2 }	2·7
8	1 }	

$n'=5$, P = 0·006.
mean: observed, 4·261.
mean: theoretical, 4·0.
S.D. : theoretical, 0·158.

	Runs.						Gaps.					
	Mean.	Departure.	S.D.	Departure ÷ S.D.	P*.	$P(\chi^2)$.	Mean.	Departure.	S.D.	Departure ÷ S.D.	P*.	$P(\chi^2)$.
(*a*) Random numbers.	2·486	0·014	0·024	0·6	0·55	0·9	3·997	0·003	0·055	0·05	0·96	0·9
(*b*) Telephone numbers.	2·493	0·007	0·029	0·2	0·84	0·5	4·005	0·005	0·058	0·09	0·93	0·9
(*c*) Brownlee's measles.	2·948	0·448	0·062	7·3	$< 10^{-10}$	$< 10^{-6}$	4·871	0·871	0·127	6·8	$< 10^{-10}$	$< 10^{-6}$
(*d*) Swedish death-rates.	2·837	0·337	0·079	4·3	2×10^{-5}	6×10^{-6}	4·698	0·698	0·112	6·23	$< 10^{-8}$	$< 10^{-6}$
(*e*) Edinburgh rainfall.	2·473	0·027	0·079	0·3	0·76	0·17	4·053	0·053	0·156	0·35	0·72	0·6
(*f*) Edinburgh temperatures.	2·552	0·052	0·075	0·7	0·48	0·20	4·082	0·082	0·156	0·49	0·62	0·38
(*g*) Ectromelia.	2·677	0·177	0·078	2·3	0·02	0·019	4·261	0·261	0·158	1·65	0·10	0·006

Comment on Tables.

Of the above series (*a*) Tippett's numbers and (*b*) the telephone numbers are definitely random in character and so ought to conform to the theory. It will be seen that the mean run in the case of (*a*) is of length 2·486 and in the case of (*b*) it is 2·493, whilst the theoretical length is 2·5 with S.D.'s of 0·024 and 0·029 respectively. For the gaps the agreement is seen to be equally good. The frequency distributions of the runs and gaps of various lengths are in complete conformity with theory ($P > 0·9$, $P = 0·9$, $P = 0·5$ and $P = 0·9$).

If we now consider (*c*) and (*d*) we see at once that they are not entirely random in character. In the case of (*c*) measles, the mean run is 2·948 and the mean gap 4·871, the deviations from the theoretical values of 2·5 and 4 being respectively seven and ten times the standard deviations, whilst the P value for the frequency distribution is less than one in a million in each case. For the Swedish death-rates (*d*) the discrepancies though not so great are also definitely significant. It may be recalled that Brownlee has shown by periodogram analysis that the measles death-rates give evidence of periodic fluctuations, whilst the Swedish death-rates show a secular fall incompatible with complete randomness. Thus, as might have been anticipated, these two series fail to conform to the criteria.

The statistics regarding rainfall and temperature in Edinburgh, (*e*) and (*f*) respectively, are seen to behave as if they were entirely random in character. We do not know whether in this case more refined analysis would demonstrate the existence of periodic components.

The last example (*g*) refers to the fluctuations of the death-rate from ectromelia in mice in an experimental epidemic described by Greenwood, Bradford Hill, Topley and Wilson (1936). It was the question as to

whether this series was purely random in character which gave rise to the present investigation. It will be seen that the average lengths of the runs and gaps are 2·677 and 4·261 respectively. The deviations from the theoretical values being 0·177 and 0·261, whereas the S.D.'s are 0·078 and 0·158 respectively. The deviations calculated from the data are approximately double the theoretical so that there would be about one chance in twenty of their occurrence on the assumption that the distribution was a random one. The values of P for the frequency distributions of the runs and gaps are 0·019 and 0·006 respectively. These results suggest that the distribution of these deaths is not altogether a random one, but the deviation from the expected is small, and too great emphasis must not be placed upon the discrepancy. It must be remembered that any non-random change in the conditions of the experiment, such, for example, as a more or less prolonged spell of hot weather would tend, to some extent, to disturb the course of the epidemic, and so produce non-random effects. The slight excess of the longer runs and gaps, if we are to attach significance to them at all, is probably the result of disturbing factors of this type.

Discussion.

The problem treated in the present paper is related to the problem of Simon Newcomb which is discussed fully by Macmahon (1915) and the closely related problem discussed by André (1884), *cf.* Netto (1901). The relationship between the function P_{nr} of André and the function x_{nr} defined in the present paper will be discussed more fully in a subsequent communication.

It should be emphasized that the criteria here proposed are *necessary* but not *sufficient* conditions that the numerical series be a random one. Deviations from randomness may be due to many causes. Some of these may be periodic, but these constitute only a small fraction of the total number of possible non-random influences.

It is well recognized that ordinary periodogram analysis is limited in its applicability. Modifications of the simple process have been proposed which partly meet this drawback (Walker, 1931). However, in such methods of approach it is assumed that a periodic process, perhaps modified to some extent in amplitude and period, is at work. The peculiarity of the present criteria is that they are based on no such assumption. Failure of a series to comply with them at once makes it out as a non-random one. At the same time it is clear that a non-random series may sometimes satisfy the criteria or at least show so small discrepancies as to be without insignificance in any series of ordinary length. Thus if to a series of

random numbers we add a second linearly increasing series, then if the rate of increase is small as compared with the range of the original numbers, the series thus obtained will in all probability satisfy the criteria, although it is not random because the later numbers in the series will tend to be greater than the earlier ones. In the same way a slow periodic superaddition may be applied to a random series in such a way that the criteria are not sensibly affected, and yet the series as a whole will have become definitely periodic.

The suggested criteria may be regarded as complementary to ordinary periodogram analysis. At the same time it is of interest to enquire whether the method can be extended in some simple way so as to detect more readily those secular or periodic changes which, as we have indicated, are likely to be missed. A simple device would be to take, not every successive number in the series as it stands, but to form supplementary series containing every 2nd, 3rd, 4th number, etc. Each such series would divide up into 2, 3, 4, 5 . . . sub-series. Thus when taking every 2nd number we would have the two sub-series, containing the odd and even numbers respectively. The test could then be applied to these sub-series, either separately or taken together. If the original series is entirely random, the runs and gaps in the sub-series should satisfy the criteria. The same should be true for the runs and gaps in the three sub-series, each including every 3rd number. Clearly in the case of a slow periodicity with perhaps a period of about ten numbers, certain of the five sub-series which are obtained by taking every 5th number should have an excessive number of runs of two, and the departure from randomness would in this way be detected.

It is to be observed that the criteria proposed in the present paper do not depend on the actual numerical magnitudes of the different numbers, but only on the existence of sets of inequalities. A run down from a_1 to a_4 merely means that $a_1 > a_2 > a_3 > a_4$. An interesting consequence is that the distributions in question, x_{nr} and p_{nr}, remain invariant under any transformation whatever of the original numbers, provided that it is a one to one transformation and that it is not chosen with reference to the order of the numbers in the series in question. This property is also possessed by any other statistical distribution which depends only on the existence of inequalities amongst the original numbers.

One obvious limitation of these criteria is that as they make use only of qualitative relationships and do not take into account the exact magnitudes of the observations, they do not make full use of all the available information. It is to be noticed, however, that there is the compensating advantage that the criteria make no assumption whatever about the law

of distribution of the observations, apart from the very general one that they are unequal. Thus, when a result indicating a significant departure from randomness is obtained, it can only be due to the presence of some factor which introduces a regularity into the system under consideration, and cannot be ascribed to the inadequacy of the theory.

In conclusion we wish to thank Professor Major Greenwood for his suggestions and assistance during the course of this investigation.

SUMMARY.

1. An expression is given for the probability that any *run* selected at random from a large series of unequal numbers arranged in random order contains 2, 3, 4, 5, etc., numbers, including the first and last. A similar expression is given for the chance that two consecutive runs (a *gap*) should contain 3, 4, 5, etc., numbers.

2. The average length of a run and of a gap are respectively $2\frac{1}{2}$ and 4 with standard deviations of $\sqrt{\frac{3}{5s_1}}$ and $\sqrt{\frac{6}{5s_2}}$, where s_1 and s_2 are the numbers of consecutive runs and gaps from which the average is calculated.

3. The results may be employed to assist in the decision as to whether any particular series is or is not a random one.

4. Examples are given of the application of these new criteria to various random and non-random series.

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XVIII.—The Benthic Amphipoda of the North-Western North Sea and Adjacent Waters. By D. S. Raitt, D.Sc., Ph.D., F.L.S., The Marine Laboratory of the Fishery Board for Scotland, Aberdeen. (With One Figure.)

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INTRODUCTION.

THE order Amphipoda, of the Crustacea Malacostraca, which consists of the 3 sub-orders Hyperiidea, Gammaridea, and Caprellidea, is highly important in the economics of the sea as a source of fish food. Hyperiids figure prominently in the diet of species which feed pelagically. Hardy (1924), for example, found them amongst the principal items in the food of East Anglian herring. Gammarids are preyed upon by bottom feeders in general. Todd (1905, 1907) and Blegvad (1916) record them in varying percentages from the stomachs of adult cod, haddock, whiting, plaice, dabs, flounders, sole, and skate. Their chief importance, however, would seem to be in the diet of fishes in post-larval and adolescent stages. Todd (1907, 1914) found Gammarids the main support of cod under 15 cm. and of plaice and dabs under 10 cm. Blegvad (1930) states that in Danish waters, plaice in their first year feed almost exclusively upon them. Clark (1922) found that in the early stages of 4 species of skate, Amphipods occurred in from 50 to 84 per cent of the stomachs examined, and Steven (1930) found the food of young stages of *R. clavata* and *R. maculata* to consist chiefly of Gammaridea. Gammarid species, as recorded by Hunt (1924), are also eaten by a variety of bottom-dwelling invertebrates.

The present study is a survey of the distribution of the order in the Scottish North Sea and adjacent waters. The material upon which it is based was obtained partly from haddock stomachs and partly by Petersen Bottom Sampler.

MATERIAL OBTAINED BY PETERSEN BOTTOM SAMPLER.

The Petersen Grab was used as part of the routine work of the Scottish Fishery Research Ship *Explorer* continuously from 1922 to 1930, during which time the hauls taken with the apparatus totalled 2144.

Each season of the year is well represented in the investigation.

Sixteen per cent of the hauls were taken in the first quarter, 27 in the second, 33 in the third, and 24 in the fourth. On the accompanying chart the statistical areas visited and the total number of hauls taken in each are shown.

The species recorded are those retained in washing the collections

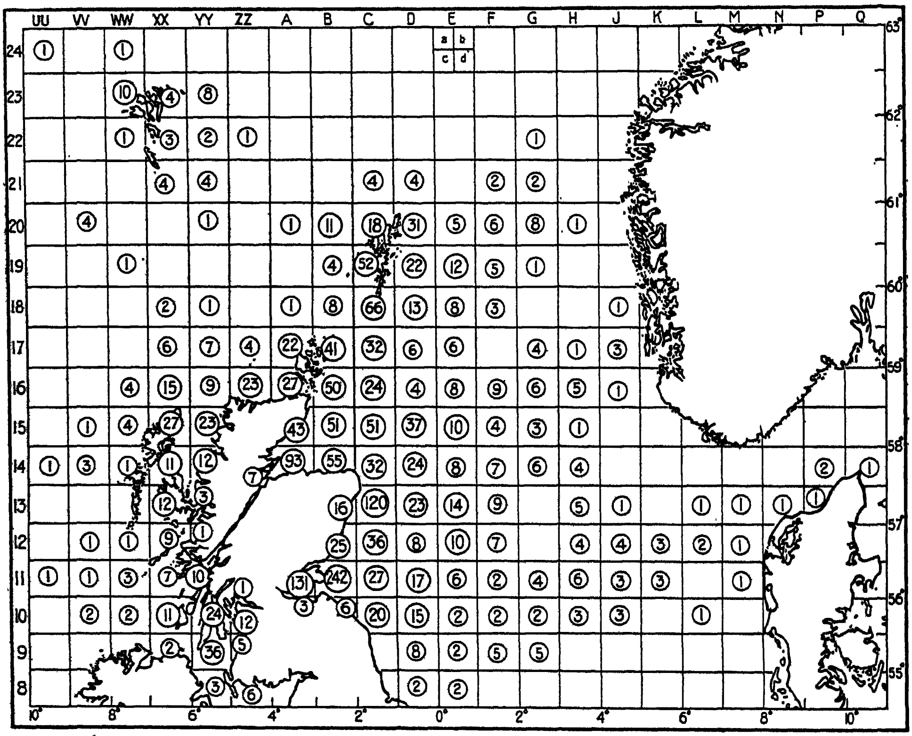

FIG. 1.—The number of Petersen Grab hauls per Statistical Area taken by the Fishery Research ship *Explorer*, 1922–30.

through a sieve with square meshes, the side of each square measuring approximately 1·5 mm.

Spårck (1935), writing upon the importance of quantitative investigation of the bottom fauna in marine biology, summarises the results obtained to date from the use of the Petersen Bottom Sampler. He states that it has made possible the distinction of waters with an abundance of bottom animals, that is fish food, from those of poor benthic fauna; considerable differences have been found, related to depth and to climatological and hydrographical conditions. It has also made possible the recognition of animal communities, characterised by certain dominant species. These communities are likewise related to depth, nature of bottom, and to the geographical factors of climate and hydrography.

Viewing the present work from similar angles, it is first to be noted that Amphipods occurred in the residues of only 481 of the hauls, but they were present in 60 per cent of the statistical areas sampled. No region of any extent was found to be devoid of the order, but considerable differences in density were found in different localities. From the average number of specimens taken per square metre of bottom in each statistical square of the North Sea, it is at once evident that coastal waters are much more densely populated than the open sea. The Moray Firth, St Andrews Bay, and the Firth of Forth are richest in Amphipoda, while the deeper central basin of the area is very much poorer. The Dogger Bank region exhibits coastal rather than open sea characteristics.

This is well illustrated when the North Sea is treated in sections, such as were recognised by Thompson (1929, fig. 1, p. 138), as productive of distinctive growth rates in the haddock. Thompson's divisions were:

1. A western area, denoted W, consisting of the whole of the shallower coastal belt from the Shetlands to the Firth of Forth, together with Scottish west coast and north-western Scottish waters.

2. A central basin, denoted C, consisting of the "Gut" and the deepest, coldest waters of the open North Sea.

3. An eastern area, denoted E, consisting of the banks and shallows to the south and east of area C.

The average frequency of Amphipods in the waters of area W was 480 per 100 sq. m., as against 215 in area C and 309 in area E. It is not suggested that there can be any direct connection between haddock growth rates and the density of a single order such as Amphipoda; nevertheless, it may be observed that area C was found by Thompson to give poor rate of growth compared with E and W. Area E is that of fastest haddock growth, and area W, while giving an intermediate rate of growth for younger age classes, supports the bulk of the stock of older fish of the species.

Stephen (1922, 1929, 1930, 1933 *a*, 1933 *b*) has already published accounts of certain of the animal groups encountered in the course of the Petersen Grab investigations of the Scottish Fishery Board in the North Sea over the years 1922 to 1926. Dealing with the mollusca of the collections (1933 *a*), he distinguishes four natural faunistic divisions within the regions:

(*a*) A Littoral Zone, extending from high-water mark to about 4 m. depth.

(*b*) A Coastal Zone, extending from 4 m. to about the 40 m. line.

(*c*) An Offshore Zone, wide in extent, made up of water of intermediate depth.

(*d*) A zone occupying the deeper north-eastern portion of the region.

The Littoral Zone, he states, is dominated by the species *Tellina tenuis*, *Macoma baltica*, and *Cardium edule*, and density of occurrence is "often very high." The Coastal Zone is dominated by *Tellina fabula*, *Nucula nitida*, *Syndosmya alba*, and *S. nitida*, and density of occurrence is "very high in many places." Many species of Lamellibranchs occur in the Offshore Zone, but no single one is dominant, and densities of occurrence are low. The North-Eastern Zone is dominated by *Thyasira flexuosa*, and density of occurrence therein is very high.

The present work does not extend into Stephen's Littoral Zone, but ample material is available for studying his Coastal, Offshore, and North-Eastern divisions. It may be taken that the Coastal Zone includes all waters of up to 40 m. depth, that the Offshore Zone includes all waters of from 40 to 100 m. depth, while the North-Eastern Zone is made up of all waters of over 100 m. depth.

Similarity has already been noted between Dogger Bank and coastal characteristics. Stephen's scheme of zoning explains the bathymetric nature of the relationship.

The average frequency of occurrence of Amphipoda in the Coastal Zone was about 540 per 100 sq. m., in the Offshore Zone about 400, and in the North-Eastern Zone about 180—that is, roughly in the proportion of 3:2:1.

The total number of specimens obtained by Petersen Grab was 1605, exclusive of a small amount of material damaged in capture and unidentifiable, and of 4 specimens of undetermined species which have been reserved for further study. Only 13 of the identified specimens were of the sub-order Hyperiidea, belonging to 3 species, 2 genera, and a single family. Only 10 specimens were of the sub-order Caprellidea, belonging to 2 species, 2 genera, and a single family. The remaining 1582 specimens were of the sub-order Gammaridea, and belonged to 74 species, 50 genera, and 19 families.

Of the species encountered in the North Sea 15 per cent occurred exclusively in the 0–40 m. zone, 26 per cent exclusively in the 40–100 m. zone, and 7 per cent exclusively in the deep-water zone. Eighteen per cent occurred in both the 0–40 and 40–100 m. zones, 9 per cent in both the 40–100 and deeper water zones, and 25 per cent occurred in all three regions.

Perhaps the most striking result of the investigations was the discovery that 55 per cent of the total North Sea specimens belonged to the one Gammarid genus *Ampelisca*. The 10 commonest North Sea species were, in descending numerical order, *Ampelisca brevicornis*, *A. tenui-*

cornis, *A. macrocephala*, *Hippomedon denticulatus*, *A. spinipes*, *Harpinia antennaria*, *Photis longicaudata*, *Bathyporeia pelagica*, *B. guilliamsoniana*, and *Westwoodilla caecula*. These comprised 70 per cent of the North Sea material. The 4 species of the genus *Ampelisca* were in themselves 52 per cent, and the single species *A. brevicornis* 25 per cent.

The dominant species of the Coastal Zone were *A. brevicornis*, 253 per 100 sq. m., *A. tenuicornis*, 61 per 100 sq. m., and *Bathyporeia guilliamsoniana*, 31 per 100 sq. m. The dominant species of the Offshore Zone were *A. tenuicornis*, 109 per 100 sq. m., *A. macrocephala*, 43 per 100 sq. m., and *A. brevicornis*, 42 per 100 sq. m. The dominant species of the North-Eastern Zone were *A. tenuicornis*, 43 per 100 sq. m., *Harpinia antennaria*, 28 per 100 sq. m., and *A. gibba*, 22 per 100 sq. m.

The Moray Firth, St Andrews Bay, and the Firth of Forth, which are the richest of the grounds surveyed so far as Amphipoda are concerned, are the principal nurseries of the plaice of Scottish waters. The Moray Firth, in addition to extensive 0–40 m. shallows, also includes a considerable area of much deeper water. The Firth of Forth and St Andrews Bay are more in the nature of estuaries. The density of Amphipods in general in the Moray Firth was about 540 per 100 sq. m., genus *Ampelisca* 370, while St Andrews Bay and the Firth of Forth, taken together, gave about 500 per 100 sq. m., genus *Ampelisca* 420. The number of different species taken in the Moray Firth was 29, as against 25 in the Firth of Forth and St Andrews Bay. Fifteen were common to both regions, 14 occurred in the north but not in the south, and 11 in the south but not in the north.

A. brevicornis and *A. tenuicornis* were by far the most numerous all over. The greater frequency of the genus *Ampelisca* in the south than in the north is a reflection of the fact that *A. macrocephala* occurs quite commonly in the Firth of Forth, but is much rarer in the Moray Firth.

A small amount of material was obtained from grounds outside the North Sea which were sampled as shown on the accompanying chart. The fauna of the west coast of Scotland is represented, together with that of the waters to the west of the Orkneys and Shetlands, and that of the Faroes. The west coast fauna would appear to be similar to that of the east of Scotland. *A. brevicornis* again predominated. The specimens from the Atlantic to the west of the Orkneys and Shetlands also correspond with those found in similar depths to the east of the two groups, except that *Orchomene humilis*, encountered for the first time, occurred to the north of Cape Wrath. Around the Faroes other 3 unfamiliar species were found. They were *Ampelisca odontoplax*,

Photis tenuicornis, and *Lembos websteri*. They occurred along with 5 species already taken in the North Sea.

MATERIAL OBTAINED FROM HADDOCK STOMACHS.

The Petersen Grab collection is supplemented by an additional 1599 specimens which were obtained from the stomachs of haddock caught by the *Explorer* between September 1926 and May 1927. A further 7 specimens of undetermined species, also secured from haddock stomachs, have been reserved for further study along with those from the Bottom Sampler collections.

Material obtained from fish stomachs cannot be taken to give any reliable indication either of the different animals living upon the grounds on which the fish were caught or of their relative numbers. The second source of specimens, nevertheless, provides valuable confirmation of the conclusions arrived at from the Petersen Grab investigations.

Haddock occurred in numbers in 69 trawl hauls taken by the *Explorer* over the period in question. Amphipods were found in the stomachs of the haddock caught in 42 of these 69 hauls. Five of the 42 records were on the west coast of Scotland, 2 were in the Atlantic to the north-west of Scotland, 1 was at Orkney, and 1 at Shetland, 8 were in the Moray Firth, 19 were from between Aberdeen and the Firth of Forth, and 6 were in the open waters of the North Sea. The depths of the hauls varied from 14 to 196 m. Eight were in from 0 to 40 m., 21 were in from 40 to 100 m., and 13 were in waters of over 100 m.

The 1599 specimens obtained were found to belong to 17 families, 33 genera, and 46 species. Only 1 specimen was of the sub-order Hyperiidea; 133 specimens, belonging to 1 family, 2 genera, and 2 species, were of the sub-order Caprellidea; and the remaining 1464 specimens, belonging to 15 families, 30 genera, and 47 species, were of the sub-order Gammaridea.

Forty-one of the species met with also occurred in the Petersen Grab. Six were encountered for the first time, none of which was of any numerical importance. The genus *Ampelisca* was again in predominance, forming 45 per cent of the total specimens secured.

The most important species, in descending numerical order, were *A. tenuicornis*, *Bathyporeia pelagica*, *B. guilliamsoniana*, *A. macrocephala*, and *A. brevicornis*. These made up 80 per cent of the total collection. They were amongst the 10 species of first importance in the Petersen Grab material, although their order was somewhat different. The advanced position of *A. tenuicornis* is mainly due to a single haul of 624 haddock taken in the Moray Firth, from which 364 specimens

were secured. The prominence of *B. pelagica* and *B. guilliamsoniana* likewise arises from a single haul taken off Aberdeen, where 794 haddock were captured.

The dominant species in 0–40 m. waters were *B. guilliamsoniana*, *B. pelagica*, and *A. brevicornis*. The two latter species, together with *A. tenuicornis*, were the coastal dominants in the Grab survey. The dominant species in 40–100 m. waters were *A. tenuicornis* and *A. macrocephala*, just as was found for the Offshore Zone in the Grab material. The dominant species in waters of over 100 m. were *Hippomedon denticulatus* and *Harpinia antennaria*, compared with *H. antennaria*, *A. tenuicornis*, and *A. gibba* in the Bottom Sampler collections.

When the species are arranged according to the zones in which they occur exclusively and to which they are common, 77 per cent fall into categories in agreement with those of the Bottom Sampler work.

No outstanding difference in species was noticeable between east coast and west coast material.

The Genus *Ampelisca*.

Thirteen species of the genus *Ampelisca* are on record as occurring in the North Sea and adjacent waters, 8 of which were encountered in the present investigations. They were, in order of importance, *brevicornis*, *tenuicornis*, *macrocephala*, *spinipes*, *typica*, *gibba*, *diadema*, and *odontoplax*.

A. brevicornis (Costa), syn. *A. laevigata* Sars, was found in depths of from 5 m. in the Firth of Forth to 140 m. south-east of the Shetlands. It is pre-eminently a shallow-water form. The Petersen Grab frequency figures give the ratio of occurrence from zone to zone as approximately 60:10:1, in a total of 402 specimens.

While the majority of the specimens were met with on sand, or on muddy sand, many were taken on heavy mud. There was no evidence of a definite preference in bottom conditions.

Depths of capture of *A. tenuicornis* Lilljeborg varied from 10 m. in the Firth of Forth to 150 m. east of the Shetlands. The Petersen Grab frequency figures give the species as most common in somewhat deeper more offshore waters than *A. brevicornis*; the ratio of occurrence from zone to zone being approximately 1·5:3:1 in a total of 268 specimens.

As with the preceding species no evidence of a preference in bottom conditions was shown. Specimens were taken on sand, sandy mud, and heavy mud.

The specimens of *A. macrocephala* Lilljeborg were taken at depths

varying from 22 m. in the Moray Firth to 140 m. west of Orkney. This species, according to the Petersen Grab frequency figures, is most common in offshore waters; the ratio of occurernce from zone to zone being approximately 1:3:1 in a total of 102 specimens.

By far the greatest number of specimens occurred on mud. This preference in bottom conditions may account for the fact that the species is much less common in the Moray Firth than in the Firth of Forth.

A. spinipes Boeck occurred at depths ranging from 10 m. in the Firth of Forth to 160 m. north of the Shetlands. This species is most common, according to the Grab data, in water of from 0 to 100 m.; the ratio of occurrence from zone to zone was approximately 5:8:1 in a total of 58 specimens. The majority occurred on sandy bottom.

Depths of capture of *A. typica* (Bate) varied from 14 m. on the east of the Shetlands to 124 m. on the Viking Bank; the ratio of frequency of occurrence in the grab from zone to zone was approximately 5:6:1 in a total of 22 specimens. Captures were for the most part upon sand.

A. gibba Sars is much rarer than any of the foregoing species. The present occurrences, 16 specimens in all, are the first from Scottish waters. Depths of capture varied from 111 m. north-east of Kinnaird Head to 148 m. north-east of Shetland. It was entirely absent from the Coastal and Offshore Zones. It appears to prefer muddy bottom conditions.

Only 3 specimens of *A. diadema* (Costa), syn. *A. assimilis* Sars, were captured.

One specimen of *A. odontoplax* Sars, a female of 10 mm., occurred in the Grab in the Faroe-Shetland Channel at a depth of 229 m. This species is known only from the west and south of Ireland at 1393 and 1484 m., and from a number of localities on the coast of Norway at depths of from 250 to 560 m.

Occurrences of ovigerous females in the different *Ampelisca* species were specially noted. They were very rare, forming only about 2·1 per cent of the total specimens of the genus. All occurred within the period May to November. Specimens showing male characteristics were even rarer. They formed only 1·8 per cent of the total material of the genus and were found only in the months over which ovigerous females were encountered.

General List of Species met with.

Chevreux and Fage (1925), Sars (1895), Stebbing (1906), and Stephensen (1915, 1918–1926, 1923–1931, 1929) are the main authorities upon which identifications have been based.

Full details of the locations of capture of the specimens have been lodged with the Department of Zoology, British Museum (Natural History), and also with the Department of Zoology, Royal Scottish Museum, Edinburgh.

In the following list of species met with, the abbreviations P.G. and H.S. indicate respectively whether the material was obtained by Petersen Grab or from haddock stomachs. Those species marked by an asterisk have also been recorded from the Tidal Zone by Elmhirst (1931).

Hyperoche medusarum Krøyer. P.G. 80 m.
Hyperia galba Montagu. H.S. 53 m.
Themisto gracilipes Norman. P.G. 72–125 m.
Themisto compressa Goes. P.G. 41–119 m.
Acidostoma obesum Bate. P.G. 21–138 m.; H.S. 53–104 m.
Aristias neglectus Hansen. P.G. 150 m.
Orchomene humilis (Costa). P.G. 50 m.
Anonyx nugax Phipps. P.G. 14 m.; H.S. 119 m.
Hippomedon denticulatus Bate. P.G. 11–162 m.; H.S. 14–140 m.
Scopelocheirus crenatus Bate. P.G. 21–80 m.; H.S. 38–120 m.
Tryphosa sarsi Bonnier. P.G. 78 m.
Tryphosites longipes Bate and Westwood. P.G. 14–147 m.; H.S. 52–137 m.
Lepidepecreum longicorne Bate. P.G. 22–93 m.; H.S. 120 m.
Tmetonyx albidus G. O. Sars. H.S. 124 m.
Tmetonyx cicada (Fabricius). P.G. 24–85 m.; H.S. 26 m.
**Ampelisca brevicornis* (Costa). P.G. 5–122 m.; H.S. 14–140 m.
Ampelisca diadema (Costa). P.G. 10–67 m.
Ampelisca gibba G. O. Sars. P.G. 111–148 m.
**Ampelisca macrocephala* Lilljeborg. P.G. 22–134 m.; H.S. 33–140 m.
Ampelisca odontoplax G. O. Sars. P.G. 229 m.
Ampelisca spinipes Boeck. P.G. 10–160 m.; H.S. 33–124 m.
**Ampelisca tenuicornis* Lilljeborg. P.G. 10–150 m.; H.S. 20–124 m.
Ampelisca typica (Bate). P.G. 14–124 M.; H.S. 46–124 m.
Byblis crassicornis Metzger. P.G. 440 m.
Byblis gaimardi Krøyer. P.G. 33–128 m.; H.S. 140 m.
Haploops tubicola Lilljeborg. P.G. 49–97 m.; H.S. 42 m.
Haploops setosa Boeck. P.G. 113–460 m.
Urothoe elegans Bate. P.G. 22–152 m.; H.S. 92 m.
**Urothoe marina* Bate. P.G. 47–100 m.; H.S. 60 m.
**Bathyporeia guilliamsoniana* Bate. P.G. 5–75 m.; H.S. 14–35 m.
**Bathyporeia pelagica* Bate. P.G. 13–120 m.; H.S. 14–120 m.
**Phoxocephalus holbolli* Krøyer. P.G. 35 m.
Harpinia antennaria Meinert. P.G. 15–170 m.; H.S. 49–123 m.
Harpinia laevis G. O. Sars. P.G. 28 m.
Leucothoe lilljeborgi Boeck. H.S. 14–26 m.
Leucothoe spinicarpa Abildgaard. P.G. 88 m.; H.S. 16–58 m.

Metopa norvegica Lilljeborg. P.G. 49–75 m.; H.S. 92–110 m.
Metopa rubrovittata G. O. Sars. P.G. 75 m.
Metopa spectabilis G. O. Sars. H.S. 71 m.
Iphimedia obesa Rathke. P.G. 14–62 m.; H.S 94 m.
Nicippe tunida Bruzelius. P.G. 100–170 m.; H.S. 196 m.
**Pontocrates arcticus* G. O. Sars. P.G. 30 m.; H.S. 140 m.
Synchelidium haplocheles Grube. P.G. 17 m.; H.S. 20 m.
Monoculodes carinatus Bate. P.G. 24–60 m.; H.S. 119 m.
Westwoodilla caecula Bate. P.G. 41–124 m.; H.S. 50–124 m.
Apherusa bispinosa Bate. P.G. 69 m.; H.S. 94 m.
Neopleustes assimilis G. O. Sars. P.G. 57 m.
Neopleustes bicuspis Krøyer. P.G. 14–104 m.; H.S. 120 m.
Neplueustes monocuspis G. O. Sars. P.G. 20–104 m.; H.S. 118 m.
Sympleustes glaber Boeck. P.G. 72 m.
Sympleustes latipes M. Sars. P.G. 66 m.
Nototropis falcatus Metzger. P.G. 22–80 m.
Nototropis norlandicus Boeck. P.G. 85 m.
**Nototropis swammerdami* Milne-Edwards. P.G. 14–22 m.
Nototropis vedlomensis Bate and Westwood. P.G. 10–98 m.; H.S. 92–120 m.
Eusirus longipes Boeck. P.G. 22 m.; H.S. 92–97 m.
Gammarellus homari (Fabricius). P.G. 24–72 m.
Eriopisa elongata Bruzelius. P.G. 264 m.
Cheirocratus assimilis Lilljeborg. H.S. 37–119 m.
Cheirocratus intermedius G. O. Sars. P.G. 36–52 m.; H.S. 42 m.
Cheirocratus sundevalli Rathke. P.G. 8–106 m.; H.S. 119 m.
Melita obtusata Montagu. P.G. 67 m.; H.S. 120 m.
Maera loveni Bruzelius. P.G. 19–98 m.; H.S. 40–58 m.
Maera oithonis Milne-Edwards. P.G. 11–126 m.; H.S. 53–110 m.
**Gammarus locusta* L. P.G. 54–145 m.; H.S. 26 m.
Dexamine spinosa (Montagu). P.G. 11 m.
Microdeutopus anomalus (Rathke). P.G. 59 m.
Aora typica Krøyer. P.G. 60 m.
Lembos longipes Lilljeborg. P.G. 28–71 m.
Lembos websteri Bate. P.G. 28 m.
Photis longicaudata Bate and Westwood. P.G. 9–120 m.
Photis tenuicornis G. O. Sars. P.G. 105 m.
Eurysteus maculatus (Johnstone). P.G. 22–76 m.
Leptocheirus hirsutimanus Bate. P.G. 88–118 m.
Protomedia fasciata Kroyer. P.G. 84 m.
Podoceropsis sophia Boeck. P.G. 22–76 m.
Amphithoe rubricata Montagu. P.G. 50 m.
Erichthonius brasiliensis Dana. P.G. 15–50 m.
Erichthonius hunteri Bate. P.G. 25–145 m.
Unciola planipes Norman. P.G. 48–74 m.; H.S. 69 m.
Siphonocetes colletti Boeck. P.G. 23–103 m.
**Corophium crassicorne* Bruzelius. P.G. 14–28 m.

Phtisica marina Slabber. P.G. 11–185 m.; H.S. 14–158 m.
Pseudoprotella phasma Montagu. H.S. 22 m.
Caprella linearis L. P.G. 86 m.

Occurrences of Species new or rare in the Area surveyed.

Hyperoche medusarum: off Aberdeen; first Scottish record.

Acidostoma obesum: N. of Faroe; first record north of Faroe; also taken in home waters where already frequently recorded.

Anonyx nugax: N. Shetland; NW. of Flannan Is.; only four previous occurrences south of Faroe, three in the south of Norway, and one in the Firth of Forth.

Tmetonyx albidus: Viking Bank; described by Sars, 1895, from several localities on the west of Norway; no other records.

Ampelisca gibba: Viking Bank; E. of Shetland; NE. of Buchanness; first records in Scottish waters.

Ampelisca odontoplax: Faroe-Shetland Channel; known only from W. and S. Iceland and Norwegian coast.

Byblis crassicornis: NW. Shetland; first British record.

Harpinia laevis: off Aberdeen; first occurrence on east of Scotland.

Metopa spectabilis: Orkney; first British record.

Nicippe tumida: Minch; SE. Orkney; Moray Firth; N. of Viking Bank; in British waters known only from Shetland and north-east of Peterhead.

Pontocrates arcticus: S. Shetland; Moray Firth; first British records.

Sympleustes glaber: Orkney; first Scottish record.

Nototropis falcatus: W. Orkney; off Peterhead; first Scottish records.

Nototropis norlandicus: E. Shetland; first occurrence outside Norway.

Cheirocratus assimilis: N. of Flannen Is.; E. of Skye; first records on west of Scotland.

Maera loveni: Clyde; Minch; Moray Firth; off St Andrews; Firth of Forth; off Firth of Forth; only one previous Scottish record, Firth of Forth.

Maera oithonis: NE. Butt of Lewis; N. Scottish coast; W., E., and SE. Orkney; W. Shetland; Moray Firth; E. of Moray Firth; off Aberdeen; off Stonehaven; off St Andrews; in British waters known only from Shetland and from Northumberland and Durham.

Lembos websteri: E. Faroe; first occurence at Faroe.

Photis tenuicornis: W. Faroe; first occurrence at Faroe.

Pseudoprotella phasma: Firth of Forth; first Scottish record.

Summary and Conclusions.

Amphipods are of importance in the economics of the sea as fish food, particularly as a main support of the young stages of cod, plaice, dabs, and skate. The present study of the order is based upon some 3000 specimens secured partly by Petersen Bottom Sampler and partly from haddock stomachs.

In common with the general results of the Petersen Grab investigations

of previous workers upon other animal groups, Amphipods were found to vary both in abundance and in species according to benthic conditions. The order is most densely distributed in coastal waters. Density is lowest in the "Gut," while the region between the "Gut" and the Dogger Bank falls into line with coastal conditions, according to depth of habitat. Bathymetric zoning of species is recognisable when division of the North Sea by the 40 and 100 m. depth contours is adopted. The ratio of frequency of occurrence of Amphipoda in general in the three zones thus formed was approximately 3:2:1. The association of species found varied from zone to zone, as also did those in especial predominance.

The specimens obtained by Petersen Grab numbered 1605, and belonged to 79 species, 74 of which were of the single sub-order Gammaridea. The genus *Ampelisca* was outstandingly abundant, forming practically 55 per cent of the total material. The species *A. brevicornis* dominated the whole collection, comprising 25 per cent of the total captures.

Additional material obtained from haddock stomachs, 1599 specimens in all, provides confirmation of the results of the Patersen Grab survey. Forty-seven species were met with from this source, only 6 of which had not been taken by Bottom Sampler. The sub-order Gammaridea was again predominant, and the genus *Ampelisca* again outstandingly abundant, forming 45 per cent of the total collection. Seventy-seven per cent of the species occurred in the same bathymetric zones as in the Grab investigations.

Zoning of species is also seen within the single genus *Ampelisca*. *A. brevicornis* is mainly a coastal form. *A. tenuicornis* and *A. macrocephala* are more offshore species. *A. spinipes* and *A. typica* and both coastal and offshore forms and *A. gibba* is confined to depths of over 100 m. *A. macrocephala* is much more frequent in the Firth of Forth than in the Moray Firth, apparently because of a favour for muddy bottom conditions. Mud is also favoured by *A. gibba*. *A. spinipes* and *A. typica* prefer sand. *A. brevicornis and A. tenuicornis* demonstrate no particular preference.

Twenty of the total of 85 species met with are new to, or rare in, the area surveyed.

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XIX.—"Spheroidal": A Mutant in *Drosophila funebris* affecting Egg Size and Shape, and Fecundity.

By **Professor F. A. E. Crew**, M.D., D.Sc., and **Charlotte Auerbach**, Ph.D., Institute of Animal Genetics, University of Edinburgh. (With Eight Graphs and Two Figures).

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INTRODUCTION.

IN the course of a study of fecundity in *Drosophila funebris* (Donald and Lamy, 1937), it was noted that a particular female was laying very few eggs, and that the shape of these was peculiar. Among her descendants other females of similarly low fecundity and laying eggs of the same abnormal shape were encountered, and it was assumed therefore that a mutation affecting fecundity and egg shape had been recognised. In this paper evidence concerning the type of heredity of these characteristics is presented, and a statistical description of the morphological aspect of the new mutation, the size and shape of the egg, as compared with those of the normal *D. funebris* egg, is given.

No mutant types of egg shape in *Drosophila* have so far been reported, though the size of the normal egg has been the subject of several investigations. Warren (1924) discovered factors for egg size in all four linkage groups of *D. melanogaster*, and studied the individual differences in egg size between females of one and the same stock and the influence of environmental and physiological factors on egg size. In this last respect his studies were continued and partly contradicted by the work of Gause (1931) and Imai (1935) on the influence of starvation and temperature respectively on egg size.

Since the first female which had been observed to lay abnormal eggs when mated to a brother yielded daughters all of whom laid normal eggs, it was assumed that possibly this new mutation, which, from the shape of the egg, received the name *spheroidal* (*sph*), was a simple Mendelian recessive. This assumption was tested by appropriate matings. To carry out such a test is, of course, difficult, since spheroidal is a property of the female which becomes visible only in the egg that she lays. The altered shape of the egg is a consequence of the genetical constitution of the mother and not of that of the embryo enclosed within the egg. No means were found to determine the genotype of laying females from

their morphology. Consequently, in order to obtain information on segregation, sufficiently large samples of females had to be cultured individually and the egg shape observed. The genetical constitution of a male had to be revealed by the eggs of his daughters by a *sph* female. The great infecundity of the *sph* females, however, rendered this test somewhat laborious, since it was not always possible to obtain a sufficient number of daughters.

The preliminary hypothesis was verified by the following matings: a *sph* female, of the supposed genetical constitution $\frac{sph}{sph}$, when mated to her brother, produced 16 tested F_1 daughters laying normal eggs. The genetic constitution of the father was assumed to be $\frac{+}{+}$, and that of the F_1, $\frac{sph}{+}$. Out of 28 tested F_2 females, 7 laid *sph* eggs. This appears to preclude the possibility of sex-linkage.

Subsequent matings were directed towards obtaining a homozygous stock. In the course of these matings, of 67 daughters of homozygous females mated to heterozygous males, 37 laid normal eggs and 30 spheroidal, which approaches the expected ratio of 1 : 1 for an autosomal backcross.

At this stage it was noted that sample matings from the wild-type culture which had been received from Professor Timoféeff-Ressovsky, and which served as a control, showed that *sph* in heterozygous condition was very widespread in the population. *Sph* is an excellent example of a mutation which can remain long undetected. In the homozygous condition it is subjected to a very strong counter-selection on account of the low fecundity; and in the heterozygous condition it has no deleterious effect. There are certain indications that the size of the eggs of heterozygous females differs very slightly from that of the normal ones. This assumption can, however, only be verified after a further purification of the stock. In the present study all females (described as normal) laying normal eggs have been pooled together, although it is possible that a few of them were actually heterozygous.

Description of the Spheroidal Egg.

The difference in shape as well as in size between normal and spheroidal eggs is very pronounced (fig. 1). The wild-type *funebris* egg is elongated and slim, the four filaments are long and bend outwards in even curves so that they spread out over the food like a four-rayed star. The spheroidal egg is distinctly shorter and broader, the filaments are short,

stiff and undulated, and spread out either not at all or only very slightly. In the shape of the egg as a whole, as well as in that of the filaments, there is a fair amount of variation in contrast to the comparative uniformity of the normal eggs. In some eggs the filaments are exceedingly rudimentary. Furthermore, the spheroidal eggs are sometimes very pale, yellowish or transparent, so that they are easily overlooked in the food. This type of egg, which seldom, if ever, hatches, is particularly frequent at the end of the laying period of a *sph* female.

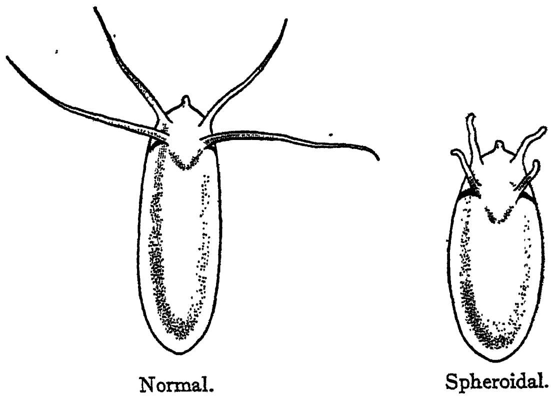


FIG. 1.

In order to gain a more accurate idea of the differences in shape and size of normal and *sph* eggs, measurements were taken of both, and from the figures obtained, information was also gained concerning individual variation within these two groups.

METHODS.

The females were kept in vials with maize meal-treacle food on paper spoons, in a density of 1–3 pairs per vial. Since it was found, in contrast to observations by Warren (1924) and Imai (1935) on *D. melanogaster*, that the age of the mother affects the size of the egg, certainly in *sph* ♀♀, and possibly also in normal, care was taken to collect both types of egg from females of varying age in order to cover the whole range of variation due to this cause. The measurements taken were length and width. Fig. 2 shows how they are defined. The figures are given in units of the scale of the micrometer eye-piece of a Zeiss microscope, one unit corresponding to 14 μ.

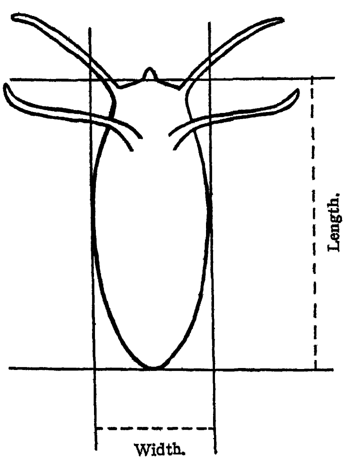


FIG. 2.

From the length l and the width w of the egg were computed (a) the

shape index, $i=\frac{100w}{l}$, (*b*) the volume, $v=\frac{4}{3}\cdot\frac{l}{2}\cdot\frac{w^2}{4}\pi=\cdot5241w^2$. The use of this formula for the volume implies that the shape is that of an ellipsoid. Grossfeld (1933) has shown that in calculating the volume of the fowl's egg in this way the error is negligible. Considering the symmetrical shape of the insect egg, it seems safe to conclude that here the error will be even smaller.

Results.

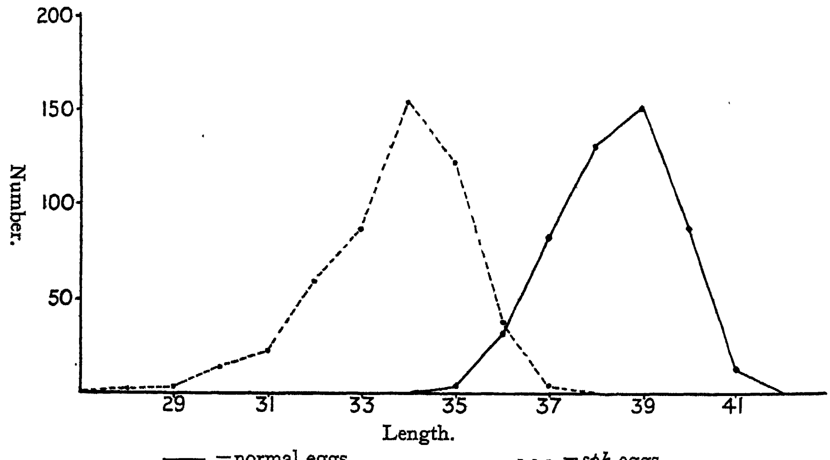
The length and width of 500 normal and 500 *sph* eggs, laid by 10 normal and about 40 *sph* females of varying ages, were measured. The following data were obtained.

Table I.—Length and Width of Normal and Spheroidal Eggs.

M_l=mean length, M_w=mean width, C.V.=coefficient of variability, σ=standard deviation.

	M_l	σ_l	C.V.$_l$	M_w	σ_w	C.V.$_w$
Normal . .	38·42	1·25	3·26	13·81	·56	4·06
Sph. . . .	33·37	1·49	4·42	15·26	·70	4·58

Graph I shows the frequency distribution of length. In either dimension there is very little overlapping. The *sph* eggs are both broader and shorter



Graph I.—Frequency distribution of length in normal and *sph* eggs.

than the normal ones. The differences between the means of normal and *sph* are, for length 5·0 ± ·09; for width − 1·5 ± ·04. Both differences are more than three times their standard error. Expressed in fractions of the means of the normal eggs, they are both about ⅛. As mentioned above, some of the normal eggs may have been laid by heterozygous females.

This possibility would tend to increase the range of variability in the frequency distributions of the normal eggs. The fact, however, that the curves are clearly unimodal indicates that if differences between wild-type and heterozygous *sph* females exist, such differences can only be very slight.

In both dimensions, *sph* has the higher coefficient of variability. The differences between the coefficients are, for length 1·21 ± ·17; for width ·52 ± ·20. Both differences are statistically significant, especially that of the variability of length, which is 7 times its standard error.

It seemed of interest to see whether the amount of variation in the dimensions of the egg was due only to random sampling and to uncontrolled influences of the environment, or whether there exist also significant differences between the performances of individual females belonging to the same stock. In order to determine possible differences between the females used in the present investigations, an analysis of variance was carried out on the length of the eggs of 10 *sph* and 9 wild-type (or heterozygous *sph*) females. The results are presented in Table II, and show that the differences are significant.

TABLE II.—EGG LENGTH IN 9 NORMAL AND 10 *SPH* FEMALES.

Variance.	D.F.	S.S.	M.S.	½ log M.S.	*z*.
Total	814	5755·15			
Between the two groups	1	4265·30			
Within normal females .	491	631·63	1·29	0·1273	1·3316
Between normal females	8	148·17	18·52	1·4589	
Within *sph* females .	305	563·96	1·8	0·2939	1·0995
Between *sph* females .	9	146·09	16·23	1·3934	

(·1 per cent. point for *z* in both cases below ·7.)

The differences between the individuals laying normal eggs may have been somewhat exaggerated by the possible inclusion of heterozygous *sph* females. The existence of such differences, however, is in agreement with the findings of other authors working on *D. melanogaster*. Warren (1924), in a graph, illustrates the differences of egg length between the individuals of two of his stocks without, however, testing their significance. Imai (1935) did not use his data for this purpose, but from his figures it can be shown that significant differences exist between the performances of his five flies reared at 23°.

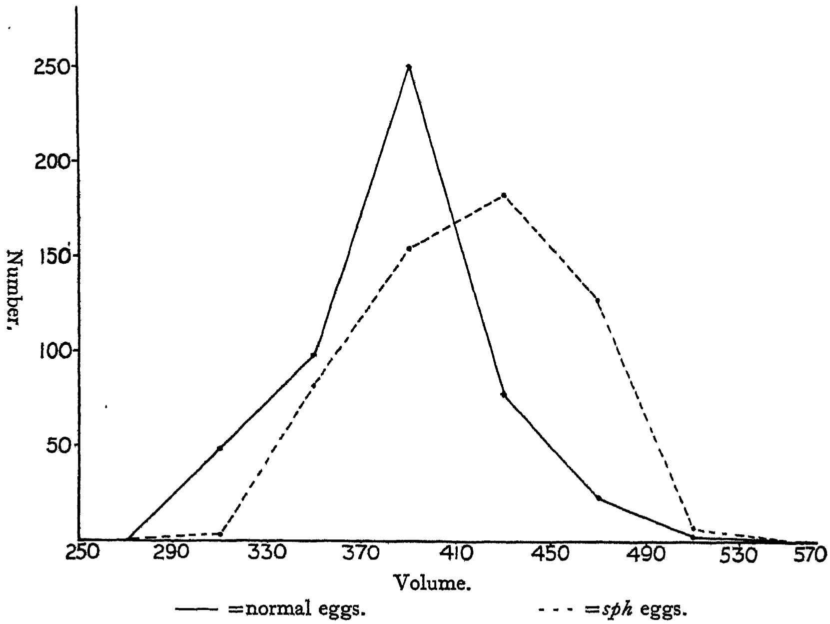
Graphs II and III represent the frequency distributions of volume and shape index respectively of normal and *sph* eggs. The respective means, standard deviations and coefficients of variability are given in Table III.

TABLE III.—VOLUME AND SHAPE INDEX OF NORMAL AND *SPH* EGGS.

(The units for mean and standard deviation of the volume are arbitrary units based on the measurements with the micrometer scale. V = volume, i = shape index.)

	M_v	σ_v	C.V.$_v$	M_i	σ_i	C.V.$_i$
Normal . . .	381	36	9·4	35·9	1·52	4·2
Sph . . .	410	37	9·0	45·3	3·66	8·1

It is seen that the *sph* eggs, though shorter than the normal, are on the average greater in volume. The difference between the mean volumes of *sph* and normal eggs is 29 ± 2·3. The difference between the mean



GRAPH II.—Frequency distribution of volume in normal and *sph* eggs.

shape indices is 9·4 ± ·18, the *sph* eggs showing the higher values. The shape index shows a very significant difference of relative variability in the two stocks. This becomes evident from Table IV, which was carried out in the manner adopted by Fabergé (1936).

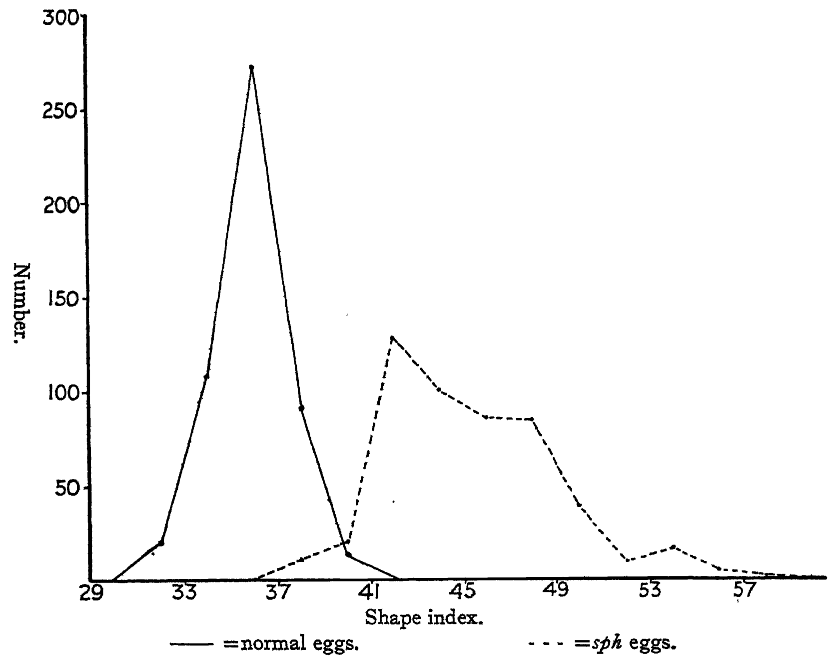
TABLE IV.—RELATIVE VARIABILITY OF SHAPE INDEX IN NORMAL AND *SPH* EGGS.

	D.F.	S.S.	M.S.	Mean².	$\frac{1000\ \text{M.S.}}{\text{Mean}^2}$.	Log.
Normal . .	499	1206	2·4	1288·8	1·9	·6419
Sph . .	499	6598	13·4	2052·1	6·5	1·8718

z = ·62 (·1 per cent. point below ·5.)

In considering the shape index, it has, however, to be noted that this is a quotient of two measured variates, both of which may be assumed to

have an approximately normal distribution. By dividing the values of width by those of length, a certain number of values of the quotient is obtained within the range of variation of the measured variates. These are the possible values of the shape index for the given range of variation of width and length. Owing to the fact that certain values of the index can be obtained by more than one combination of numerator and denominator, these possible values are not distributed evenly over the range of variation of the shape index, but form a unimodal curve, as shown

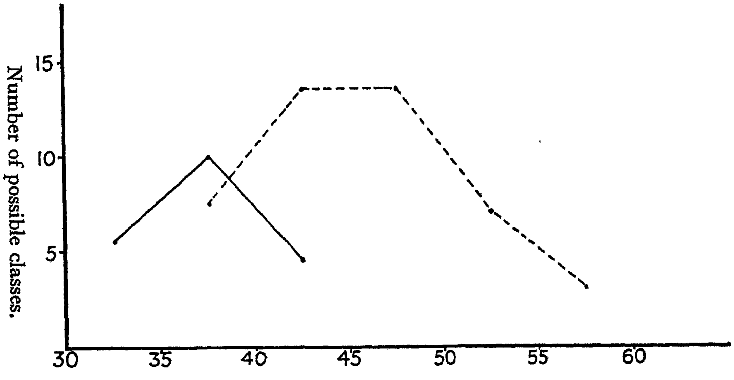


GRAPH III.—Frequency distribution of shape index in normal and *sph* eggs.

in graph IV. Therefore the possibility for any one fly to fall into an interval of the shape index containing the peak of the curve is higher than that of falling into some other interval. To account for this peculiarity, a graph of "weighted" frequency distribution of the shape index was constructed in such a way that the actual frequencies within any one interval of the abscissa were divided by the number of possible classes within this interval (graph V). As shown by comparison of graphs III and V, and also by computation of the coefficients of skewness and excess for both kinds of representation, this process renders the curve for the normal eggs almost normal, whereas that for the *sph* eggs becomes less skew, but at the same time exceedingly platycurtic. An explanation for this feature might be sought for in a possible negative correlation between

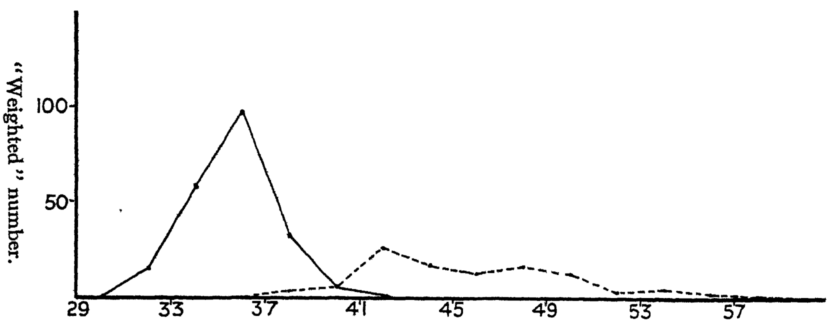
length and width in *sph* eggs, for such a correlation would tend to spread out the distribution of the quotients.

It was, indeed, found that the correlation coefficient between length and width in the *sph* eggs was $-\cdot 4 \pm \cdot 04$, whereas in the normal eggs there



GRAPH IV.—Frequency distribution of the possible classes of shape index in normal and *sph* eggs.

could be shown a positive correlation with the coefficient $\cdot 3 \pm \cdot 04$. Graph VI shows the regression curves of width on length and the converse for *sph* and normal eggs. In the units used in the graph the coefficients of regression of length on width are ·70 for normal and $-\cdot 87$ for *sph* eggs, the



GRAPH V.—Distribution of the "weighted" frequencies of shape index in normal and *sph* eggs.

coefficients of regression of width on length, ·13 and $-\cdot 18$ for normal and *sph* eggs respectively.

The regression curves for *sph* are very nearly straight lines. The deviations from a straight regression line are without significance when

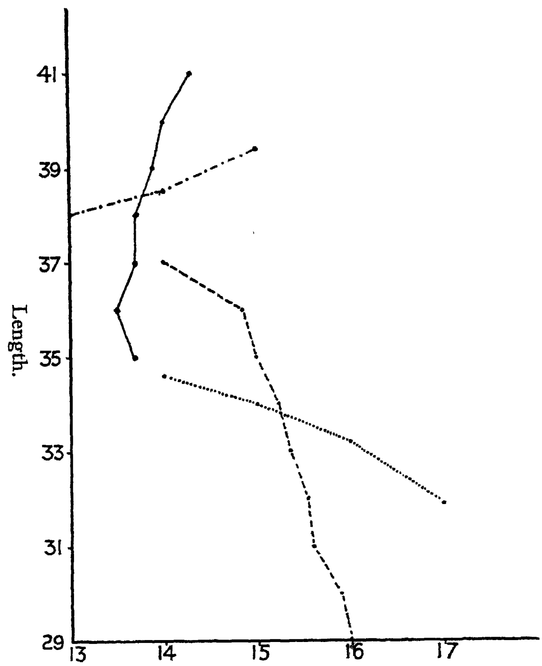
compared with random variation, as shown by Table V for the regression of length on width.

TABLE V.—SIGNIFICANCE OF DEVIATIONS FROM STRAIGHT LINE REGRESSION OF LENGTH ON WIDTH IN *SPH* EGGS.

Variance due to	D.F.	S.S.	M.S.	½ log M.S.
Linear regression . .	1	169·37		
Deviations from it . .	2	5·55	2·78	·5117
Variation within groups of same width	496	915·63	1·9	·3209
Total	499	1090·55		$z = \cdot 19$

(1 per cent. point > ·7.)

The variance due to deviations from a straight regression line of width on length in *sph* eggs was even smaller than that due to random variation



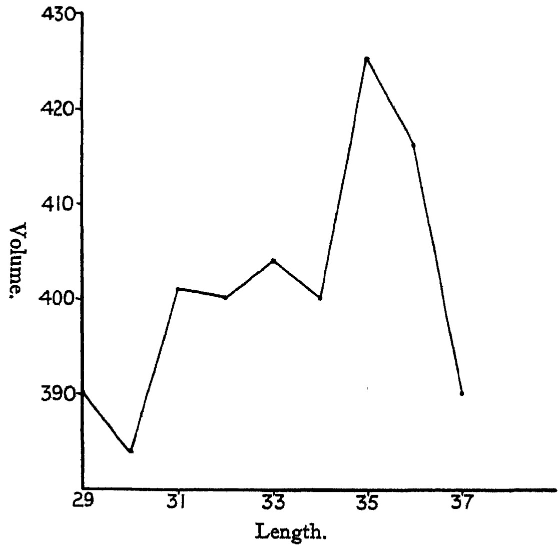
—— =regression of width on length in normal eggs. - - - =regression of width on length in *sph* eggs. —·—·— =regression of length on width in normal eggs. . . . =regression of length on width in *sph* eggs.

GRAPH VI.—Regression of length on width and the converse in normal and *sph* eggs.

within groups of the same length, so there can be no doubt of a straight regression line giving the best fit. The discrepancy at the left end of the curve is due to 3 eggs only, and therefore without much weight.

In spite of the fact that the normal eggs have a positive regression of

width on length, the *sph* a negative one, both types of egg show a positive regression of shape index on volume. The coefficients of correlation are ·5 ± ·03 for normal; ·4 ± ·04 for *sph*. The coefficients of regression, expressed in the units of the graph, are ·69 for normal, ·56 for *sph*. This means that in both types of egg the larger eggs are proportionately broader than the small ones. In *sph* the curve is again approximately



GRAPH VII.—Regression of volume on length in *sph* eggs.

fitted by a straight line, the discrepancy at its left end being due to 6 eggs only. The deviations from a straight regression line are of some significance, but still the chief part of the variance between groups of the same volume is due to linear regression, as shown by Table VI.

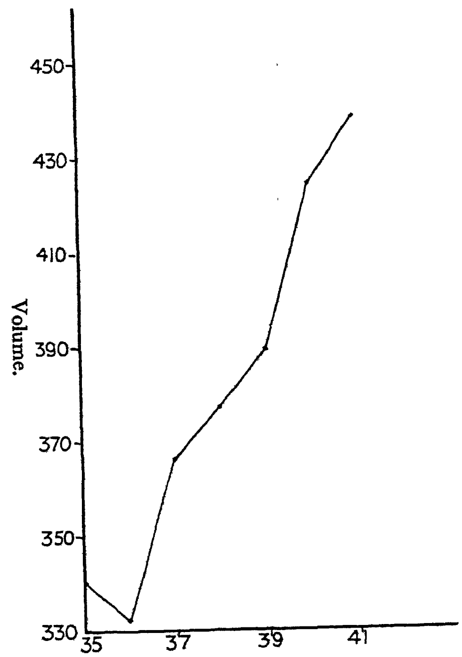
TABLE VI.—REGRESSION OF SHAPE INDEX ON VOLUME IN *SPH* EGGS.

Variance due to	D.F.	S.S.	M.S.	½ log.
Linear regression . . .	1	63·56		
Deviations from it . .	2	12·842	6·421	·9279
Variation within groups of same volume . .	496	341·19	·69	$\bar{1}$·8145
Total	499	417·592		

$z = 1{\cdot}12$ (1 per cent. point below ·8).

In view of the fact that in all studies on the size of the egg of *Drosophila* the length has been taken as a measure of size, it seemed pertinent to examine the degree of correlation between volume and linear dimensions. Actually a positive correlation of only ·67 ± ·02 was found to exist between length and volume of the normal eggs. Correlation between width and

volume is higher, its coefficient having the value of $\cdot 89 \pm \cdot 01$. In the *sph* eggs the coefficient of correlation between width and volume is $\cdot 78 \pm \cdot 02$; that is, almost as high as in normal eggs. On the other hand, the coefficient of correlation between length and volume is only $\cdot 22 \pm \cdot 04$. Hence, in these eggs, length is certainly not a suitable indicator of size. The nature of this surprisingly low correlation becomes clearer through construction of the regression curve (graph VII). This curve appears to consist of five distant sections. The first section comprises the small number—18 out of 500—of extremely short eggs; these have the lowest volume. Then follows a very marked rise as the eggs reach the length of 31. Between the lengths 31–34 no correlation between length and volume seems to exist. This group contains the bulk of the *sph* eggs, 320 out of 500. There is a second rapid rise in volume when the eggs attain the length 35 (121 eggs). After this, the curve shows a definite negative regression of volume on length. It is noticeable that in the otherwise positive regression curve of the normal eggs (graph VIII), negative regression occurs in respect of the shortest eggs, which in their length show overlapping with the *sph* eggs (graph I). However, as the volume of the eggs with length 35 represents the average of only 4 eggs, not much weight can be attributed to this observation.



GRAPH VIII.—Regression of volume on length in normal eggs.

FECUNDITY OF *SPH* FEMALES.

The fecundity of *sph* females is very low. Whereas a normal female produces well over 1000 eggs, the total output of *sph* females ranges mostly between 30 and 50. Also the curve of fecundity differs very much in the two stocks. Whereas a normal female gradually reaches a maximum, then exhibits progressively lower fecundity, a *sph* female usually starts with her maximum performance and then follows a more or less

rapid decrease in fecundity which soon is reduced to a daily performance of one or two eggs, interrupted by days without any eggs at all.

As the studies concerning the fecundity and fertility of *sph* are still in progress, the present paper includes only the tabulated results on the total and average daily fecundity of 4 normal and 4 *sph* females each, kept under standardised conditions, the normal females being observed until their death, the *sph* ones for three weeks, when their laying period can be stated with certainty to have passed (Table VII).

TABLE VII.—SAMPLE OF LAYING PERFORMANCES OF NORMAL AND *SPH* FEMALES.

Female.	A_6	A_7	A_9	A_{10}	A_{18}	A_{29}	A_{34}	A_{46}
	Normal				*Sph*			
Total number of eggs laid	1786	2328	1503	1747	49	33	15	51
Length of laying period in days	39	33	23	31	8	6	4	6
Average number of eggs laid per day . .	45·8	70·6	65·3	56·3	6·1	5·5	3·8	8·5
Number of actual laying days. . . .	33	33	23	30	6	4	4	6

DISCUSSION.

This mutation would seem to provide an attractive material for use in studies of the action of the gene. Egg size and shape in the insect are quantitative characters, and lend themselves better to exact treatment than do those of a purely qualitative nature. The *sph* eggs differ markedly from normal in size, and especially in shape, and thus the mutant and the normal can readily be contrasted. The negative regression of length on width, the abnormal curve of regression of volume on length suggest that not only are the processes of development of *sph* eggs fundamentally different from those operating in the development of normal eggs, but also exhibit unusual fluctuations in relation to egg size. Since these processes are of short duration and relatively uncomplicated, it should easily be possible to detect the onset of the deflection from the normal in the development of the *sph* egg and therefore to gain a clear idea of its nature. *Sph* eggs are much more variable than are normal in respect of length and shape, and since it has been noted that they are affected much more readily by a physiological factor, such as the age of the mother, than are normal eggs, it is to be expected that they will also be more susceptible to altered conditions and therefore provide a useful material

for studying the effect upon the expression of a genetical character of varying environmental conditions during development.

The *sph* female is characterised by an extremely low fecundity. It is practically certain that the altered shape of the egg and this low fecundity are pleiotropic effects of the same gene, this modifying some process underlying both. It should be possible to determine whether or not in the insect some kind of connection exists between egg size or shape and fecundity for the reason that both egg shape and fecundity ultimately refer back to the formation of primordia in the apical chambers of the ovarioles. The eggs develop from definite numbers of these primordia enclosed within an egg chamber where one of the cells develops at the expense of the others into the ovum. It has been suggested (Donald and Lamy, 1937) that fecundity depends upon the rate at which the primordia are formed in the apical chambers and upon the rate at which they develop into mature eggs. The recognition of the *sph* egg has provided material which can be satisfactorily used in a study of the possible parallelism between egg shape and fecundity, and of the details of egg formation and egg laying in *Drosophila*.

Though the process of egg formation in the fowl differs fundamentally from that in insects, it is not without interest to compare the results now obtained with those that relate to the fowl. Curtis (1914) found the coefficients of variation in length, width and weight (taken as an indication of volume) of the same order as those obtained in the present study. He also found a slight positive correlation between length and width, but a negative one, significant in about half of his birds, between size and shape index. This finding is confirmed by the studies of Pearl and Curtis (1916) on dwarf eggs. They also found positive correlation between length and width, and negative between width and shape index. Although the dwarf eggs of the fowl compare with the *sph* eggs in that both are variations of shape, the dwarf egg does not differ from the normal egg in the type of correlation between the various parameters. Moreover, the dwarf egg is not a mutation but a variation due to a temporary disturbance in the reproductive processes of the hen. Marble (1930) found a non-linear regression of egg weight on annual production in the fowl, and others (Bennion and Warren, 1933) have shown the existence in the fowl of a correlation between production and egg size.

SUMMARY.

A new mutation in *D. funebris* has been recognised. It is distinguished by extremely low fecundity of the female and abnormal shape of the egg. From the shape of the egg it has been named spheroidal (*sph*).

Sph eggs are broader and shorter than normal and possess rudimentary filaments. In respect of width, length and especially of shape index, *sph* eggs have a significantly greater variability than normal. The correlation between length and width of the egg is positive in normal, negative in *sph* eggs. In both normal and *sph* eggs there exists a positive regression of volume on shape index. Width and volume are highly correlated in both types, but whereas in normal the correlation between volume and length is also positive, and strong, in the *sph* eggs it is very slight, and the curve of regression is extremely irregular.

Normal and *sph* females differ not only in respect of total fecundity, but also in respect of the curve of the laying performance from day to day.

The new mutant provides a suitable material for studies of the time and mode of gene action, and of the processes of egg formation and egg laying in *Drosophila*.

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XX.—Studies in Practical Mathematics. II. The Evaluation of the Latent Roots and Latent Vectors of a Matrix.

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1. Introductory.

In many branches of applied mathematics problems arise which require for their solution a knowledge of the latent roots of a matrix A, sometimes only the root of greatest modulus but often the second and other roots as well, and the corresponding latent vectors. A few examples, among many that might be cited, are problems in the dynamical theory of oscillations, problems of conditioned maxima and minima, problems of correlation between statistical variables, the determination of the principal axes of quadrics, and the solution of differential or other operational equations. It is important, therefore, to have a choice of methods for obtaining latent roots and latent vectors without undue labour, and the object of the present paper is to augment the existing store of such methods.

The present paper may also be regarded as in several respects an extension and a sequel to two earlier papers (Aitken, 1925, 1931); the first of these was concerned with the solution of algebraic equations by the process of forming convergent sequences of quotients of symmetric functions of the coefficients, the second examined the method of matrix-powering as applied to matrices in general and to the companion matrix of an algebraic equation (*i.e.* the matrix in "rational canonical form" for which the equation is characteristic) in particular. In resuming these studies we take the opportunity of acknowledging the stimulus received from some related work of Hotelling (1933, 1936 *a*, *b*) on finding the latent roots and latent vectors of a symmetric matrix by successive operations on an arbitrary vector.

When the matrix A is not symmetric there are two kinds of latent vectors to be considered, row vectors u, satisfying $uA = \lambda u$ (or, in transposed form, $A'u' = \lambda u'$), and column vectors x, satisfying $Ax = \lambda x$. In addition there may be complex roots, or multiple roots real or complex which may also be associated with non-linear elementary divisors of the characteristic determinant. We extend the method of the arbitrary

vector to meet these various complications; and we wish to draw special attention to the processes for obtaining a greatly enhanced convergence of the sequences, not only towards latent roots, but also towards latent vectors. These processes are so effective that it is usually better not to spend time in carrying out a first sequence to a long series of terms, but to stop at a comparatively early stage and apply the accelerating methods. (Other procedures, such as preliminary matrix-squaring, or alteration of diagonal elements, will be suggested as occasion arises.) Under appropriate treatment a few crudely convergent first terms may be made to yield valuable information not merely with respect to the greatest root, but also with respect to the smaller roots and the vectors associated with them.

The methods here described have been tested on varied numerical material, and the conclusion seems to be that, given the service of a good machine, the evaluation of latent roots and latent vectors of matrices as ordinarily encountered need not be regarded as formidable, except in the genuinely troublesome case of roots of not equal, but *almost* equal moduli. The simplest cases are, fortunately, common in practice, namely, those in which A is symmetric (or Hermitian) and often positive definite as well; in such cases the roots are real, or real and positive, and the elementary divisors are linear, so that the complications of § 5 below do not arise; in addition the property of symmetry has its usual effect of halving the labour required.

The examples of §§ 2, 4, 5, and 6 involve simpler elements in the matrices than would normally occur in practice. This is intentional; the examples have been specially constructed to show within a short sequence the different types of convergence at the stage of becoming perceptible. In these sections also a close determination of latent roots and latent vectors to within a prescribed range of error has not been attempted; that has been reserved for the later sections devoted to the more powerful processes.

PART I.

2. The Method of Repeated Transformation of an Arbitrary Vector.

Let A be a real square matrix of order n, which for the present is not considered to be symmetric or in any way specialized. We shall denote the transposed matrix by A', the latent roots, taken always in descending order of absolute magnitude, by $\lambda_1, \lambda_2, \ldots, \lambda_n$, the corresponding latent row vectors by $u_1, u_2, \ldots, u_n$, the latent column vectors by $x_1, x_2, \ldots, x_n$,

and arbitrary vectors of order n by v, or $v_1, v_2, \ldots, v_p$, and w. Let us take an arbitrary real vector v and premultiply it by A in the usual row-by-column way. Then it may happen (but this is very unlikely) that the vector Av is an exact multiple of v, so that $Av = \lambda v$. This would indicate that λ was a latent root of A, not necessarily the greatest, and that v was its latent column vector. In such a case we should record λ and v for future reference, and should then discard v in favour of some other vector w for which $Aw \neq \mu w$.

It is actually inadvisable to begin operations with a vector v which produces at the first step a vector Av *too nearly* proportional to v, for if the factor of approximate proportionality is not the greatest root λ_1, there will be a stage of indecision during which the influence of the smaller root is dying away while as yet the dominance of the greater is not fully asserted.

Choosing, then, a suitable vector v, we premultiply it by A and repeat this operation, thus obtaining the vector sequence v, Av, $A(Av) = A^2v$, $A(A^2v) = A^3v$, and so on, until corresponding elements of successive vectors in

$$\ldots, \quad A^tv, \quad A^{t+1}v, \quad A^{t+2}v, \quad \ldots$$

are seen to show either (i) an approach, possibly crude and ill-defined, to a geometric progression; or (ii) a persistent fluctuation in sign and magnitude, indicating that the two greatest roots are a conjugate complex pair. In forming this first vector sequence (a convenient and rapid process by machine) we shall not approximate by suppressing or drastically forcing the later digits of elements, however unreliable or even misleading these may seem in the earlier stages. On the contrary we shall record and use all elements to at least as many digits as we propose to retain in the final results, for in these apparently idle digits resides the material for greatly improved approximation, awaiting extraction by the methods of §§ 7, 8. Lastly, if A is of order n, it is better to have $t > n$; that is, to perform the equivalent of more than n operations on v before using the sequence to determine λ_1.

Convergence may be inordinately slow. In this case, if the tests for non-linearity of elementary divisors (§ 5) do not point to this as the cause, it may be advisable to subject A to matrix-squaring a few times, obtaining, for example, A^2, A^4, A^8 or even A^{16}, then to operate on v a few times with A^8 or A^{16}, and finally to operate with A itself as many more times as seem necessary. (Howland, 1928; Aitken, 1931; Hotelling, 1936.) A further point is the necessity for the adequate checking of A^tv at each stage; for this purpose the check described in § 6, based on the identity $(A'v)'A^tv = v'A^{t+1}v$, will be found rapid and efficient.

The examples given below show: (i) a normal and moderately rapid approach to a geometrical progression in corresponding elements;

(ii) fluctuations in sign and magnitude due to cosine factors, indicating a pair of complex roots; (iii) a slow approach to a geometrical progression, due to a double root with a quadratic elementary divisor; (iv) fluctuations, superficially indistinguishable from those of (ii), caused by a double pair of complex roots, with quadratic elementary divisors.

Example 1.

$$A=\begin{bmatrix}2&3&2\\4&3&5\\3&2&9\end{bmatrix},\qquad A'=\begin{bmatrix}2&4&3\\3&3&2\\2&5&9\end{bmatrix},\qquad v=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

The vector sequence $A^t v$, checked at each stage as described in § 6, begins as follows:—

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}7\\12\\14\end{bmatrix}\begin{bmatrix}78\\134\\171\end{bmatrix}\begin{bmatrix}900\\1569\\2041\end{bmatrix}\begin{bmatrix}10589\\18512\\24207\end{bmatrix}\begin{bmatrix}125128\\218927\\286654\end{bmatrix}\begin{bmatrix}1480345\\2590563\\3393124\end{bmatrix}.$$

The vector sequence $(A')^t v$ is likewise:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}9\\8\\16\end{bmatrix}\begin{bmatrix}98\\83\\202\end{bmatrix}\begin{bmatrix}1134\\947\\2429\end{bmatrix}\begin{bmatrix}13343\\11101\\28864\end{bmatrix}\begin{bmatrix}157682\\131060\\341967\end{bmatrix}\begin{bmatrix}1865505\\1550160\\4048367\end{bmatrix}.$$

The latent roots of A (and of A') are actually

$$\lambda_1=11{\cdot}8353625,\qquad \lambda_2=3{\cdot}029287,\qquad \lambda_3=-0{\cdot}864649.$$

The ratios of the elements of the last vector to those of the last but one in the two sequences above give good approximations to λ_1, namely:

$$\begin{aligned}&1480345/125128=11{\cdot}8306, && 1865505/157682=11{\cdot}8308,\\ &2590563/218927=11{\cdot}8330, && 1550160/131060=11{\cdot}8279,\\ &3393124/286654=11{\cdot}8370, && 4048367/341967=11{\cdot}8385.\end{aligned}$$

Example 2.

$$A=\begin{bmatrix}3&1&-2\\-3&-2&2\\4&-2&2\end{bmatrix},\qquad A'=\begin{bmatrix}3&-3&4\\1&-2&-2\\-2&2&2\end{bmatrix},\qquad v=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

After the first five terms the vector sequence $A^t v$ continues thus:

$$\begin{bmatrix}253\\-336\\-1218\end{bmatrix}\begin{bmatrix}2859\\-2523\\-752\end{bmatrix}\begin{bmatrix}7558\\-5035\\14978\end{bmatrix}\begin{bmatrix}-12317\\17352\\70258\end{bmatrix}\begin{bmatrix}-160115\\142763\\56544\end{bmatrix}\begin{bmatrix}-450670\\307907\\-812898\end{bmatrix}.$$

The corresponding terms of the sequence $(A')^t v$ are:

$$\begin{bmatrix}-1589\\-230\\518\end{bmatrix}\begin{bmatrix}-2005\\-2165\\3754\end{bmatrix}\begin{bmatrix}15496\\-5183\\7188\end{bmatrix}\begin{bmatrix}90789\\11486\\-26982\end{bmatrix}\begin{bmatrix}129981\\121781\\-212570\end{bmatrix}\begin{bmatrix}-825680\\311559\\-441540\end{bmatrix}.$$

In these no approach to a geometrical progression is visible, only a general but irregular increase in magnitude of the elements. The latent

roots are actually λ_1 and $\lambda_2 = 1{\cdot}974617 \pm 3{\cdot}293878i$, $\lambda_3 = -0{\cdot}949234$, the two of greatest modulus being conjugate complex.

Example 3.

$$A = \begin{bmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & \cdot \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{bmatrix}, \quad A' = \begin{bmatrix} 6 & 4 & 4 & 4 \\ -3 & 2 & -2 & 2 \\ 4 & 4 & 3 & 3 \\ 1 & \cdot & 1 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The sequence $A^t v$ begins thus:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 8 \\ 10 \\ 6 \\ 10 \end{bmatrix} \begin{bmatrix} 52 \\ 76 \\ 40 \\ 80 \end{bmatrix} \begin{bmatrix} 324 \\ 520 \\ 256 \\ 560 \end{bmatrix} \begin{bmatrix} 1968 \\ 3360 \\ 1584 \\ 3664 \end{bmatrix} \begin{bmatrix} 11728 \\ 20928 \\ 9568 \\ 23008 \end{bmatrix} \begin{bmatrix} 68864 \\ 127040 \\ 56768 \\ 140480 \end{bmatrix}.$$

The corresponding vectors $(A')^t v$ are:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 18 \\ -1 \\ 14 \\ 3 \end{bmatrix} \begin{bmatrix} 172 \\ -78 \\ 119 \\ 35 \end{bmatrix} \begin{bmatrix} 1336 \\ -840 \\ 838 \\ 326 \end{bmatrix} \begin{bmatrix} 9312 \\ -6712 \\ 5476 \\ 2498 \end{bmatrix} \begin{bmatrix} 60920 \\ -47316 \\ 34322 \\ 17286 \end{bmatrix} \begin{bmatrix} 382688 \\ -311464 \\ 209240 \\ 112528 \end{bmatrix}.$$

The latent roots are actually $\lambda_1 = \lambda_2 = 5{\cdot}236068$, $\lambda_3 = \lambda_4 = 0{\cdot}763932$, and these equal pairs are each associated with a quadratic elementary divisor. The three ratios derived from the leading elements of the last four vectors in the sequences are respectively:

$$\begin{aligned} 1968/324 &= 6{\cdot}0741, & 9312/1336 &= 6{\cdot}9701, \\ 11728/1968 &= 5{\cdot}9593, & 60920/9312 &= 6{\cdot}5421, \\ 68864/11728 &= 5{\cdot}8718, & 382688/60920 &= 6{\cdot}2818. \end{aligned}$$

The ratios of other elements show the same slowness of convergence towards $\lambda_1 = 5{\cdot}236068$.

Example 4.

$$A = \begin{bmatrix} 15 & 11 & 6 & -9 & -15 \\ 1 & 3 & 9 & -3 & -8 \\ 7 & 6 & 6 & -3 & -11 \\ 7 & 7 & 5 & -3 & -11 \\ 17 & 12 & 5 & -10 & -16 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

After the first three terms, the sequence $A^t v$ continues thus:

$$\begin{bmatrix} -178 \\ -130 \\ -154 \\ -169 \\ -178 \end{bmatrix} \begin{bmatrix} -833 \\ -23 \\ -485 \\ -461 \\ -818 \end{bmatrix} \begin{bmatrix} 761 \\ 2660 \\ 1502 \\ 1964 \\ 836 \end{bmatrix} \begin{bmatrix} 19471 \\ 9679 \\ 15211 \\ 16369 \\ 19351 \end{bmatrix} \begin{bmatrix} 52214 \\ -18508 \\ 23669 \\ 18137 \\ 49904 \end{bmatrix} \begin{bmatrix} -190157 \\ -243932 \\ -206891 \\ -249068 \\ -195947 \end{bmatrix} \begin{bmatrix} -1596136 \\ -469192 \\ -1133416 \\ -1170457 \\ -1568476 \end{bmatrix}.$$

The outward features, a general increase accompanied by fluctuations in sign and magnitude, are the same as in Example 2. It requires the methods of § 5 to discover in what respect, namely, the nature of the elementary divisors, the cases are not comparable.

3. Classification of the Types of Vector Sequence.

An arbitrary column vector v of n elements may be expressed linearly and uniquely in terms of the latent vectors of A, thus,

$$v = a_1x_1 + a_2x_2 + \ldots + a_nx_n, \qquad (1)$$

where the a_j are scalar constants. In particular cases some of the a_j may be zero, but in practice this will be unlikely to occur, especially since the latent roots will usually be irrational while in practice the chosen arbitrary vector will almost always be rational.

The coefficients a_j can be expressed simply by using the fact that latent row vectors are orthogonal to latent column vectors—that is, $u_ix_j = 0$, $i \neq j$. For by premultiplying (1) by u_j we have

$$u_jv = a_ju_jx_j = a_j,$$

provided that the vectors are normalized so that $u_ix_i = 1$.

After t premultiplications by A (or their equivalent, by matrix-squaring and short-cutting for a time with A^8 or A^{16}) we shall therefore obtain the vector

$$A^tv = a_1\lambda_1{}^tx_1 + a_2\lambda_2{}^tx_2 + \ldots + a_n\lambda_n{}^tx_n. \qquad (2)$$

The elements of A^tv are evidently polynomial in the latent roots of A, but the precise form of the polynomial will depend on the canonical nature of A. The various cases must be reviewed and distinguished.

Case I.—Roots real, single or multiple, elementary divisors linear.

In this case A is reducible to canonical form $H\Lambda H^{-1}$, where H is non-singular, its columns being the latent column vectors of A, and Λ is a purely diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its diagonal elements. Any element of A^tv, that is of $H\Lambda^tH^{-1}v$, is therefore linear in $\lambda_1{}^t, \lambda_2{}^t, \ldots, \lambda_n{}^t$, and so is of the form

$$f(t) = c_1\lambda_1{}^t + c_2\lambda_2{}^t + \ldots + c_n\lambda_n{}^t. \qquad (3)$$

If equalities exist among the roots, naturally the number of independent terms in (3) will be reduced, but its form will remain the same. For example, if $\lambda_1 = \lambda_2 = \ldots = \lambda_k$, we shall have

$$f(t) = c_1\lambda_1{}^t + c_{k+1}\lambda_{k+1}^t + \ldots + c_n\lambda_n{}^t. \qquad (4)$$

Case II.—Conjugate complex roots, possibly multiple, linear elementary divisors.

Suppose that a pair of roots λ_k, λ_{k+1} are complex conjugates $r_ke^{i\theta_k}$, $r_ke^{-i\theta_k}$. Then the terms $c_k\lambda_k{}^t + c_{k+1}\lambda_{k+1}^t$ of $f(t)$ in (3) of Case I take the form

$$r_k{}^t(c_k \cos t\theta_k + c_{k+1} \sin t\theta_k) \quad \text{or} \quad c_kr_k{}^t \cos (t\theta_k + \epsilon_k), \qquad (5)$$

where ϵ_k is some angle, independent of t, which we shall later eliminate.

Case III.—Multiple real roots, non-linear elementary divisors.

Suppose again that λ is a real root of multiplicity m, corresponding to one or more elementary divisors, the highest exponent of these being p. Then the canonical matrix Λ contains submatrices $\lambda I + U$ (Turnbull and Aitken, 1932, pp. 60–63) as below, of highest order p, and so Λ^t contains the corresponding $(\lambda I + U)^t$, thus:

$$\lambda I + U = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}, \quad (\lambda I + U)^t = \begin{bmatrix} \lambda^t & t\lambda^{t-1} & t_{(2)}\lambda^{t-2} & \cdots & t_{(p-1)}\lambda^{t-p+1} \\ & \lambda^t & t\lambda^{t-1} & \cdots & t_{(p-2)}\lambda^{t-p+2} \\ & & \lambda^t & \cdots & t_{(p-3)}\lambda^{t-p+3} \\ & & & \ddots & \vdots \\ & & & & \lambda^t \end{bmatrix}. \quad (6)$$

Here U is the "auxiliary unit matrix," with units in the first superdiagonal and zeros in all other places, and $t_{(r)}$ denotes the binomial coefficient often written ${}^tC_r$. It follows (Aitken, 1931, p. 86) on multiplying out $A^t v$, that is $H\Lambda^t H^{-1}v$, that the groups of terms of $f(t)$ which in Case I would be of the form

$$c_k\lambda_k^t + c_{k+1}\lambda_{k+1}^t + \ . \ . \ . \ + c_{k+p-1}\lambda_{k+p-1}^t$$

now assume, on being added together, the following form:—

$$c_k\lambda_k^t + c_{k+1}t\lambda_k^{t-1} + c_{k+2}t_{(2)}\lambda_k^{t-2} + \ . \ . \ . \ + c_{k+p-1}t_{(p-1)}\lambda_k^{t-p+1}, \quad . \quad (7)$$

where the intrusive binomial coefficients are the distinguishing feature.

Case IV.—Multiple complex pairs of roots, non-linear elementary divisors.

If in Case III instead of a real root λ we have a complex pair $re^{i\theta}$, $re^{-i\theta}$, each member of the pair being multiple and of maximum exponent p of elementary divisors, we find by Case III that the groups of terms which in Case I would be

$$c_k\lambda_k^t + c_{k+1}\lambda_{k+1}^t + \ . \ . \ . \ + c_{k+2p-1}\lambda_{k+2p-1}^t$$

and the like can be merged together in the form

$$c_k r_k^t \cos(t\theta_k + \epsilon_k) + c_{k+1}t r_k^{t-1}\cos(\overline{t-1}\,\theta_k + \epsilon_{k+1}) + \ . \ . \ .$$
$$+ c_{k+p-1}t_{(p-1)}r_k^{t-p+1}\cos(\overline{t-p+1}\,\theta_k + \epsilon_{k+p-1}), \quad (8)$$

where the angles ϵ_k, ϵ_{k+1}, . . . are independent of t.

It is of course possible to have terms which conform to all or any of Cases I, II, III and IV, occurring at the same time in $f(t)$.

One further sub-case of less importance may be mentioned, that of *zero roots*, multiple, with non-linear elementary divisors. This case offers no practical difficulty, for the rank of the corresponding canonical submatrix in Λ (which is of the form U) is reduced by 1 whenever t is

increased by 1. Consequently if, as suggested in § 2, we proceed until $t > n$, all canonical submatrices in Λ for zero latent roots will have been reduced to nullity; hence the form of $f(t)$ will be unaffected by them.

It is now necessary to consider Cases I, II, III and IV in order, examining the behaviour, as t increases, of the vector sequence $A^t v$.

4. The Case of Linear Elementary Divisors.

Case I.—

$$\text{(i)} \ |\lambda_1| > |\lambda_2|; \qquad \text{(ii)} \ \lambda_1 = \lambda_2 = \ldots = \lambda_k, \quad |\lambda_k| > |\lambda_{k+1}|.$$

(i) Assuming $c_1 \neq 0$, we have, by § 3 (3),

$$f(t+1) - \lambda_1 f(t) = c_2(\lambda_2 - \lambda_1)\lambda_2^t + c_3(\lambda_3 - \lambda_1)\lambda_3^t + \ldots + c_n(\lambda_n - \lambda_1)\lambda_n^t. \qquad (1)$$

It follows readily that

$$f(t+1) - \lambda_1 f(t) = O\,|\lambda_2|^t \qquad (2)$$

and that

$$f(t+1)/f(t) = \lambda_1 + O\,|\lambda_2/\lambda_1|^t \qquad (3)$$

$$= \lambda_1 + \frac{c_2}{c_1}(\lambda_2 - \lambda_1)(\lambda_2/\lambda_1)^t + O\,|\lambda_3/\lambda_1|^t + O\,|\lambda_2/\lambda_1|^{2t}. \qquad (4)$$

Hence by increasing t we can make the second term on the right-hand side of (3) less than any assigned deviation from λ_1. The rapidity of convergence to λ_1 thus depends ultimately on the magnitude of $|\lambda_2/\lambda_1|$; even if this (which we shall find means of estimating) should be as large as $\frac{1}{2}$ or even more, a dominance of λ_1^t sufficient for the purpose is soon effected. If λ_1 is a k-fold root, convergence will depend similarly on $|\lambda_{k+1}/\lambda_k|$, provided $c_k \neq 0$.

If (as is possible but unlikely) the first non-vanishing coefficient in § 3 (3) is c_k, and if $|\lambda_k| > |\lambda_{k+1}|$, $c_{k+1} \neq 0$, naturally $f(t+1)/f(t)$ will tend not to λ_1 but to λ_k, with rapidity depending ultimately on the magnitude of $|\lambda_{k+1}/\lambda_k|$. Again, if the elementary divisors for any root smaller than λ_1 are non-linear, this will not affect the rapidity of convergence of $f(t+1)/f(t)$ towards λ_1.

To summarize this case, the distinctive requisites are the separation of the root or multiple roots of greatest magnitude from the next greatest root or roots, and the linearity of the elementary divisors corresponding to the two greatest roots or sets of roots.

It will next be shown that in this case the vector $A^t v$ converges (provided $c_1 \neq 0$) to the latent vector x_1, in the sense that the ratios of its elements converge to the ratios of those of x_1. (The elements of a latent vector are to be regarded as homogeneous co-ordinates; their ratios are fixed, but the vector itself is indefinite to the extent of an arbitrary multiplier.) For by § 3 (4) we have

$$\begin{aligned} A^t v &= \lambda_1{}^t[a_1 x_1 + a_2(\lambda_2/\lambda_1)^t x_2 + \ \ldots \ + a_n(\lambda_n/\lambda_1)^t x_n] \\ &= a_1\lambda_1{}^t[x_1 + \{O \mid \lambda_2/\lambda_1 \mid^t\} x_2] \quad . \quad . \quad . \quad . \quad . \quad . \quad (5) \\ &= a_1\lambda_1{}^t[x_1 + a_2(\lambda_2/\lambda_1)^t x_2 + \{O \mid \lambda_3/\lambda_1 \mid^t\} x_3], \quad . \quad . \quad . \quad (6) \end{aligned}$$

provided $a_1 \neq 0$. Since $\mid \lambda_1 \mid > \mid \lambda_2 \mid$, it follows at once that $A^t v$ is tending to x_1, with scalar factor $a_1\lambda_1{}^t$.

If λ_1 is k-fold, and $\mid \lambda_1 \mid > \mid \lambda_{k+1} \mid$, a similar result is obtained.

If λ_1 is multiple we obtain only one of the latent vectors corresponding to λ_1. The others may be found either by the method of "deflation" of A (§ 9) or possibly by taking further arbitrary vectors v_2, v_3, . . . and testing afterwards the linear independence of the latent vectors obtained from $A^t v_2$, $A^t v_3$, . . .

The vectors $A^t v$, $A^{t+1} v$, $A^{t+2} v$, . . . may be brought to standard form, with a view to their comparison, either by division of each by its leading element or some other assigned element, or else by normalizing them so that the sum of the squares of their elements is unity. Taking the first course with § 2, Example 1, and dividing the last three vectors in the sequence $A^t v$ each by its leading element, we obtain

$$\begin{bmatrix} 1\cdot0000 \\ 1\cdot7482 \\ 2\cdot2861 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000 \\ 1\cdot7496 \\ 2\cdot2909 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000 \\ 1\cdot7500 \\ 2\cdot2921 \end{bmatrix}, \qquad x_1 = \begin{bmatrix} 1\cdot0000 \\ 1\cdot7501 \\ 2\cdot2925 \end{bmatrix},$$

the actual latent vector x_1 being shown to four places for comparison. The corresponding vectors in the sequence $(A')^t v$ are

$$\begin{bmatrix} 1\cdot0000 \\ 0\cdot8320 \\ 2\cdot1632 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000 \\ 0\cdot8312 \\ 2\cdot1687 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000 \\ 0\cdot8310 \\ 2\cdot1701 \end{bmatrix}, \qquad u'_1 = \begin{bmatrix} 1\cdot0000 \\ 0\cdot8309 \\ 2\cdot1706 \end{bmatrix}.$$

The discrimination of multiple latent roots will be taken up later in § 7.

Case II.—Suppose

$$\lambda_1 = r_1 e^{i\theta_1}, \qquad \lambda_2 = r_1 e^{-i\theta_1}, \qquad \mid \lambda_1 \mid > \mid \lambda_3 \mid.$$

In this case, as we have seen, the terms of $f(t)$ involving λ_1 are of the form given in § 3 (5), and the presence of the factor $\cos(t\theta_1 + \epsilon_1)$ causes fluctuations of $f(t)$ as t increases. We may, however, determine r_1 by the method of § 7 below, and thence (*cf.* Aitken, 1925, p. 295) determine $\cos \theta_1$ by the approximate elimination of t and ϵ_1 from three corresponding elements in consecutive vectors as follows:—

Let two operations δ_r and μ_r be defined by

$$\delta_r f(t) = \tfrac{1}{2}\{rf(t-1) - r^{-1}f(t+1)\}, \qquad \mu_r f(t) = \tfrac{1}{2}\{rf(t-1) + r^{-1}f(t+1)\}. \quad (7)$$

These operators, which obey the ordinary laws of algebra, are a slight extension of the mean central differencing and the central averaging operators of the Calculus of Finite Differences.

Then, except always for terms $O\,|\,\lambda_3/\lambda_1\,|^t$, we have

$$\cos\theta_1 = \mu_r f(t)/f(t), \qquad r = r_1, \quad . \quad . \quad . \quad . \qquad (8)$$

since

$$\cos(\overline{t+1}\theta_1+\epsilon_1)+\cos(\overline{t-1}\theta_1+\epsilon_1) = 2\cos(t\theta+\epsilon_1)\cos\theta_1.$$

As an illustration, take § 2, Example 2. The leading elements in the last five vectors A^tv are

$$2859, \qquad 7558, \qquad -12317, \qquad -160115, \qquad -450670.$$

By the method of k-ary products of roots (§ 7) we may find

$$r_1^2 = 14{\cdot}74874. \qquad \text{Hence } r_1 = 3{\cdot}84041, \quad r_1^{-1} = 0{\cdot}260389.$$

The five leading elements given above yield three values of $\mu_r f(t)$, namely, 3886·3, $-6333{\cdot}2$, and -82326. Hence for $\mu_r f(t)/f(t)$ we have the values

$$3886{\cdot}3/7558 = 0{\cdot}51420,$$
$$6333{\cdot}2/12317 = 0{\cdot}51418,$$
$$82326/160115 = 0{\cdot}51417.$$

Hence, approximately,

$$\cos\theta_1 = 0{\cdot}5142, \quad \sin\theta_1 = 0{\cdot}8577, \qquad \text{roots } 1{\cdot}9747 \pm 3{\cdot}2939i.$$

Also, since the sum of the latent roots is the trace (sum of diagonal elements) of A, namely 3, the third latent root is $3 - 3{\cdot}9494 = -0{\cdot}9494$.

(The actual values are 0·514168, 0·857689, roots $1{\cdot}974617 \pm 3{\cdot}293878i$, $-0{\cdot}949234$.)

It will next be shown that the latent vector A^tv tends to the *real part* of x_1, or rather of x_1 multiplied by a certain complex factor. Let $x_1 = x + iy$. Then, apart from terms of order $|\,\lambda_3/\lambda_1\,|^t$, we have

$$\begin{aligned} A^tv &= a_1 r_1^t e^{it\theta_1}(x+iy) + a_2 r_1^t e^{-it\theta_1}(x-iy) \\ &= (a_1+a_2) r_1^t (x\cos t\theta_1 - y\sin t\theta_1), \end{aligned}$$

since A^tv is real; and this is the real part of $x+iy$ multiplied by $(a_1+a_2)r_1^t e^{it\theta_1}$.

The imaginary part of the latent vector with this multiplier, namely, $i(a_1+a_2)r_1^t(x\sin t\theta_1 + y\cos t\theta_1)$, can be obtained as follows. Perform the operation $\delta_r A^tv$, with $r = r_1$, and divide the resulting vector by $\sin\theta_1$. We obtain

$$\begin{aligned} \delta_r A^tv/\sin\theta_1 &= (r_1 A^{t-1}v - r_1^{-1}A^{t+1}v)/2\sin\theta_1 \\ &= (a_1+a_2)r_1^t(x\sin t\theta_1 + y\cos t\theta_1), \end{aligned}$$

except, as before, for terms of order $|\,\lambda_3/\lambda_1\,|^t$. Thus the complete latent vector is obtained.

To illustrate this we may take again the last four vectors A^tv of § 2, Example 2. Performing the operation $\delta_r A^tv$, with $r = 3{\cdot}84041$, and $\sin\theta = 0{\cdot}85769$, we obtain

$$\delta_r A^t v = \begin{bmatrix} 35359 \\ -28255 \\ 21399 \end{bmatrix} \quad \begin{bmatrix} 35024 \\ -6768\cdot4 \\ 240745 \end{bmatrix},$$

whence

$$\delta_r \dot{A}^t v/\sin\theta = \begin{bmatrix} 41226 \\ -32943 \\ 24950 \end{bmatrix} \quad \begin{bmatrix} 40835 \\ -7891\cdot4 \\ 280690 \end{bmatrix}.$$

Thus, taking the two middle vectors $A^t v$ for the real parts, we obtain the two vectors

$$\begin{bmatrix} -12317+41226i \\ 17352-32943i \\ 70258+24950i \end{bmatrix} \quad \begin{bmatrix} -160115+40835i \\ 142763-7891\cdot4i \\ 56544+280690i \end{bmatrix},$$

and dividing them through by their leading elements for comparison with each other and with the actual latent vector x_1, we obtain

$$\begin{bmatrix} 1\cdot00000 \\ -0\cdot84905-0\cdot16723i \\ 0\cdot08817-1\cdot73056i \end{bmatrix} \quad \begin{bmatrix} 1\cdot00000 \\ -0\cdot84898-0\cdot16723i \\ 0\cdot08821-1\cdot73056i \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1\cdot00000 \\ -0\cdot84899-0\cdot16722i \\ 0\cdot08820-1\cdot73055i \end{bmatrix}.$$

It appears from the investigation of this section that the determination of the modulus, the amplitude and the latent vector corresponding to the greatest root, when it is complex and perhaps multiple, presents no special difficulty, provided that the root is sufficiently separated in magnitude from the next greatest root or roots, and that when it is multiple the elementary divisors corresponding to it are linear.

5. The Case of Non-linear Elementary Divisors.

The presence of non-linear elementary divisors of the characteristic determinant $|A-\lambda I|$ introduces less tractable features, which quickly force themselves upon the notice of the computer.

Let us suppose that by the means of discrimination developed in § 7 it has been ascertained that the greatest root is real and also multiple. For illustration, let us suppose that it is a double root, $\lambda_1=\lambda_2$, and that we have been able to determine $\lambda_1\lambda_2$, and therefore λ_1, with sufficient accuracy, the sign being fixed by other considerations. Then the problem is: by means of the value of λ_1 to analyse further the vector sequence $A^t v$ (which in this case, as we have already seen, is poorly convergent), and by this analysis to obtain (i) verification that λ_1 actually is the greatest root, (ii) information concerning p, the maximum exponent of elementary divisors.

Case III.—From § 3 (7) we know that the terms involving λ_1 in the elements of $A^t v$ are of the form

$$f(t) = c_1\lambda_1{}^t + c_2 t\lambda_1{}^{t-1} + c_3 t_{(r)}\lambda_1{}^{t-2} + \ldots + c_p t_{(p-1)}\lambda_1{}^{t-p+1}. \qquad (1)$$

If $|\lambda_1| > |\lambda_{p+1}|$, the ratio $f(t+1)/f(t)$ tends to λ_1, but not rapidly enough

for practical needs. Let us introduce the operation of λ-*differencing* with respect to t, defined by

$$\Delta_\lambda f(t) = f(t+1) - \lambda f(t). \qquad (\lambda = \lambda_1). \qquad (2)$$

Then from (1) we have (*cf.* Aitken, 1931, p. 87)

$$\Delta_\lambda^{p-1} f(t) = c_p \lambda_1^{t-p+1}, \qquad \Delta_\lambda^p f(t) = 0, \qquad (3)$$

except for terms of smaller order, as usual. This process of differencing, therefore, gives an adequate test for the multiplicity of λ_1, and a means of determining at the same time the value of p. In fact we construct a λ_1-difference table from corresponding elements $f(t)$, $f(t+1)$, . . .; the $(p-1)^{\text{th}}$ differences should show an approximate geometrical progression of common ratio λ_1, and the p^{th} differences should be negligible.

Let us take the leading elements in the vector sequence $A^t v$ in § 2, Example 3, the value of λ_1 having been found by § 7 to be 5·236068. The table of λ_1-differences is as follows:—

$f(t)$.	Δ_λ.	Δ_λ^2.	$\Delta_\lambda(\lambda=\lambda_3)$.
1			
	2·76		
8		−4·34	
	10·11		2·10
52		−1·22	
	51·72		1·63
324		0·70	
	271·51		1·25
1968		1·78	
	1423·42		0·91
11728		2·27	
	7455·39		
68864			

The second λ_1-differences are tending to become negligible compared with the first λ_1-differences. We conclude that $p=2$. At the same time the first λ_1-differences appear to be tending towards a geometrical progression of common ratio λ_1.

$\Delta_\lambda f(t+1)/\Delta_\lambda f(t)$.	$f(t+1)/f(t)$.
271·51/51·72 = 5·2496,	1968/324 = 6·0741,
1423·42/271·51 = 5·2426,	11728/1968 = 5·9593,
7455·39/1423·42 = 5·2377,	68864/11728 = 5·8718.

We give their ratios above; and on the right we have placed, for comparison, the much more slowly convergent ratios of the undifferenced $f(t)$.

The second λ_1-differences in the above table will eventually tend very slowly towards a ratio given by $\lambda_3 = 0·763932$. Performing λ_3-differencing upon them in their turn (compare § 9, Example 3), as we have shown in the column added to the table given above, we obtain a geometric progression of common ratio tending to $\lambda_3 : 1$. The inference (§ 9) is that λ_3 is also a double root and corresponds to an elementary divisor of exponent 2.

The operation of λ-differencing can also be applied to the vectors *as a whole*, the $(p-1)^{\text{th}}$ λ_1-difference of $A^t v$ yielding a sequence tending with geometric rapidity towards x_1. For since each element of $A^t v$ is of the form $f(t)$ indicated by § 3 (1), it follows that $A^t v$ itself may be expressed as a linear combination of vectors in that form, as follows:—

$$A^t v = \lambda_1{}^t v_1 + t\lambda_1{}^{t-1} v_2 + \ldots + t_{(p-1)}\lambda_1{}^{t-p+1} v_p + O\mid \lambda_{m+1}/\lambda_1 \mid^t. \quad . \quad (4)$$

Hence

$$\Delta_\lambda{}^{p-1} A^t v = \lambda_1{}^{t-p+1} v_p,$$
$$\Delta_\lambda{}^{p-1} A^{t+1} v \equiv A \,.\, \Delta_\lambda{}^{p-1} A^t v = \lambda_1 \Delta_\lambda{}^{p-1} A^t v, \quad . \quad . \quad . \quad (5)$$

except for terms of small order. It follows that $\Delta_\lambda{}^{p-1} A^t v$ tends to a latent vector of A corresponding to λ_1.

As an illustration, let us take the last four vectors of the sequence for A in § 2, Example 3, and construct from them three vectors of first λ_1-differences, given that $\lambda_1 = 5{\cdot}236068$. We obtain the vectors below:

$$\begin{bmatrix} 271{\cdot}51 \\ 637{\cdot}24 \\ 243{\cdot}57 \\ 731{\cdot}80 \end{bmatrix} \quad \begin{bmatrix} 1423{\cdot}42 \\ 3334{\cdot}81 \\ 1274{\cdot}07 \\ 3823{\cdot}05 \end{bmatrix}. \quad \begin{bmatrix} 7455{\cdot}4 \\ 17459{\cdot}6 \\ 6669{\cdot}3 \\ 20008{\cdot}5 \end{bmatrix}.$$

Dividing these by their leading elements, we obtain

$$\begin{bmatrix} 1{\cdot}0000 \\ 2{\cdot}3470 \\ 0{\cdot}8971 \\ 2{\cdot}6953 \end{bmatrix} \quad \begin{bmatrix} 1{\cdot}0000 \\ 2{\cdot}3428 \\ 0{\cdot}8951 \\ 2{\cdot}6858 \end{bmatrix} \quad \begin{bmatrix} 1{\cdot}0000 \\ 2{\cdot}3419 \\ 0{\cdot}8946 \\ 2{\cdot}6838 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1{\cdot}0000 \\ 2{\cdot}3417 \\ 0{\cdot}8944 \\ 2{\cdot}6833 \end{bmatrix}, \quad A^6 v = \begin{bmatrix} 1{\cdot}0000 \\ 1{\cdot}8448 \\ 0{\cdot}8245 \\ 2{\cdot}0400 \end{bmatrix},$$

the actual latent vector x_1 being shown for comparison, as well as the last vector of the sequence $A^t v$ used, which is still very remote from x_1.

Case IV.—Let us suppose finally that the greatest root is one of a complex pair, each member of which is of multiplicity m and has non-linear elementary divisors of maximum exponent p. By the method of § 7 (evaluation of k-ary products of latent roots) the modulus r_1 can be found. The problem is therefore to determine the amplitude, and the value of p.

In this case the terms of $f(t)$, apart from those of order $\mid \lambda_{2m+1}/\lambda_1 \mid^t$, constitute the polynomial given in § 3 (8). Now this polynomial is the solution of the difference equation (*e.g.* Milne-Thomson, 1933, p. 387)

$$(r_1{}^2 - 2r_1 \cos\theta_1 \,.\, \mathrm{E} + \mathrm{E}^2)^p f(t) = 0, \quad . \quad . \quad . \quad (6)$$

where E is the operator of the calculus of Finite Differences defined by $\mathrm{E}f(t) = f(t+1)$. We may write this in the form

$$(\mu_r - \cos\theta)^p f(t+p) = 0, \quad . \quad . \quad . \quad . \quad (7)$$

and when this is expanded we may regard the result as an equation of degree p in $\cos\theta$, with coefficients linear in the elements $f(t)$. There

appears to be no alternative but to solve this equation by one or other of the accepted methods, and to solve corresponding equations for the cases where the operands in (6) are $f(t), f(t+1), f(t+2), \ldots$, in order to judge the convergence of the solutions towards $\cos\theta$. Fortunately (see next paragraph) p, which cannot exceed half the order of A, will not usually be large.

Further, we must determine the minimal value p for which these solutions tend to agreement. This is found as follows. The condition of consistency of $p+1$ equations of the form (7) above, the operands involving consecutive values of t, is that the determinant of the coefficients of $\cos^p\theta$, $p\cos^{p-1}\theta$, $p_{(2)}\cos^{p-2}\theta, \ldots, p\cos\theta$, 1 in those equations must vanish, apart from terms of order $|\lambda_{2m+1}/\lambda_1|^t$; and if p is minimal, no determinant of like form but of lower order can tend to vanish, in this sense. Let us call the rank of this matrix of coefficients, as obtained by neglecting the terms of smaller order, the *asymptotic rank* of the matrix as $t\to\infty$. Then the criterion is as follows:—

The asymptotic rank of the matrix (transposed for convenience)

$$T = \begin{bmatrix} f(t) & f(t+1) & f(t+2) & \ldots\ldots \\ \mu_r f(t) & \mu_r f(t+1) & \mu_r f(t+2) & \ldots\ldots \\ \mu_r^2 f(t) & \mu_r^2 f(t+1) & \mu_r^2 f(t+2) & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \quad . \quad . \quad (8)$$

is the maximal exponent p *of elementary divisors for* λ_1.

In practice the procedure will be: construct the matrix T by operating with $\mu_r, \mu_r^2, \ldots$, on corresponding elements in the vector sequence, and condense it (as in methods for evaluation determinants) by Chiò's or any similar method (Whittaker and Robinson, 1929, pp. 70–72; Aitken, 1937, pp. 172–181) until minors of small order relatively to their predecessors appear simultaneously. This will be at the p^{th} stage of condensation. The most convenient procedure will be to compute minors, at each stage of condensation, which involve *consecutive* rows and have in common the column farthest to the right (for greatest accuracy), and to stop at the stage, the $(p-1)^{\text{th}}$, when the rows of minors obtained show approximate proportionality.

Below is shown the relevant part of T, as constructed from the leading elements of vectors $A^t v$ in § 2, Example 4, with $r_1 = 3{\cdot}87298$. The elements of the right-hand column are italicized; all the minors of order 2 obtained at the first stage of condensation contain elements from this column. Inspection of these two rows of minors will show the proportionality to which we have referred; in this case the factor of proportionality, which is shown in the added row of quotients of minors, is

tending to $\cos^2\theta_1$. (In the general case of minors from $p-1$ consecutive rows of T, the exponent of elementary divisors being p, this factor can be proved by (10), *infra*, to tend towards $\cos^p\theta_1$.)

f	7	−178	−833	761	19471	52214	−190157
$\mu_r f$	−7·4878	−93·984	−246·45	900·60	8214·5	13156·2	−104948
$\mu_r^2 f$	−6·6333	−46·317	−65·732	583·24	3442·5	2358·6	−52769
I.		$-21585^{(7)}$	$40557^{(8)}$	$91390^{(8)}$	$-48140^{(9)}$	$-29780^{(10)}$	
		$-30103^{(6)}$	$61065^{(7)}$	$13686^{(8)}$	$-72187^{(8)}$	$-44671^{(9)}$	
Quotients		0·13946	0·15057	0·14976	0·14995	0·15000	

(The raised index after the entries denotes the number of significant digits before the decimal point.)

Now taking the equation (for $p=2$)

$$(\mu_r^2 - 2\mu_r \cos\theta + \cos^2\theta) f(t) = 0$$

and using the values of $f(t)$, $\mu_r f(t)$, $\mu_r^2 f(t)$ given in the last three columns of T above, we have three quadratic equations in $\cos\theta_1$:

$$\begin{aligned} 19471z^2 - 16429z + 3442{\cdot}5 &= 0, &\quad \text{giving } z &= 0{\cdot}38747. \\ 52214z^2 - 26312{\cdot}4z + 2358{\cdot}6 &= 0, &\quad \ldots\ z &= 0{\cdot}38730. \\ 190157z^2 - 209896z + 5276{\cdot}9 &= 0, &\quad \ldots\ z &= 0{\cdot}38730. \end{aligned}$$

We conclude that, approximately,

$$\cos\theta_1 = 0{\cdot}3873, \qquad \sin\theta_1 = 0{\cdot}9220, \qquad r_1 = 3{\cdot}8730,$$
$$\lambda_1 = \lambda_2 = 1{\cdot}5000 + 3{\cdot}571i, \qquad \lambda_3 = \lambda_4 = 1{\cdot}5000 - 3{\cdot}571i,$$

and $p=2$ for each of λ_1 and λ_3. Also, from the trace of A,

$$\lambda_5 = 5 - 6{\cdot}0000 = -1{\cdot}0000.$$

(The actual values are 0·387298, 0·921954, 3·87298, $1{\cdot}5 \pm 3{\cdot}57071i$, -1.)

Finally, from the knowledge of r_1, $\cos\theta_1$, and p we may determine the latent vector x_1, as the analogy of Case III suggests. For since the elements $f(t)$ in the present case satisfy the difference equation (6) above, which may be written in terms of the operator μ_r as

$$(\mu_r - \cos\theta_1)^p f(t) = 0, \qquad \qquad (9)$$

it follows that

$$F(t) \equiv (\mu_r - \cos\theta_1)^{p-1} f(t) \qquad \qquad (10)$$

satisfies the difference equation

$$(\mu_r - \cos\theta) F(t) = 0. \qquad \qquad (11)$$

But this is the difference equation satisfied by the $f(t)$ of Case II, the case of a pair of complex conjugate roots with linear elementary divisors. Hence, just as $p-1$ operations of λ-differencing reduce sequences belonging to Case III to the tractable form presented by real roots with linear elementary divisors, so $p-1$ operations $\mu_r - \cos\theta_1$ upon vectors $A^t v$

conforming to Case IV will produce a vector sequence which can be treated exactly as in Case II of § 4.

To illustrate this, let us take the last six vectors of § 2, Example 4, and use them to construct four vectors of the form $(\mu_r - \cos\theta_1)A^t v$, given (§ 7) that $r_1 = 3\cdot87298$, $r_1^{-1} = 0\cdot258199$, $\cos\theta = 0\cdot387298$, $\sin\theta = 0\cdot921954$. These four vectors (which are used again, with more significant digits, in § 9, Example 5) are:

$$\begin{bmatrix} 605\cdot86 \\ 174\cdot80 \\ 442\cdot81 \\ 459\cdot85 \\ 590\cdot37 \end{bmatrix} \quad \begin{bmatrix} 673\cdot39 \\ -986\cdot97 \\ 73\cdot07 \\ -194\cdot94 \\ 566\cdot88 \end{bmatrix} \quad \begin{bmatrix} -7066\cdot15 \\ -5580\cdot10 \\ -6420\cdot53 \\ -7480\cdot57 \\ -7151\cdot36 \end{bmatrix} \quad \begin{bmatrix} -31301\cdot0 \\ -1938\cdot6 \\ -20360\cdot2 \\ -19519\cdot8 \\ -29960\cdot9 \end{bmatrix}.$$

Forming the imaginary parts, exactly as in Case II, by the further operations $\delta_r F(t)/\sin\theta$, and dividing the complete vectors through by their leading elements, for comparison with each other and with the actual latent vector x_1, we obtain

$$\begin{bmatrix} 1\cdot00000 \\ 0\cdot34709 + 0\cdot53965i \\ 0\cdot75164 + 0\cdot19145i \\ 0\cdot79405 + 0\cdot32256i \\ 0\cdot97876 + 0\cdot04076i \end{bmatrix} \quad \begin{bmatrix} 1\cdot00000 \\ 0\cdot34696 + 0\cdot53962i \\ 0\cdot75157 + 0\cdot19143i \\ 0\cdot79400 + 0\cdot32256i \\ 0\cdot97866 + 0\cdot04071i \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1\cdot00000 \\ 0\cdot34698 + 0\cdot53959i \\ 0\cdot75158 + 0\cdot19142i \\ 0\cdot79401 + 0\cdot32255i \\ 0\cdot97868 + 0\cdot04069i \end{bmatrix}.$$

The remaining latent vectors for multiple complex roots can be found either by the method of "deflation" of A (§ 9) or by choosing new arbitrary vectors v_2, v_3, . . . and testing afterwards the linear independence of the latent vectors obtained by their use.

PART II.

6. The Construction and Use of Sequences of Bilinear Forms.

We pass now to methods of more rapid approximation, specially valuable when the matrix A is symmetric or Hermitian.

Let us suppose that we have constructed a number of terms of a vector sequence $A^t v$, and also of the sequence $(A')^t v$, the second sequence being of course the same as the first if A is symmetric, and the complex conjugate of the first if A is Hermitian. Let us next form the sum of products of corresponding elements in $A^t v$ and $(A')^t v$. This gives the scalar or inner product of the vectors, $[(A')^t v]' A^t v$ or $v' A^{2t} v$, a bilinear form. In the same way the sum of products of corresponding elements of $(A')^{t+1} v$ and $A^t v$ gives $v' A^{2t+1} v$, which is also equal to the sum of products of corresponding elements of $A^{t+1} v$ and $(A')^t v$. These sums of products

are very easily computed. If A is symmetric they are quadratic forms, sums of squares of elements in $A^t v$, and sums of products of corresponding elements in adjacent vectors $A^t v$ and $A^{t+1}v$. (Or, equally well, $A^{t-1}v$ and $A^{t+2}v$, and so on, which might be used in practice as a check.) Further, considering the canonical form of A, and writing $v'A^{2t}v$ as $v'H\Lambda^{2t}H^{-1}v$, we see that these bilinear (or quadratic) forms are polynomials of the form $f(2t)$ and $f(2t_{+1})$, where $f(t)$ falls under Cases I, II, III and IV of § 3. Hence, for the purpose of determining latent roots (but not latent vectors) the vector sequences $A^t v$ and $(A')^t v$ can be used in this way to provide a convenient sequence of scalars at a more advanced stage of convergence, equivalent in fact to twice as many operations with A or A' as were used in forming the vectors. This is so advantageous in the case of *symmetric* matrices that the elements of the vectors themselves would not be used in the manner of the earlier sections of this paper for finding latent roots; the sums of squares and products of their elements would be formed at once and used instead.

Even in the unsymmetric case a short sequence of $(A')^t v$, which does not take long to compute, can be used with advantage to form scalar products with a longer sequence $A^t v$. It is often easier to form these scalars than to carry the vectors $A^t v$ a corresponding distance farther. In any case $A'v$ should always be found, and used, if v is $\{1, 1, \ldots, 1\}$, in the following excellent check on $A^{t+1}v$, namely: *the sum of the elements of $A^{t+1}v$ is equal to the sum of products of corresponding elements of $A'v$ and $A^t v$. This brief and convenient check should be applied at every stage in forming $A^t v$.*

Example 1.—The last three vectors in the sequences $A^t v$ and $(A')^t v$ of § 2, Example 1, give us five sums of products, their ratios being:

$$123738 73^{(11)}/10455016^{(10)} = 11{\cdot}835346,$$
$$14644921^{(12)}/12373873^{(11)} = 11{\cdot}835357,$$
$$17332794^{(13)}/14644921^{(12)} = 11{\cdot}835362,$$
$$20513989^{(14)}/17332794^{(13)} = 11{\cdot}835362.$$

Recalling that $\lambda_1 = 11{\cdot}8353625$, we may observe the distinct improvement upon the results of § 3.

For a second example we shall take a symmetric matrix.

Example 2.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 5 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The sequence begins thus:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} \begin{bmatrix} 25 \\ 34 \\ 63 \end{bmatrix} \begin{bmatrix} 167 \\ 237 \\ 433 \end{bmatrix} \begin{bmatrix} 1130 \\ 1647 \\ 2973 \end{bmatrix} \begin{bmatrix} 7689 \\ 11404 \\ 20419 \end{bmatrix} \begin{bmatrix} 52501 \\ 78765 \\ 140281 \end{bmatrix}.$$

The sums of squares and products for the last three vectors give five scalars, their ratios being:

$$
\begin{aligned}
88176645/12828238 &= 6{\cdot}87364,\\
606107498/88176645 &= 6{\cdot}87379,\\
4166313988/606107498 &= 6{\cdot}87389,\\
28639039187/4166313988 &= 6{\cdot}87395.
\end{aligned}
$$

The actual value of λ_1 is 6·87408, the approximation in so short a sequence being good, considering (p. 293) that $|\lambda_2/\lambda_1|$ is about 2/3.

It is evident that since these sums of products are of the form $f(2t)$ and $f(2t_{+1})$ they will have exactly the same properties and use for determining complex roots and multiple roots with non-linear elementary divisors as the elements $f(t)$ of $A^t v$ had. This being so, it is unnecessary to cover similar ground, and we shall merely give one more example involving complex roots.

Example 3.—Taking the last three vectors of the sequences $A^t v$ and $(A')^t v$ of § 2, Example 2, we have five sums of corresponding products

$$-28146444^{(10)},\quad -14422575^{(11)},\quad -15445645^{(11)},\quad 15171641^{(12)},\quad 82696739^{(12)}.$$

Taking them in threes and performing $\mu_r f(t)/f(t)$ with $r=3{\cdot}840409$, we obtain

$$
\begin{aligned}
7415631/14422575 &= 0{\cdot}5141683,\\
7941655/15445645 &= 0{\cdot}5141679,\\
7800779/15171641 &= 0{\cdot}5141684.
\end{aligned}
$$

These give approximations to $\cos\theta_1$, the actual value of which (compare § 4) is 0·51416828.

7. The Evaluation of the Product of the k Greatest Roots.

In the nature of the results obtained, the work of the first half of this section resembles the corresponding part of a former paper of the author (Aitken, 1925, pp. 290–292) concerning the solution of algebraic equations, but the methods of proof employed are more direct.

(i) We require the "theorem of multiplication of determinantal arrays" extended to more than two arrays. Denote the determinant formed from the rows $\alpha_1, \alpha_2, \ldots, \alpha_k$, columns $\beta_1, \beta_2, \ldots, \beta_k$ of an array or matrix T by

$$T\begin{pmatrix}\alpha_1 & \alpha_2 & \ldots & \alpha_k\\ \beta_1 & \beta_2 & \ldots & \beta_k\end{pmatrix}.$$

Then by successive applications of the classical theorem for two arrays we have the general theorem of this kind:

If

$$T = MN \ldots RS,$$

where M, N, . . ., R, S are matrices of order $k\times m$, $m\times n$, . . ., $r\times s$, $s\times k$ respectively (so that T is of order $k\times k$), then the determinant

$$|T|=\sum_{\alpha}\sum_{\beta}\;.\;.\;\sum_{\rho}M\begin{pmatrix}1\;2\;.\;.\;k\\ \alpha_1\alpha_2\;.\;.\;\alpha_k\end{pmatrix}N\begin{pmatrix}\alpha_1,\,\alpha_2,\,.\;.,\,\alpha_k\\ \beta_1,\,\beta_2,\,.\;.,\,\beta_k\end{pmatrix}\;.\;.\;S\begin{pmatrix}\rho_1\rho_2\;.\;.\;\rho_k\\ 1\;2\;.\;.\;k\end{pmatrix},\qquad(1)$$

where capital letters followed by bracketed indices denote determinants formed from the corresponding rows (upper indices) and columns (lower indices) of the respective matrices, and where the summations are over all sets of k columns taken from the columns of M, N, . . ., R independently.

By means of this theorem we may deduce the value of a persymmetric determinant of order k (Aitken, 1925, p. 291) with elements of the form $f(t)$.

Theorem.—Let

$$f(t)=c_1\lambda_1^{\,t}+c_2\lambda_2^{\,t}+\;.\;.\;.\;+c_n\lambda_n^{\,t},\qquad \lambda_1\neq\lambda_2\neq\;.\;.\;.\;\neq\lambda_n.$$

Let

$$f_k(t)=\begin{vmatrix}f(t+k-1) & f(t+k-2) & \cdots & f(t)\\ f(t+k-2) & f(t+k-3) & \cdots & f(t-1)\\ \vdots & \vdots & & \\ f(t) & f(t-1) & \cdots & f(t-k+1)\end{vmatrix}.$$

Then

$$f_k(t)=\Sigma c_\alpha c_\beta\;.\;.\;.\;c_\kappa(\lambda_\alpha\lambda_\beta\;.\;.\;.\;\lambda_\kappa)^{t-k+1}\{\Delta(\lambda_\alpha,\,\lambda_\beta,\,.\;.\;.,\,\lambda_\kappa)\}^2,\qquad(2)$$

where the summation is over all sets of k different indices α, β, . . ., κ taken from 1, 2, . . ., n and where $\Delta(\lambda_1,\,\lambda_2,\,.\;.\;.,\,\lambda_\kappa)$ denotes the difference-product of λ_α, λ_β, . . ., λ_κ.

Proof.—By inspection the matrix $[f_k(t)]$ is equal to the product of three matrices $K'CL$, where

$$K'=\begin{bmatrix}\lambda_1^{\,t} & \lambda_2^{\,t} & \cdots & \lambda_n^{\,t}\\ \lambda_1^{\,t-1} & \lambda_2^{\,t-1} & \cdots & \lambda_n^{\,t-1}\\ \vdots & \vdots & & \vdots\\ \lambda_1^{\,t-k+1} & \lambda_2^{\,t-k+1} & \cdots & \lambda_n^{\,t-k+1}\end{bmatrix},\quad C=\begin{bmatrix}c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n\end{bmatrix},\quad L=\begin{bmatrix}\lambda_1^{\,k-1} & \lambda_1^{\,k-2} & \cdots & 1\\ \lambda_2^{\,k-1} & \lambda_2^{\,k-2} & \cdots & 1\\ \vdots & \vdots & & \vdots\\ \lambda_n^{\,k-1} & \lambda_n^{\,k-2} & \cdots & 1\end{bmatrix}.\qquad(3)$$

Using the theorem on the determinant of a product of matrices, and observing that, since all k-line minors of C vanish other than diagonal ones, the double summation of that theorem reduces a single summation, we see that $f_k(t)$ is equal to the sum of products of determinants taken from sets of k columns α, β, . . ., κ of K' and the k corresponding rows of L, with factors taken from the corresponding diagonal elements c_α, c_β, . . ., c_κ of C. These determinants, apart from factors $\lambda_\alpha^{\,t-k+1}$ and so

on from columns of K', are of simple alternant type and so are difference-products of the k-ary sets $\lambda_\alpha, \lambda_\beta, \ldots, \lambda_\kappa$. The theorem is thus established.

The corresponding theorem for the *confluent* case of coalescing arguments (Aitken, 1925, p. 292) is not so simply enunciated. The matrices K' and L become *confluent* alternants, with differentiated columns and rows (Muir, 1923, pp. 178, 201), and C is no longer purely diagonal. On the right-hand side products appear as well as squares, the coefficients of the products, minors of C, are more complicated, and while some of the alternants are ordinary difference-products and others are *confluent* difference-products (Turnbull and Aitken, pp. 60, 63), others again, being minors taken from *non-consecutive* columns or rows in the suites of differentiated columns or rows for each repeated root, are of anomalous nature, and cannot be called "difference-products" under any legitimate extension of the term.

Theorem.—Let

$$\lambda_1=\lambda_2=\ldots=\lambda_p, \qquad \lambda_{p+1}=\lambda_{p+2}=\ldots=\lambda_{p+q-1}, \qquad \lambda_1\neq\lambda_{p+1},$$

and so on. Let

$$f(t)=c_1\lambda_1{}^t+c_2t\lambda_1{}^{t-1}+\ldots+c_pt_{(p-1)}\lambda_1{}^{t-p+1}$$
$$+c_{p+1}\lambda_{p+1}^t+c_{p+2}t\lambda_{p+1}^{t-1}+\ldots+c_{p+q-1}t_{(q-1)}\lambda_{p+1}^{t-q+1}+\ldots,$$

and let $f_k(t)$, $k\geqslant p$, $k\geqslant q$, $\ldots$, be a persymmetric determinant of order k, constructed as in the previous theorem. Then $f_k(t)$ is equal to

$$\sum\sum C\begin{pmatrix}\alpha' & \beta' & \ldots & \kappa' \\ \alpha & \beta & \ldots & \kappa\end{pmatrix}(\lambda_\alpha\lambda_\beta\ldots\lambda_\kappa)^{t-k+1-d}K'(\alpha',\beta',\ldots,\kappa')L(\alpha,\beta,\ldots,\kappa), \tag{4}$$

where K' and L denote confluent alternant matrices (partitioned to show the suites of differentiated columns or rows) and C is as indicated below. (For convenience in printing we have taken the special case of $t=4$, $k=3$, λ_1 triple, λ_4 double, λ_6 single, for illustration.) The form of C, with triangular persymmetric submatrices placed in isolation across the diagonal, will be noticed. The summation is over all sets of k columns of K' and k rows of L, the products $\lambda_\alpha\lambda_\beta\ldots\lambda_\kappa$ are chosen from $(\lambda_1, \lambda_1, \ldots \lambda_1, \lambda_{p+1}, \lambda_{p+1}, \ldots\lambda_{p+1}, \ldots \lambda_n)$, and if the columns $\alpha, \beta, \ldots, \kappa$ contain any differentiated ones, the index $t-k+1$ has a defect d which it is not important to specify.

$$\left[\begin{array}{ccc:cc:c} \lambda_1{}^4 & 4\lambda_1{}^3 & 6\lambda_1{}^2 & \lambda_4{}^4 & 4\lambda_4{}^3 & \lambda_6{}^4 \\ \lambda_1{}^3 & 3\lambda_1{}^2 & 3\lambda_1 & \lambda_4{}^3 & 3\lambda_4{}^2 & \lambda_6{}^3 \\ \lambda_1{}^2 & 2\lambda_1 & 1 & \lambda_4{}^2 & 2\lambda_4 & \lambda_6{}^2 \end{array}\right] \left[\begin{array}{cccccc} c_1 & c_2 & c_3 & & & \\ c_2 & c_3 & \cdot & & & \\ c_3 & \cdot & \cdot & & & \\ & & & c_4 & c_5 & \\ & & & c_5 & \cdot & \\ & & & & & c_6 \end{array}\right] \left[\begin{array}{ccc} \lambda_1{}^2 & \lambda_1 & 1 \\ 2\lambda_1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \hline \lambda_4{}^2 & \lambda_4 & 1 \\ 2\lambda_4 & 1 & \cdot \\ \hline \lambda_6{}^2 & \lambda_6 & 1 \end{array}\right]. \tag{5}$$

Proof.—The matrix $[f_k(t)]$ is again equal to the product $K'CL$, as follows at once by multiplying out, observing the isolation of the submatrices of C, and using the simplest properties of binomial coefficients, namely, Vandermonde's identity:

$$(t+r)_{(s)} = t_{(s)} + rt_{(s-1)} + r_{(2)}t_{(s-2)} + \ . \ . \ . \ + r_{(s)}.$$

For example, the product in (5) above gives

$$\begin{bmatrix} f(6) & f(5) & f(4) \\ f(5) & f(4) & f(3) \\ f(4) & f(3) & f(2) \end{bmatrix},$$

where

$$f(t) = c_1\lambda_1{}^t + c_2 t\lambda_1{}^{t-1} + c_3 t_{(2)}\lambda_1{}^{t-2} + c_4\lambda_4{}^t + c_5 t\lambda_4{}^{t-1} + c_6\lambda_6{}^t,$$

as is readily verified.

The theorem now follows by applying the fundamental theorem concerning the determinant of a product of matrices.

The important feature for our purpose, both in the non-confluent and the confluent cases, is that $f_k(t)$ is polynomial in the k-ary products of the λ's. Those coefficients of k-ary products which involve ordinary or confluent difference-products are independent of t; those which involve the anomalous minors change their form when t changes. However, provided $|\lambda_k| > |\lambda_{k+1}|$, the term containing the product of the *k greatest* roots must derive of necessity from the *first k* columns of K' and the *first k* rows of L. The coefficient of $\lambda_1\lambda_2 \ . \ . \ . \ \lambda_k$ in it therefore involves the square of a difference-product (ordinary or confluent as the case may be), and so is independent of t. It follows that

$$f_k(t+1)/f_k(t) = \lambda_1\lambda_2 \ . \ . \ . \ \lambda_k + O\,|\lambda_{k+1}/\lambda_k|^t, \qquad . \quad . \qquad (6)$$

so that by construction of persymmetric determinants $f_k(t)$ from corresponding elements in the successive vectors of the sequence $A^t v$ we can determine products of latent roots, and hence, by division, individual latent roots or powers of roots, or, where the roots are complex, squares or even powers of their moduli. The determinants $f_k(t)$ are easily constructed (Aitken, 1925, p. 293; 1931, p. 80) by making systematic repeated use of the identity:

$$f_{k+1}(t) = [\{f_k(t)\}^2 - f_k(t-1)f_k(t+1)]/f_{k-1}(t). \qquad . \quad . \qquad (7)$$

The theory here, and the practical formulation that emerges, are almost a duplication of the results of the two former papers (1925, 1931) cited. But the fact that we have now a *vector* sequence $A^t v$ and no longer only a scalar sequence leads to the following alternative method for obtaining k-ary products of latent roots, which is equally convenient.

(ii) Let the vectors v, Av, A^2v, . . . of the sequence $A^t v$ be regarded as columns of a matrix V, of n rows but indefinitely many columns.

Then A^tV will be the matrix with columns A^tv, $A^{t+1}v$, $A^{t+2}v$, . . .; that is, the first t columns of V have disappeared, and all the others have moved t places to the left. Now let us consider the kth compound matrices $V^{(k)}$ and $(AV)^{(k)}$. The columns of $V^{(k)}$ contain minors of order k taken from the vector sequence A^tv regarded as an array; the columns of $(A^tV)^{(k)}$ contain minors taken in the same way from vectors t places farther to the right in A^tv. In virtue of the identity of compound matrices

$$(A^tV)^{(k)} = [A^{(k)}]^t V^{(k)} \quad . \quad . \quad . \quad . \quad (8)$$

we may look upon the successive columns of minors of order k formed from the column vectors A^tv as vectors in a vector sequence $[A^{(k)}]^t v^{(k)}$ obtained by repeated operation with the compound matrix $A^{(k)}$. It follows at once, by §§ 3, 4, 5, that the properties of this kth derived sequence will be the same, in respect of the latent roots of $A^{(k)}$, as those of A^tv are in respect of the latent roots of A. But by the theorem of Rados (Muir, 1923, pp. 215, 217) the latent roots of $A^{(k)}$ are the k-ary products of the latent roots of A. Hence by constructing minors from our first sequence A^tv (the best way, illustrated in Example 2 below, will be to construct them successively from consecutive vectors, which is equivalent to the process of Dodgson for evaluating determinants) we may find k-ary products of latent roots of A as quotients of consecutive minors, with the same approximation as in the earlier sections.

The same special cases, under the heads of complex, multiple with non-linear elementary divisors and so on, arise for k-ary products found in this way. It is therefore unnecessary to treat them in detail.

As illustrations, we shall find r_1^2 for § 2, Example 2, using both of the above methods on the last six vectors of the sequence $(A')^tv$.

Example 1.—The leading elements of the vectors are shown in the first row. The second row contains the persymmetric determinants of order 2, each constructed by squaring the entry above it and subtracting the product of entries on either side of that.

$f(t)$	-1589	-2005	15496	90789	129981	-825680
$f_2(t)$		$2864317^{(8}$	$4221580^{(9}$	$6228457^{(10}$	$9185772^{(11}$.	

The four values $f_2(t)$ now give the three ratios:

$$42215800/2864317 = 14{\cdot}73852,$$
$$62284570/4221580 = 14{\cdot}75385,$$
$$91857720/6228457 = 14{\cdot}74807.$$

The actual value of r_1^2 is 14·748744.

It will be found interesting to obtain $\lambda_1\lambda_2$ in this way for § 2, Example 1, as well. The true value is 35·85271.

Example 2.—The first two rows of the vectors are shown, and below them the determinants of order 2 formed from consecutive vectors.

V	-1589	-2005	15496	90789	129981	-825680
	-230	-2165	-5183	11486	121781	311559
$V^{(2)}$	2979035	$4394076^{(8}$	$6485464^{(9}$	$9563413^{(10}$	$14104889^{(11}$.	

The five determinants of $V^{(2)}$ give the four ratios:

$$\begin{aligned}
4394076^{(8}/2979035 &= 14{\cdot}75000,\\
6485464^{(9}/4394076^{(8} &= 14{\cdot}75956,\\
9563413^{(10}/6485464^{(9} &= 14{\cdot}74592,\\
14104889^{(12}/9563413^{(10} &= 14{\cdot}74880.
\end{aligned}$$

It happens in this example that the approximation by this second method is slightly better than by the other; but this is quite accidental, and merely shows the advantage of having alternatives. The methods are almost equivalent in the time and labour required, which is not great in either case.

8. The δ^2-Process for Accelerating Convergence.

All of the sequences used in the preceding sections, whether to obtain latent roots, k-ary products of latent roots or latent vectors, yield sequences of superior convergence when subjected to the following process, which we have used before (Aitken, 1925, p. 300) in a more special application.

We consider the case, common in practice (as when A is symmetric and has different roots), where $|\lambda_1| > |\lambda_2| > |\lambda_3|$, and where λ_3 is real and the corresponding elementary divisor is linear. Let us take § 4 (4) and express λ in terms of the three ratios

$$\phi(t-1) \equiv f(t)/f(t-1), \qquad \phi(t), \qquad \phi(t+1),$$

treating their differences as if in exact geometrical progression. We obtain

$$\begin{vmatrix} \phi(t+1) & \phi(t) \\ \phi(t) & \phi(t-1) \end{vmatrix} \div \{\phi(t+1) - 2\phi(t) + \phi(t-1)\} = \lambda_1 + O\,|\,\lambda_3/\lambda_1\,|^t + O\,|\,\lambda_2/\lambda_1\,|^{2t}. \quad (1)$$

The numerator on the left-hand side is a persymmetric determinant $\phi_2(t)$ of the second order with elements $\phi(t)$; the denominator is the second central difference $\delta^2\phi(t)$; the error committed, $O\,|\,\lambda_3/\lambda_1\,|^t$ or $O\,|\,\lambda_2/\lambda_1\,|^{2t}$, whichever is the greater, is often much less than $O\,|\,\lambda_2/\lambda_1\,|^t$, the error of $\phi(t)$ itself, so that convergence towards λ_1 is improved. We shall denote the process of forming $\phi_2(t)$ and dividing by $\delta^2\phi(t)$ by $P\phi(t)$, and for brevity we shall call it the "δ^2-process." For practical computation it may be remembered by the following *memoria technica: product of*

outers minus square of middle, divided by sum of outers minus double of middle. When the roots are all real (as when A is symmetric) it may be proved that a *second* operation, giving $P^2\phi(t)$, yields a still more convergent sequence, and so on indefinitely; but in most cases the first operation is so effective that there is no need to repeat it.

In practice the operation P admits of the following very useful simplification. In (1) above put $\phi(t)+c$ instead of $\phi(t)$, where c is a constant. We find at once

$$P\{\phi(t)+c\}=c+P\phi(t). \qquad (2)$$

It follows that before applying the process we may remove any convenient constant from $\phi(t-1)$, $\phi(t)$, and $\phi(t+1)$, and the most convenient is, of course, the part constituted by the digits which these ratios possess in common. Thus the δ^2-process can be applied to the later differing digits only. (This very useful property was overlooked in the paper of 1925.)

The process can be applied, for reasons similar to those already given, not merely to the ratios $\phi(t)$, but also to those of the bilinear and quadratic forms (sums of squares and products) used in § 6, the persymmetric determinants used in § 7 for finding k-ary products of latent roots, and the minors of A^tV used there also. The sufficient conditions for successful application are $|\lambda_k| > |\lambda_{k+1}| > |\lambda_{k+2}$, where λ_{k+2} is real and (preferably though not indispensably) associated with linear elementary divisors. When λ_{k+2} is one of a complex pair the process is not available. Modifications appropriate to such a case have been given (Aitken, 1925, p. 302) and still apply here, but to pursue them would carry us too far into detail.

It will be sufficient to give one example, with a few comments.

Example 1.—Let us take the ratios of sums of squares and sums of products of elements of A^tv in § 6, Example 2, and use the δ^2-process to obtain a closer approximation to λ_1. To nine places of decimals the ratios of these sums are as below, the work being arranged in columns explained by the headings.

ϕ.	$\delta\phi$.	$\delta^2\phi$.	ϕ_2.	$\phi_2/\delta^2\phi$.
6·873636504				
	150744			
787248		51756	556667	107557
	98988			
886236		33989	365564	107554
	64999			
951235				

We conclude that $\lambda_1 = 6{\cdot}8740755$ approximately. Since the actual value is 6·87407553, and since the ratios of the differences $\delta\phi(t)$ (as also

of $\delta^2\phi(t)$) show that λ_2/λ_1 is almost 2/3, the power of the method is sufficiently evident.

A few comments and cautions may be made. Since the derived sequence $P\phi(t)$ is more rapidly convergent than the original one, $\phi(t)$, it is quite possible for $P\phi(t)$ to show a poorer approximation to λ_1 in its earlier terms than $\phi(t)$, but to redeem this very quickly by acquiring a much better approximation a few terms later. The computer will not be misled by such delayed action. He will notice when the *first differences* $\delta\phi(t)$ are tending perceptibly to a geometric progression; it is then that the δ^2-process will begin to produce good results. The common ratio of this progression, by § 4 (4), is λ_2/λ_1, and will thus provide a good preliminary estimate of λ_2. Thus in the above example the three first differences $\delta\phi(t)$ gives the two ratios $98988/150744=0{\cdot}65666$ and $64999/98988=0{\cdot}65664$, leaving little doubt that $\lambda_2=6{\cdot}874\times0{\cdot}6566=4{\cdot}5135$ approximately. (Actually it is 4·5135435.) Such knowledge concerning λ_2/λ_1 is very useful; for example if λ_2/λ_1 is negative, the sequence $\phi(t)$ will oscillate (§ 10) about the value λ_1, and this will set boundaries to λ_1 on either side. Similar considerations, with appropriate change of terms, hold for k-ary products of latent roots.

The δ^2-process is also applicable, under the same conditions as before, for constructing from the vector sequence A^tv a derived sequence tending with superior convergence to the latent vector x_1. There are two principal ways of doing this, according as λ_1 has already been determined with accuracy or not.

(i) Taking the less favourable case, let us suppose that we have only a set of consecutive terms of the sequence A^tv. Let each vector be divided through by an assigned element, such as the leading element. Let this element be $f(t)$ and any other element be $g(t)$, where by § 3 (3) we have

$$f(t)=c_1\lambda_1^t+c_2\lambda_2^t+c_3\lambda_3^t+\ldots, \qquad (3)$$

$$g(t)=b_1\lambda_1^t+b_2\lambda_2^t+b_3\lambda_3^t+\ldots \qquad (4)$$

Then the division reduces $f(t)$ to unity and $g(t)$ to $g(t)/f(t)$, where

$$g(t)/f(t)=b_1/c_1+(b_2c_1-b_1c_2)(\lambda_2/\lambda_1)^t/c_1^2+O\mid\lambda_3/\lambda_1\mid^t, \qquad (5)$$

as follows from (3) and (4). Elimination of $(b_2c_1-b_1c_2)/c_1^2$ and λ_2/λ_1 between

$$g(t-1)/f(t-1), \qquad g(t)/f(t), \qquad g(t+1)/f(t+1)$$

now gives

$$P\{g(t)/f(t)\}=b_1/c_1+O\mid\lambda_3/\lambda_1\mid^t+O\mid\lambda_2/\lambda_1\mid^{2t}, \qquad (6)$$

just as in (1). Thus when the δ^2-process is applied to *all* the elements of a vector A^tv we obtain, except for terms of small order, the coefficients of λ_1^t in the polynomial expressions for those elements, divided by c_1. These coefficients, by § 3 (2), constitute the latent vector x_1.

Example 2.—Let us take the last four vectors of the sequence $(A')^t v$ in § 2, Example 1, and divide them through by their leading elements. We obtain the following vectors:—

$$\begin{bmatrix} 1\cdot0000000 \\ 0\cdot8350970 \\ 2\cdot1419753 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8319718 \\ 2\cdot1632317 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8311665 \\ 2\cdot1687130 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8309600 \\ 2\cdot1701185 \end{bmatrix}.$$

Taking the corresponding elements three at a time and applying the δ^2-process, we obtain the following two improved vectors, the actual latent vector x_1 being also shown for comparison:—

$$\begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308870 \\ 2\cdot1706176 \end{bmatrix} \quad \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308888 \\ 2\cdot1706032 \end{bmatrix}, \qquad x_1 = \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308889 \\ 2\cdot1706023 \end{bmatrix}.$$

(*Note.*—The computed latent vector x_1 should always be checked by forming Ax_1 and verifying, for *all* its elements, that it is accurately equal to $\lambda_1 x_1$. This checks both x_1 and λ_1.)

The improvement in the approximation is evident, especially in the second of the two derived vectors; but when λ_1 has been found we can do much better than this.

(ii) Suppose that λ_1 has been determined with sufficient accuracy. Let us make $A^{t-1}v$, $A^t v$ and $A^{t+1}v$ comparable in magnitude by multiplying them respectively by λ_1, 1 and λ_1^{-1} or by any suitable multiples of these, and let us then apply the δ^2-process to their elements. It may be proved, almost exactly as in (6) above, that this gives a vector tending to x_1 with smaller error.

Example 3.—Let us again take the last four vectors $(A')^t v$ of § 2, Example 1, and divide them through respectively by 1, λ_1, λ_1^2, and λ_1^3, where $\lambda_1 = 11\cdot8353625$. We obtain the reduced vectors:

$$\begin{bmatrix} 1134 \\ 947 \\ 2429 \end{bmatrix} \quad \begin{bmatrix} 1127\cdot3841 \\ 937\cdot9518 \\ 2438\cdot7931 \end{bmatrix} \quad \begin{bmatrix} 1125\cdot6905 \\ 935\cdot6362 \\ 2441\cdot2995 \end{bmatrix} \quad \begin{bmatrix} 1125\cdot2570 \\ 935\cdot0435 \\ 2441\cdot9411 \end{bmatrix}.$$

Taking corresponding elements three at a time and applying the δ^2-process, we obtain the two vectors below, which we have divided through by their leading elements (the prefixed factor) for comparison with x_1:

$$1125\cdot1078 \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308891 \\ 2\cdot1706023 \end{bmatrix}, \qquad 1125\cdot1079 \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308888 \\ 2\cdot1706023 \end{bmatrix}, \qquad x_1 = \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308889 \\ 2\cdot1706023 \end{bmatrix}.$$

The greatly enhanced approximation, due to our use of an accurate value of λ_1, is evident.

When latent roots are real the δ^2-process can be applied a second time,

and a third time, and so on. It is not necessary to give examples. In fact a second application is often too powerful to be really serviceable, for in an example such as the one just given it may well give 12-digit or 15-digit accuracy, which is much more than will be required; on the other hand, the digits which would have to be retained in the elements of vectors, in the ratios $\phi(t)$, and so on, would tax the capacity of the register in most calculating machines. The method is best kept in reserve for really refractory cases, such as latent roots difficult to separate.

9. Determination of Remaining Latent Roots and Vectors.

The determination of the remaining latent roots and vectors will now be considered; in many applications they are not required to so great an accuracy as λ_1, u_1, and x_1. The latent roots cause no trouble; they can be found from the k-ary products of latent roots, as evaluated by either of the methods of § 7. We have a choice of methods for evaluating the latent vectors, based on the assumption that λ_1, u_1, and x_1 have already been determined with sufficient accuracy.

The first method, which may be described as *deflation*, consists in removing entirely from A the part due to λ_1; it has the effect of replacing A by a deflated matrix A_1 which has the same latent roots and latent vectors as A, except that λ_1 is replaced by zero.

Let u_1 and x_1 be normalized so that their scalar product $u_1x_1=1$. *Reversing* their order, let us form their *matrix* product x_1u_1, which will be a matrix of order n and rank 1, and from this construct

$$A_1=A-\lambda_1x_1u_1. \qquad (1)$$

Then it may be shown that A_1 has the same latent vectors and latent roots as A, except that λ_1 is replaced by zero.

Proof.

$$\begin{aligned} A_1x_1&=Ax_1-\lambda_1x_1u_1x_1\\ &=\lambda_1x_1-\lambda_1x_1=0, \qquad \text{since } u_1x_1=1. \end{aligned}$$

Also

$$\begin{aligned} A_1x_j&=Ax_j-\lambda_1x_1u_1x_j\\ &=\lambda_jx_j, \qquad j\neq 1, \qquad \text{since } u_1x_j=0, \end{aligned}$$

the vectors being orthogonal. The result is thus established.

The following alternative proof shows more clearly the meaning of this process of deflation. Consider first the case where the canonical form of A is purely diagonal. Then we have $A=H\Lambda H^{-1}$, where Λ is of the diagonal form mentioned in § 3, with latent roots in the diagonal. From the theory of matrices it is known that the columns of H are the latent column vectors x_j and the rows of H^{-1} are the (suitably normalized)

row vectors u_i. Let us *partition* H according to its columns, H^{-1} according to its rows, and multiply out $H\Lambda H^{-1}$ as a product (Turnbull and Aitken, pp. 5–6) of partitioned matrices. The result is

$$A = \lambda_1 x_1 u_1 + \lambda_2 x_2 u_2 + \ldots + \lambda_n x_n u_n, \qquad (2)$$

which shows the dissection of A into matrix constituents corresponding to the respective latent roots. Deflation corresponds to the removal of the first constituent.

It may be noted (compare Turnbull, 1927, p. 126, Theorem VI) that $x_i u_i$ is a product of $n-1$ factors $(A-\lambda_j I)/(\lambda_i - \lambda_j), j \neq i$, and so is polynomial in A, of degree $n-1$. Its latent roots are 1, 0, 0, . . ., 0.

When the elementary divisors are non-linear, the dissection of A into matrix constituents involves additional terms. The case of a triple root λ_1 with an elementary divisor of exponent $p=3$ will serve to illustrate the modification. The constituent part of A due to λ_1 is then

$$[x_1 \quad x_2 \quad x_3] \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda_1 x_1 u_1 + (\lambda_1 x_2 + x_1) u_2 + (\lambda_1 x_3 + x_2) u_3. \qquad (3)$$

For a latent root λ_j with an elementary divisor of exponent p it is

$$\lambda_j x_j u_j + (\lambda_j x_{j+1} + x_j) u_{j+1} + \ldots + (\lambda_j x_{j+p-1} + x_{j+p-2}) u_{j+p-1}. \qquad (4)$$

In the linear case, therefore, to evaluate the latent vector or vectors corresponding to λ_2, a sufficient procedure will be to deflate A, replacing it by $A_1 \equiv A - \lambda_1 x_1 u_1$, and then to construct and use a vector sequence $A_1{}^t v$, which will play in respect of λ_2, x_2, and u_2 a rôle corresponding to that already played by $A^t v$ in respect of λ_1, x_1, and u_1. The formation of the sequence $A_1{}^t v$ will be simplified in practice by the following considerations. When the elementary divisors of A are linear we obtain at once, since $A^t = H\Lambda^t H^{-1}$, the constituents of A^t, namely,

$$A^t = \lambda_1{}^t x_1 u_1 + \lambda_2{}^t x_2 u_2 + \ldots + \lambda_n{}^t x_n u_n, \qquad (5)$$

analogous to (2). In the same way, by considering that the canonical form of A_1 is simply Λ with λ_1 replaced by zero, we have at once

$$(A - \lambda_1 x_1 u_1)^t = A^t - \lambda_1{}^t x_1 u_1, \qquad (6)$$

as might also have been proved (compare Hotelling, 1936 *a*, p. 30), step by step from (1), by using at each stage the orthogonal properties of u_i and x_j. For the same reason it is true more generally that

$$(A - \lambda_i x_i u_i - \ldots - \lambda_k x_k u_k)^t = A^t - \lambda_i{}^t x_i u_i - \ldots - \lambda_k{}^t x_k u_k. \qquad (7)$$

As is to be expected, the corresponding result for the case of non-linear elementary divisors is less simple. By partitioning H and H^{-1} as before and considering the form of Λ^t as given in § 3 (6), we find that

the constituents of A^t corresponding to a root λ_j with an associated elementary divisor of exponent p are

$$\lambda_j{}^t x_j u_j + (\lambda_j{}^t x_{j+1} + t\lambda_j{}^{t-1} x_j) u_{j+1} + \dots$$
$$+ (\lambda_j{}^t x_{j+p-1} + t\lambda_j{}^{t-1} x_{j+p-2} + \dots \quad + t_{(p-1)}(\lambda_j{}^{t-p+1} x_j) u_{j+p-1}. \qquad (8)$$

In theory it would seem from (6) above that the procedure to adopt when λ_1 is associated with linear elementary divisors would be simple. We should construct from each vector $A^t v$ of the first sequence the deflated vector $A^t v - \lambda_1{}^t x_1 u_1 v$, where the vector $x_1 u_1 v$ would be computed once and for all, not, of course, by forming the matrix $x_1 u_1$ and operating with it on v, but by forming the scalar $u_1 v$ and multiplying the elements of x_1 by it. This would give us the sequence $A_1{}^t v$, which we should treat exactly as $A^t v$ had already been treated. The procedure is sound, and we shall exemplify it; but we shall find in the course of doing so that its scope and accuracy are limited by a disability inherent in the technique of practical arithmetic.

Example 1.—To find x_2 from the sequence $A^t v$ of § 2, Example 1, given that

$$x_1 = \begin{bmatrix} 1\cdot0000000 \\ 1\cdot7500932 \\ 2\cdot2925415 \end{bmatrix}, \qquad u_1{}' = \begin{bmatrix} 1\cdot0000000 \\ 0\cdot8308889 \\ 2\cdot1706023 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$\lambda_1 = 11\cdot8353625$ and (from $\lambda_1\lambda_2$ by § 7) $\lambda_2 = 3\cdot0292865$.

The sum of products of corresponding elements of u_1 and x_1 gives the scalar $u_1 x_1 = 7\cdot4303289$. The sum of the elements of u_1 gives the scalar $u_1 v = 4\cdot0014912$. Normalizing, we must reduce $u_1 x_1$ to 1; this will reduce $u_1 v$ to $4\cdot0014912/7\cdot4303289 = 0\cdot53853487$.

So, multiplying the elements of x_1 by this factor, we have the normalized vector

$$x_1 u_1 v = \begin{bmatrix} 0\cdot53853487 \\ 0\cdot94248621 \\ 1\cdot23461354 \end{bmatrix}; \qquad \text{also} \qquad \begin{aligned} \lambda_1{}^4 &= 19621\cdot231, \\ \lambda_1{}^5 &= 232224\cdot39, \\ \lambda_1{}^6 &= 2748459\cdot8. \end{aligned}$$

Now multiply $x_1 u_1 v$ in turn by $\lambda_1{}^4$, $\lambda_1{}^5$, $\lambda_1{}^6$ and subtract the resulting vectors from the last three vectors $A^t v$ ($t = 4$, 5, 6) of § 2, Example 1. We obtain the deflated vectors $A_1{}^t v$, namely:

$$\begin{bmatrix} 22\cdot2829 \\ 19\cdot2604 \\ -17\cdot6375 \end{bmatrix} \quad \begin{bmatrix} 67\cdot068 \\ 58\cdot715 \\ -53\cdot376 \end{bmatrix} \quad \begin{bmatrix} 203\cdot56 \\ 177\cdot54 \\ -161\cdot68 \end{bmatrix} . \quad . \quad . \qquad (9)$$

Dividing these through by their leading elements for comparison with the actual latent vector x_2, and also constructing $P \,.\, A^t v$, we have

$$\begin{bmatrix} 1\cdot00000 \\ 0\cdot86436 \\ -0\cdot79153 \end{bmatrix} \begin{bmatrix} 1\cdot00000 \\ 0\cdot87545 \\ -0\cdot79585 \end{bmatrix} \begin{bmatrix} 1\cdot00000 \\ 0\cdot87218 \\ -0\cdot79426 \end{bmatrix}, \quad PA_1{}^t v = \begin{bmatrix} 1\cdot0000 \\ 0\cdot8729 \\ -0\cdot7947 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1\cdot000000 \\ 0\cdot873019 \\ -0\cdot794885 \end{bmatrix}.$$

This has worked well enough, and the δ^2-process has given fair accuracy in x_2, but it is to be noticed that though we have been using 8 places of decimals in our data, we have been able to keep fewer and fewer digits in the elements of $A^t{}_1v$ as t increases. In fact the last digits of the elements that we have kept are quite uncertain. Take, for example, the first element of the third vector in (9). It is obtained from

$$1480345-0\cdot53853487\,(274845\cdot98)=1480345-1480141\cdot44,$$

where even the second decimal of the subtrahend is doubtful. We are in fact obtaining each element as the difference of two almost equal numbers, an operation which cannot be avoided here and which invariably entails the loss of some significant digits. The position becomes still worse as t increases. For example, the leading element of $A^{10}v$ is 29042215314, a number of 11 digits, while that of $A_1{}^{10}v$ is in the region of 17140, of 5 digits only. Hence, working to 8-figure accuracy as before, we should hardly be able to claim even two digits of accuracy for our evaluation of the leading element of $A_1{}^{10}v$.

The remedies are: (i) to restrict the deflating process to the *earlier* terms $A^{10}v$, and for improvement use the δ^2-process; (ii) to ignore altogether the short-cut suggested by (6) above, and to construct $A_1=A-\lambda_1x_1u_1$ and operate successively on v, just as was done with A in the first place. This is playing for safety, but it avoids any question of taking the difference of large and almost equal numbers.

This second course had better be taken in any case when the elementary divisors associated with λ_1 are *non-linear*, of exponent $p>1$; for the deflation of A^tv by (8) above contains too many pitfalls. We shall construct $A_1=A-\lambda_1x_1u_1$ as before. This is equivalent to removing the elements λ_1 from the diagonal of the corresponding submatrix in the canonical form Λ of § 3, (6), while *leaving the units in the superdiagonal.* The submatrix will therefore take the form U, of order p. Now $U^t=0$, $t\geqslant p$. Hence, if we take $t>n$, as was advocated in § 1, the corresponding constituent for λ_1 will be entirely removed from $A_1{}^tv$. The sequence $A_1{}^tv$ will then yield, by the analysis of the preceding sections of this paper, either the latent vectors corresponding to other elementary divisors associated with λ_1, or those corresponding to λ_2.

After what has been said, the second and subsequent deflations

$$A_2=A_1-\lambda_2x_2u_2=A-\lambda_1x_1u_1-\lambda_2x_2u_2,$$
$$A_3=A_2-\lambda_3x_3u_3=A-\lambda_1x_1u_1-\lambda_2x_2u_2-\lambda_3x_3u_3,$$

and so on, and their construction and use in finding the remaining vectors, do not call for further explanation; neither does the double deflation

which removes two conjugate complex roots, namely,

$$A_2 = A - \lambda_1 x_1 u_1 - \bar{\lambda}_1 \bar{x}_1 \bar{u}_1, \qquad . \quad . \quad . \quad . \quad (10)$$

where the bar denotes the complex conjugate.

Finally, the case when A is symmetric (Hotelling, 1933, 1936) is the simplest of all. Then u_1 is the vector x_1 transposed, the normalisation is performed by dividing by the sum of *squared* elements of x_1, *i.e.* the scalar $x_1'x_1$, and the k^{th} deflated matrix is

$$A_k = A - \lambda_1 x_1 x_1' - \lambda_2 x_2 x_2' - \ . \ . \ . \ - \lambda_k x_k x_k'. \qquad . \quad . \quad (11)$$

Example 2.—To construct A_1 from A as given in § 6, Example 2, given that

$$\lambda_1 = 6{\cdot}8740755, \qquad x_1 = \begin{bmatrix} 1{\cdot}000000 \\ 1{\cdot}533596 \\ 2{\cdot}703836 \end{bmatrix}.$$

We have

$$\lambda_1/(\text{sum of squares in } x_1) = 6{\cdot}8740755/10{\cdot}662646 = 0{\cdot}6446876.$$

Multiplying the elements of x_1 by this and constructing a multiplication table as below, only the upper part of which need be written, because of symmetry, we obtain the normalized $\lambda_1 x_1 u_1$. Subtracting the elements of this matrix from those of A we obtain A_1.

$$\begin{array}{r|c} & \underline{0{\cdot}6446876 \quad 0{\cdot}9886904 \quad 1{\cdot}743130} \\ \begin{array}{r} 1{\cdot}000000 \\ 1{\cdot}533596 \\ 2{\cdot}703836 \end{array} & \begin{bmatrix} 0{\cdot}644688 & 0{\cdot}988690 & 1{\cdot}743130 \\ & 1{\cdot}516252 & 2{\cdot}673257 \\ & & 4{\cdot}713138 \end{bmatrix} = \lambda_1 x_1 u_1. \end{array}$$

(Check by summing the diagonal elements.)

Hence

$$A_1 = \begin{bmatrix} 2{\cdot}35531 & -1{\cdot}98869 & 0{\cdot}25687 \\ -1{\cdot}98869 & 2{\cdot}48375 & -0{\cdot}67326 \\ 0{\cdot}25687 & -0{\cdot}67326 & 0{\cdot}28686 \end{bmatrix}.$$

The vector sequence $A_1{}^t v$ can now be constructed.

(ii) The second method for finding the remaining latent vectors depends on the use of λ-differencing and cognate operations.

Consider once again Case I of § 3, and perform λ_1-differencing once upon the elements of $A^t v$. We obtain the vector $\Delta_\lambda A^t v$, $\lambda = \lambda_1$, and its elements, by § 3 (3), are of the form

$$\Delta_\lambda f(t) = c_2(\lambda_2 - \lambda_1)\lambda_2{}^t + \ . \ . \ . \ + c_n(\lambda_n - \lambda_1)\lambda_n{}^t, \qquad . \quad . \quad (12)$$

while the vector itself, by § 3 (2), takes the form

$$\Delta_\lambda A^t v = a_2(\lambda_2 - \lambda_1)\lambda_2{}^t x_2 + \ . \ . \ . \ + a_n(\lambda_n - \lambda_1)\lambda_n{}^t x_n. \quad . \quad . \quad (13)$$

Now in (12) and (13) the factors $(\lambda_2 - \lambda_1)$, . . ., $(\lambda_n - \lambda_1)$ are independent of t and so may be absorbed in the coefficients c_i and a_i. When this is done λ_1 is effectively removed, and analysis of $\Delta_\lambda A^t v$ will therefore yield

the second latent root and latent vectors. This appears to be simpler than deflation, but it will often be subject to the same difficulty in technique that was noticed before, since the λ-differences will be the differences of large and almost equal numbers. None the less it is a useful method.

Example 3.—To find x_2 from the last four vectors of the sequence $A^t v$ of § 2, Example 1, given that $\lambda_1 = 11{\cdot}8353625$.

Multiplying each vector by λ_1 and subtracting it from its successor, we obtain the three vectors $\Delta_\lambda A^t v$:

$$\begin{bmatrix} -62{\cdot}826 \\ -57{\cdot}684 \\ 51{\cdot}025 \end{bmatrix} \begin{bmatrix} -196{\cdot}65 \\ -169{\cdot}23 \\ 155{\cdot}38 \end{bmatrix} \begin{bmatrix} -590{\cdot}24 \\ -517{\cdot}40 \\ 470{\cdot}00 \end{bmatrix}.$$

Dividing these through by their leading elements and applying the δ^2-process we obtain a close approximation to x_2.

$$\begin{bmatrix} 1{\cdot}0000 \\ 0{\cdot}9182 \\ -0{\cdot}8122 \end{bmatrix} \begin{bmatrix} 1{\cdot}0000 \\ 0{\cdot}8606 \\ -0{\cdot}7901 \end{bmatrix} \begin{bmatrix} 1{\cdot}0000 \\ 0{\cdot}8766 \\ -0{\cdot}7963 \end{bmatrix}, \quad P\Delta_\lambda A^t v = \begin{bmatrix} 1{\cdot}0000 \\ 0{\cdot}8731 \\ -0{\cdot}7949 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1{\cdot}000000 \\ 0{\cdot}873019 \\ -0{\cdot}794885 \end{bmatrix}.$$

It is not surprising that the δ^2 process gives so good a result here; the λ_1-differencing has removed λ_1, the δ^2-process eliminates λ_3, and there are no other latent roots except λ_2.

When, as in Case III of § 3, λ_1 is real, and the elementary divisors are non-linear, the process of λ_1-differencing must be performed on $A^t v$ *more than once*; in fact, if the greatest exponent of elementary divisors is p, we must form $\Delta_\lambda^p A^t v$, $\lambda = \lambda_1$. It then follows from § 3 (7) that λ_1 is removed from the elements, except for factors independent of t which can be absorbed into the coefficients. The sequence $\Delta_\lambda^p A^t v$ can then be analysed for the remaining latent vectors.

Example 4.—To find x_3 from the last six vectors of the sequence $A^t v$ of § 2, Example 3, given that $\lambda_1 = \lambda_2 = 5{\cdot}2360680$ and that $p = 2$.

We form the four vectors $\Delta_\lambda^2 A^t v = A^{t+2} v - 2\lambda_1 \Delta^{t+1} v + \lambda_1{}^2 A^t v$:

$$\begin{bmatrix} -1{\cdot}2198 \\ -1{\cdot}7183 \\ 1{\cdot}6130 \\ -3{\cdot}6068 \end{bmatrix} \begin{bmatrix} 0{\cdot}6812 \\ -1{\cdot}8637 \\ -0{\cdot}2105 \\ -7{\cdot}0835 \end{bmatrix} \begin{bmatrix} 1{\cdot}7525 \\ -1{\cdot}8448 \\ -1{\cdot}2630 \\ -8{\cdot}7178 \end{bmatrix} \begin{bmatrix} 2{\cdot}2799 \\ -1{\cdot}7313 \\ -1{\cdot}8070 \\ -9{\cdot}1862 \end{bmatrix}.$$

These show so slow a convergence that the test for non-linearity of elementary divisors corresponding to λ_3 is suggested. By the methods of § 7 we find $\lambda_3 = 0{\cdot}763932$, and we subject the above vectors to λ_3-differencing. This gives the three vectors

$$\begin{bmatrix} 1{\cdot}6130 \\ -0{\cdot}5510 \\ -1{\cdot}4427 \\ -4{\cdot}3281 \end{bmatrix} \begin{bmatrix} 1{\cdot}2321 \\ -0{\cdot}4211 \\ -1{\cdot}1022 \\ -3{\cdot}3065 \end{bmatrix} \begin{bmatrix} 0{\cdot}9411 \\ -0{\cdot}3220 \\ -0{\cdot}8422 \\ -2{\cdot}5264 \end{bmatrix},$$

corresponding elements of which show the desired proportionality of 0·764 : 1. Dividing out by leading elements for comparison with x_3 we obtain

$$\begin{bmatrix} 1\cdot0000 \\ -0\cdot3416 \\ -0\cdot8944 \\ -2\cdot6833 \end{bmatrix} \begin{bmatrix} 1\cdot0000 \\ -0\cdot3418 \\ -0\cdot8946 \\ -2\cdot6836 \end{bmatrix} \begin{bmatrix} 1\cdot0000 \\ -0\cdot3422 \\ -0\cdot8949 \\ -2\cdot6845 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} 1\cdot000000 \\ -0\cdot341641 \\ -0\cdot894427 \\ -2\cdot683282 \end{bmatrix}.$$

Here the first vector, obtained from the earlier $A^t v$, is the best approximation; the others deteriorate. This is due to the progressive loss of significant digits through the cause previously mentioned; accuracy cannot be recovered in these later vectors except by using more than 8 digits in λ_1.

Finally, as may be expected, in the case of complex roots the operation $(\mu_r - \cos\theta)$ replaces that of λ-differencing. In fact, when the greatest two roots λ_1 and λ_2 are complex conjugates $r_1 e^{i\theta_1}$, $r_1 e^{-i\theta_1}$, we replace the vector sequence $A^t v$ by $(\mu_r - \cos\theta_1)A^t v$. It is not difficult to prove from § 3 (5) that this effectively removes those roots from the elements $f(t)$; analysis of the derived sequence then proceeds as before. If corresponding to each of the complex conjugates there is an elementary divisor of maximal exponent p, we replace $A^t v$ by $(\mu_r - \cos\theta_1)^p A^t v$; this follows in the same way from § 3 (8) or from § 5 (6).

Example 5.—To find the latent vector x_3, given the first five vectors $A^t v$ shown in § 2, Example 2, and $r_1 = 3\cdot84040936$, $r_1^{-1} = 0\cdot26038891$, $\cos\theta_1 = 0\cdot51416828$.

Taken three at a time and subjected to $\mu_r - \cos\theta_1$, the five given vectors yield the following three vectors:—

$$\begin{bmatrix} -0\cdot18549 \\ -3\cdot47138 \\ -2\cdot10191 \end{bmatrix} \begin{bmatrix} 0\cdot17597 \\ 3\cdot2954 \\ 1\cdot9970 \end{bmatrix} \begin{bmatrix} -0\cdot17069 \\ -3\cdot1247 \\ -1\cdot8929 \end{bmatrix}.$$

Dividing these by their largest element in each case and comparing with the actual latent vector x_3, we have

$$\begin{bmatrix} 0\cdot05343 \\ 1\cdot00000 \\ 0\cdot60550 \end{bmatrix} \begin{bmatrix} 0\cdot05340 \\ 1\cdot00000 \\ 0\cdot60600 \end{bmatrix} \begin{bmatrix} 0\cdot05463 \\ 1\cdot00000 \\ 0\cdot60579 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} 0\cdot053481 \\ 1\cdot000000 \\ 0\cdot605605 \end{bmatrix}.$$

Once again, and for the same reason as before, the earliest vector in the sequence gives good approximation, and the later ones deteriorate.

Example 6.—We shall remove the two equal pairs of complex roots in § 2, Example 4, by forming $(\mu_r - \cos\theta)^2 A^t v$, given $r = 3\cdot87298335$, $r^{-1} = 0\cdot25819889$, $\cos\theta = 0\cdot38729833$. This compound operation would normally be carried out in two stages, the vectors at the first stage being used (§ 5, at end) to find the complex latent vector x_1.

The first operation gives the five vectors

$$\begin{bmatrix} 76{\cdot}168668 \\ 100{\cdot}568468 \\ 83{\cdot}527338 \\ 104{\cdot}828747 \\ 80{\cdot}041651 \end{bmatrix} \begin{bmatrix} 605{\cdot}86370 \\ 174{\cdot}80066 \\ 442{\cdot}81110 \\ 459{\cdot}85223 \\ 590{\cdot}37177 \end{bmatrix} \begin{bmatrix} 673{\cdot}3828 \\ -986{\cdot}9652 \\ 73{\cdot}0704 \\ -194{\cdot}9401 \\ 566{\cdot}8758 \end{bmatrix} \begin{bmatrix} -7066{\cdot}129 \\ -5580{\cdot}065 \\ -6420{\cdot}503 \\ -7480{\cdot}538 \\ -7151{\cdot}334 \end{bmatrix} \begin{bmatrix} -31300{\cdot}81 \\ -1938{\cdot}56 \\ -20360{\cdot}02 \\ -19519{\cdot}58 \\ -29960{\cdot}75 \end{bmatrix}.$$

The second operation gives the three vectors

$$-\begin{bmatrix} 0{\cdot}2167 \\ 0{\cdot}3667 \\ 0{\cdot}3167 \\ 0{\cdot}2667 \\ 0{\cdot}4667 \end{bmatrix}, \quad \begin{bmatrix} 0{\cdot}2166 \\ 0{\cdot}3667 \\ 0{\cdot}3166 \\ 0{\cdot}2667 \\ 0{\cdot}4667 \end{bmatrix}, \quad -\begin{bmatrix} 0{\cdot}2171 \\ 0{\cdot}3671 \\ 0{\cdot}3170 \\ 0{\cdot}2670 \\ 0{\cdot}4662 \end{bmatrix}, \quad x_5 = \begin{bmatrix} 13 \\ 22 \\ 19 \\ 16 \\ 16 \end{bmatrix},$$

which, multiplying through by 60, we may compare with x_5. The value $\lambda_5 = -1$ is made plausible. The third vector shows a lessened approximation to x_5, due (as before) to the loss of significant digits.

10. Concluding Observations.

In conclusion we may mention some special applications and some considerations bearing on practical computation.

(i) A common problem, which is more general than that of finding the latent roots and latent vectors of A, is that of determining the roots of the equation $|A - \lambda B| = 0$, and the vectors x satisfying $Ax = \lambda Bx$, where (usually) A and B are symmetric and B is positive definite. Without such restrictions on A and B, but assuming only that B is non-singular, we may solve this problem by evaluating $B^{-1}A$ (Aitken, 1937) and finding its latent roots and latent vectors; these will be the required roots and vectors. It is to be observed that even when A and B are both symmetric the matrix $B^{-1}A$ is not in general symmetric.

(ii) The solution of algebraic equations has already been mentioned. With an algebraic equation

$$x^n - a_1x^{n-1} - a_2x^{n-2} - \ldots - a_n = 0 \quad . \quad . \quad . \quad (1)$$

is associated the "companion matrix"

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ 1 & . & . & \ldots & . & . \\ . & 1 & . & \ldots & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & . & \ldots & 1 & . \end{bmatrix} \quad . \quad . \quad . \quad (2)$$

for which it is characteristic. The latent roots of A are then the roots of the equation (1), and can therefore be found by operating on a vector v and using the vector sequence A^tv in the manner of the present paper. The method can easily be shown to be equivalent to that of D. Bernoulli

as recently extended (Aitken, 1925, 1931), and would be inferior to the root-squaring process (Whittaker and Robinson, 1929, p. 106, Brodetsky and Smeal, 1923) were it not for the great power of the δ^2-process, which should be applied as soon as possible.

The latent vectors of the companion matrix A of the equation (1) constitute an alternant matrix L', where L is of the form given in § 7 (3), or, where there are multiple roots, of the confluent or differentiated form given in § 7 (5).

(iii) Finally, the question of accuracy and remainder terms. There are computers of robust confidence who would base their estimate of a latent root or latent vector more on the instinctive convictions of probability than on the punctilious scruples of analysis; there are also some of more diffident spirit who, if presented with such a sequence of approximations to λ as 11·835362545, 11·835362537, 11·835362535, would like to know for certain that λ lies between 10 and 15. We must find restraints for the first kind, and guiding principles for the second.

Fortunately, the test of verification *a posteriori* removes most of the difficulties. The sequences $A^t v$ provide both the latent root and the latent vector, and the computer should always assure himself after these have been evaluated that they do actually satisfy $Ax = \lambda x$ to the number of digits retained. Also, during the formation of the sequence $A^t v$, he can and should verify, or by a slight modification in one or more elements of v actually ensure, that some of the ratios $\phi(t)$ of corresponding elements in consecutive vectors are steadily increasing with t (λ being supposed real), while others are steadily decreasing, the reason being that for some elements $f(t)$ of $A^t v$ the c_2/c_1 of § 4 (4) is positive, for others negative. Thus in § 2, Example 1, a study of the three sequences $\phi(t)$ for the three elements of $A^t v$ will show that two of these sequences are rising towards the root $\lambda_1 = 11{\cdot}8353625$, while the remaining one is falling towards it; this leaves no doubt as to the delimitation of the root. Again, if λ_1 and λ_2 are real and of opposite sign, the sequence $\phi(t)$ will oscillate, and this will also give a delimitation. If λ_1 and λ_2 are of the same sign, and if it is desired to obtain an oscillating sequence, this may often be done by using, instead of A, the matrix $A - aI$, where a, which must be such that $|\lambda_1| > |a| > |\lambda_2|$, is a suitably chosen constant. This "change of origin" is also useful in cases of slow convergence. The latent roots of $A - aI$ are $\lambda_i - a$, where λ_i is a latent root of A, and the latent vectors are the same as those of A. Finally, in all cases it will be well, by examining (§ 8) the first differences $\delta\phi(t)$, to form an estimate of the second greatest latent root at an early stage.

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XXI.—On the Immature Stages of some Scottish and other Psyllidæ.

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CONTENTS.

	PAGE
Introduction	305
General Morphology of Psyllid Nymphs	306
List of Psyllid Nymphs of which Descriptions are Available	308
Description of the Nymphs of Individual Species	310
1. *Psyllia försteri* Flor.	310
2. *Psyllia alni* L.	312
3. *Psyllopsis fraxinicola* Först.	314
4. *Psyllopsis discrepans* Flor.	314
5. *Psyllia buxi* L.	316
6. *Psyllia melanoneura* Först.	316
7. *Psyllia pyricola* Först.	319
8. *Psyllia mali* Schmidb., race *mali*	319
8*a*. *Psyllia mali* Schmidb., race *peregrina*	320
9. *Psyllia ambigua* Först.	322
10. *Trioza urticæ* L.	323
11. *Aphalara nebulosa* Zett.	324
12. *Eurhinocola eucalypti* Mask.	326
13. *Trichopsylla walkeri* Thom.	328
Summary	328
Acknowledgments	329
References to Literature	330

INTRODUCTION.

THE nymphs that are here described belong to those species the biology of which has been discussed in a previous paper (Lal, 1934 *b*). The plan followed was the one originally adopted by Ferris (1923), in which descriptions and illustrations of all morphological features important in the identification of the species of the nymph, apart from the adult, were prepared from careful examination of specimens mounted for microscopic observation. The importance of such a study has been increasingly appreciated in the last decade, both from the economic and the systematic points of view. Harmful species of Psyllidæ are mainly injurious because of the feeding activities of their immature stages, and there are but two species known, *Psyllia pyricola* Först. and *Paratrioza cockerelli* Sulc., in which the adults have also been found to be capable of causing injury. On the other hand, the specific determination of immature stages has been rendered difficult owing to the fact (first) that the published descriptions of individual species of nymphs are often found to be applicable

to species other than those to which the particular description refers; and (second) that sometimes the nymphs of two closely allied species are distinguished by such marked differences as to warrant the adults being assigned to different genera and even subfamilies. Such cases have been mentioned by Crawford (1919), Ferris (1928 *a*), Husain and Nath (1927), and have been summarised by Rahman (1932) in the introductory part of his paper. The present descriptions have, therefore, been prepared with the avowed objects of facilitating the determination of individual species, where comparison with allied species is impossible owing to lack of available material, and also of assessing the value of the various nymphal characters with a view to establishing a classification based on these as contrasted with one based upon the characters of the adults.

General Morphology of Psyllid Nymphs.

The nymphs of Psyllidæ are usually of small size, 1–4 mm. in length, with the body dorso-ventrally flattened. The dorsum, chiefly of the thorax, is sculptured with special areas of sclerotisation—a term which has been used throughout this paper in preference to "chitinisation" in accordance with the views of Ferris and Chamberlin (1928), Snodgrass (1933), and others. A greater or lesser amount of pigment occurs in these areas, so that the individual nymphs may be either dark or light coloured, depending on the degree of pigmentation. The coloration of the nymphs of many species—for example, *Psyllopsis discrepans* Flor. and *Aphalara nebulosa* Zett.—might well be regarded as protective but for the fact that resemblance to the environment is often nullified by the white waxy secretion which exudes from the pores of the circumanal rings and renders the individual conspicuous.

The secretion of a white waxy substance is almost a universal feature of the nymphs of Psyllidæ. The only exception has been recorded by Uichanco (1921), who noted that *Paurocephala kleinhofiæ* Uich., a species from the Philippine Islands, produced no secretion in any of its stages. As a rule, the secretion is exuded through the sets of pores surrounding the anal aperture. Sometimes there are additional pores, situated generally on the surface of the body, as in *Psyllopsis fraxinicola* Först., which supplement the secretion of the circumanal pores. In the nymph of *Pachypsylla venusta* O.S. the rings of circumanal pores are absent, but as the habits of the nymphs are unknown, nothing can be said of the secretory activity of this species.

The physical nature of the secretion in the different groups of species of the family differs considerably. In some species, such as *Psyllia mali*

Schmidb., *P. buxi* L., and *Aphalara nebulosa* Zett., etc., the secretion takes the form of white waxy droplets, ribbons, or tubes surrounded by a coating of denser texture. In the nymphs of *Trioza urticæ* L., and adults of all species which possess wax glands, the secretion takes the form of little pellets of waxy matter. In other species such as *Psyllia alni* L., *P. försteri* Flor., and *Psyllopsis fraxinicola*, besides the waxy paste-like substance there is a huge mass of fluffy cottony exudation, which entirely covers the nymph in a meshwork of extremely fine threads. More uniformly regular structures are also secreted by the nymphs of many species of *Trioza* and *Psyllopsis*, which take the form of long stiff shafts, arising from the bases of small marginal setæ, which they severally enclose. These structures are fragile, break away easily, and are of varying lengths, the longest easily exceeding the length of the body. *Psyllia pyricola* Först. is probably the only species which secretes a transparent liquid instead of a waxy substance, in which the injurious fungus *Cladosporium herbarium* is said to grow.

Chemically, the nature of these secretions does not seem to vary much. They are insoluble in water, mineral acids, caustic potash solution, ether, and chloroform. Ethyl-alcohol dissolves them more readily when they are freshly secreted than when they have been exposed on plants for some time. The shaft-like secretion arising from the bases of the marginal setæ withstands the action of alcohol longest. The fine thread-like and silky white mass secreted by the ash Psyllids (*Psyllopsis fraxinicola* and *P. discrepans*) is, however, only slightly soluble even in alcohol.

The developing nymph furnishes no external indication of its sex until the last instar, when the genitalia of the adult may be seen through the cuticle. Awati (1915) remarks that the parts of the reproductive system are readily seen in the 4th and 5th instars, in which the external sexual differences become evident, and the males (*i.e.* nymphs which are destined to become males) are shorter than the females. This last criterion holds good in certain species only. In *Psyllopsis fraxinicola*, nymphs which are destined to become males are, as a rule, more elongated than those destined to be females, in which the abdomen is markedly round and broad.

A number of setæ, often of more than one kind, generally invest the body of Psyllid nymphs. These have, unfortunately, been given different names by different authors. For instance, the "dagger-like" setæ of the antennæ, wing-pads, legs, and abdomen in *Paurocephala fremontæ*, as described by Klyver (1931), are not different from the "sharply pointed setæ" described by Ferris (1928 *a*) in *Synoza floccosa* Fer. On the other hand, the setæ on the abdominal margin of *Euphyllura arbuti*

Schw., figured and termed by Ferris and Hyatt (1923) as lanceolate, are obviously not the same as the marginal setæ on the apical fourth of the abdomen in *Tenaphalara elongata* Crawf., also called lanceolate by Rahman (1932). This may easily result in confusion, and for this reason the names of the chief types of setæ mentioned in this paper are given below, together with short descriptions or explanations necessary to distinguish them:

1. Setæ: small hair-like bristles.
2. Simple setæ: larger than 1, elongated or curved but not structurally modified otherwise.
3. Secta-setæ: small, broad, stout setæ, with sharply cut distal ends, occurring characteristically in the nymphs of the subfamily Triozinæ.
4. Spear-shaped setæ: very much like secta-setæ, but with the distal ends not sharply cut but pointed.
5. Lanceolate setæ: of usually the same size as secta-setæ, long, oval, tapering at both ends, and sharply pointed distally.
6. Dagger-shaped setæ: long, stout setæ like the blade of a dagger, straight not curved.
7. Spatulate setæ: various modifications of long simple setæ, with bluntly or broadly ending extremities.
8. Ring-based setæ: simple setæ with the bases implanted in circular ring-like structures.

The nymphs of Psyllidæ have been differentiated into three forms, chiefly on the basis of the shape of the wing-pads and the degree of their projection from the body contour. According to Ferris (1925), the first is the Triozine form, in which the wing-pads are produced anteriorly at the humeral angle, and are otherwise so arranged that their margin is more or less continuous with that of head and abdomen. The second is the Psylline form, in which the wing-pads are not produced anteriorly at the humeral angle, and project prominently from the contour of the body. A third form, Pauropsylline, intermediate between these two, has been rightly created by Rahman (1932), in which the fore wing-pads are not produced anteriorly at the humeral angle but show a tendency in that direction. Their outer margins are, however, in line with the general contour of the body.

List of Psyllid Nymphs of which Descriptions are Available.

Rahman (1932) has given a list of species, the nymphs of which have been described in detail. This list is not only out of date, but also contains several omissions, which are made good in the following table:—

Subfamily.	Species.	Nymphal Type.	Author and Date.
Pauropsyllinæ.	1. *Pauropsylla tuberculata* Crawf.	Pauropsylline.	Rahman, 1932.
,,	2. *P. depressa* Crawf.	,,	,, ,,
,,	3. *Microceropsylla nigra* Crawf.	Triozine.	Boselli, 1930 *a*.
,,	4. *Paurocephala chonchaiensis* Bos.	Psylline.	,, 1929 *a*.
,,	5. *P. fremontæ* Klyv.	,,	Klyver, 1931.
Carsidarinæ.	6. *Freysuila cohahuayanæ* Fer.	,,	Ferris, 1928 *a*.
,,	7. *Carsidara gigantea* Crawf.	,,	,, ,,
,,	8. *Tenaphalara elongata* Crawf.	,,	Rahman, 1932.
,,	9. *Synoza floccosa* Fer.	,,	Ferris, 1928 *a*.
,,	10. *Homotoma ficus* L.	Triozine.	Boselli, 1929 *a*.
,,	11. *Mesohomotoma lineaticollis* End.	Psylline.	,, 1930 *a*.
Psyllinæ.	12. *Euphyllura arbuti* Schw.	,,	Ferris and Hyatt, 1923.
,,	13. *Psyllopsis fraxinicola* Först.	,,	Ferris, 1923.
,,	14.**Psyllia alni* L.	,,	,, 1925.
,,	15.**P. buxi* L.	,,	,, 1926.
,,	16. *Pachypsylla venusta* O.S.	,,	,, ,,
,,	17. *Euphalerus gallicola* Fer.	,,	,, 1928 *a*.
,,	18. *Euphyllura arctostaphyli* Schw.	,,	,, ,,
,,	19. *Arytaina punctipennis* Crawf.	,,	Rahman, 1932.
,,	20. *Diaphorina citri* Kuw.	Triozine.	Husain and Nath, 1927.
,,	21. *Pachypsylla celtidie-mamma* Rly.	Psylline.	Boselli, 1929 *b*.
,,	22. *Euceropsylla russoi* Bos.	,,	,, 1929 *d*.
,,	23. *Psyllia toroenensis* Kuw.	,,	,, 1930 *a*.
,,	24. *Rhinocola succincta* Heg.	,,	,, 1930 *b*.
,,	25.**Psyllia mali* Schmidb.	,,	Klyver, 1931.
,,	26. *P. pyricola* Först.	,,	,, ,,
,,	27. *P. uncata* Fer. and Klyv.	,,	Ferris and Klyver, 1932.
,,	28. *P. albizziæ* Fer. and Klyv.	,,	,, ,,
,,	29. *P. acaciæ* Mask.	,,	,, ,,
,,	30. *Ctenarytaina thysanura* Fer. and Klyv.	,,	,, ,,
,,	31. *Psyllia försteri* Flor.	,,	Present paper.
,,	32. *P. ambigua* Först.	,,	,, ,,
,,	33. *P. melanoneura* Först.	,,	,, ,,
,,	34. *P. mali* Schmibd., race *peregrina* Först.	,,	,, ,,
,,	35. *Psyllopsis discrepans* Flor.	,,	,, ,,
,,	36. *Eurhinocola eucalypti* Mask.	,,	,, ,,
Liviinæ.	37. *Aphalara calthæ* L.	Pauropsylline (Psylline type, Klyver).	Klyver, 1930.

* Indicates species of which only the last-instar nymphs were described in the papers mentioned, the remaining stages being described in the present work.

Subfamily.	Species.	Nymphal Type.	Author and Date.
Liviinae.	38. *A. martini* van D.	Pauropsylline (Psylline type, Ferris).	Ferris, 1924.
,,	39. *A. nebulosa* Zett.	Pauropsylline.	Present paper.
Triozinæ.	40. *Ceropsylla sideroxyli* Rly.	Triozine.	Ferris, 1923.
,,	41.* *Trioza urticæ* L.	,,	,, ,,
,,	42. *Paratrioza cockerelli* Sulc.	,,	,, ,,
,,	43. *Phyllopecta diospyri* Ashm.	,,	,, 1926.
,,	44. *Leuronota michoacana* Fer.	Pauropsylline (Triozine type, Ferris).	,, 1928 *b*.
,,	45. *Trioza albifrons* Crawf.	Triozine.	Klyver, 1930.
,,	46. *T. fletcheri* Crawf.	,,	Rahman, 1932.
,,	47. *Spanioza galii aspinovelutina* Sulc.	,,	Boselli, 1929 *c*.
,,	48. *S. taiwanica* Bos.	,,	,, 1930 *a*.
,,	49. *S. erythreæ* Del Guer.	,,	,, 1930 *c*.
,,	50. *Cecidotrioza sozanica* Bos.	,,	,, 1930 *a*.
,,	51. *Egeirotrioza ceardi* De Berg, var. *euphratica*.	,,	,, 1931 *a*.
,,	52. *Kuwayama lavateræ* van D.	,,	Ferris, 1924.
,,	53. *Trichopsylla walkeri* Thoms.	,,	Present paper.
,,	54. *Powellia vitreoradiata* Mask.	,,	Ferris and Klyver, 1932.

Description of the Nymphs of Individual Species.

1. *Psyllia försteri* Flor.

Hosts.—*Alnus glutinosa* and *A. incana*.

Locality.—Royal Botanic Garden and Boghall, Edinburgh.

Fifth-stage Nymph (fig. 1, A): Length, 2·4 mm. Body yellowish green, eyes yellowish red, sclerotised areas brownish; antennal segments 1, 2, and 3, wing-pads, and legs greenish yellow, rest of antennæ progressively black. *Form*: Psylline. Sclerotisation on dorsum extending to a pair of elongate ocular areas on head, a number of small areas on thorax, four pairs of transverse areas on anterior half, and to entire posterior half of abdomen. Ventrally, sclerotised areas include a small part of apical abdomen, four pairs of submedian and three pairs of marginal abdominal areas. Dorsum with many bluntly pointed ring-based setæ (fig. 1, D) and in places smaller but stouter setæ of same kind. Venter beset with similar setæ but with sharply pointed ends. *Head* as broad as abdomen, anterior margin with a number of setæ. Antennæ (fig. 1, E) more than twice as long as width of head, of nine segments, third longest, weakly jointed in middle. *Thorax* broadest next to head. Wing-pads uniformly oval with several stout spatulate setæ and about eight on costal margin of anterior, and two of same kind at apical end of posterior pair. Legs long, beset with numerous stout simple setæ of various lengths, femora reaching beyond body margin; trochanter absent; tibio-tarsal articulation distinct; pulvillus petiolate, cleft in middle with two

* Indicates species of which only the last-instar nymphs were described in the papers mentioned, the remaining stages being described in the present work.

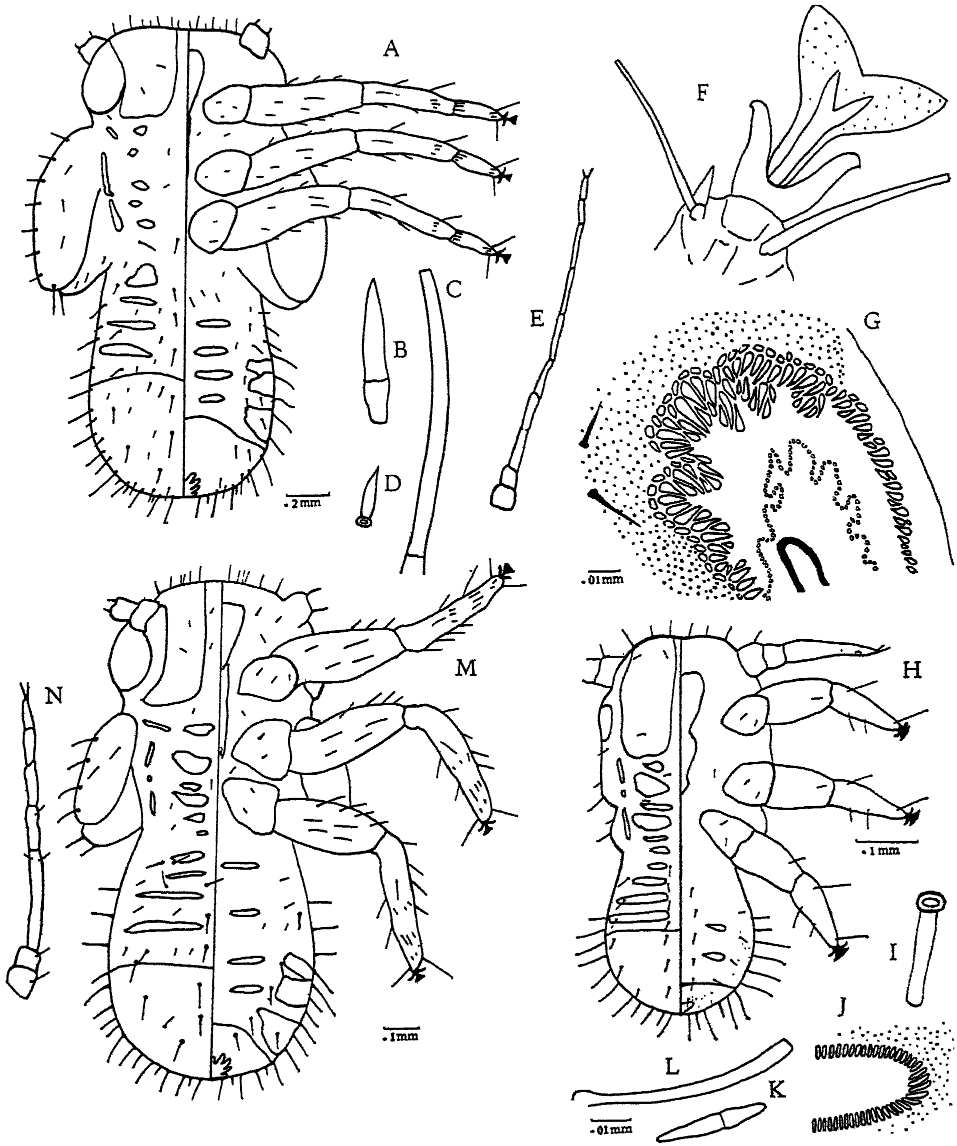

FIG. 1.—*Psyllia försteri* Flor. A, fifth-stage nymph; B, spear-shaped seta of the same; C, spatulate seta of the same; D, ring-based seta of the same; E, antenna of fifth-stage nymph; F, apex of tarsus of fifth-stage nymph; G, portion of circumanal pore-rings of fifth-stage nymph, with round pores surrounding them; H, second-stage nymph; I, bluntly pointed ring-based seta of second-stage nymph; J, portion of outer circumanal pore-ring of second-stage nymph; K, spear-shaped seta of second-stage nymph; L, spatulate seta of second-stage nymph; M, fourth-stage nymph; N, antenna of fourth-stage nymph.

claws at base (fig. 1, F). *Abdomen* round and elongate, margin with a number of long spatulate (fig. 1, C) and a few smaller spear-shaped setæ (fig. 1, B). Anal opening a short distance from apex with two sets of pore-rings (fig. 1, G). Inner ring consisting of a single row of round pores disposed zigzag, outer ring of long, pointed oval pores also set zigzag and mostly three-layered. Just beyond this layer of outer pores, and for a short distance near each margin, ventral body-wall studded with numerous round pores. FOURTH STAGE (fig. 1, M): Length, 1·3 mm. Colour same as in previous-stage nymphs. Differs from it in larger size of certain sclerotised areas on thorax, in having seven-segmented antennæ (fig. 1, N), and in absence of tibio-tarsal articulation. THIRD STAGE: Length, ·94 mm. Body greenish yellow, eyes and sclerotised areas brown, antennal tips dark, rest of antennæ, wing-pads, and legs yellowish brown. Differs from fourth-stage nymph in possessing four-segmented antennæ and relatively much smaller wing-pads. SECOND STAGE (fig. 1, H): Length, ·71 mm. Body yellow, eyes red, sclerotised areas brown, legs and antennæ brownish yellow, antennal tips dark. Differs from third-stage nymph in absence of marginal and two pairs of submedian sclerotised areas ventrally, in possessing three-segmented antennæ and very rudimentary wing-pads with a seta at apical ends of each, and in absence of inner ring of circumanal pores, outer ring (fig. 1, J) consisting of a single row of elongated slit-like pores not sinuately disposed. Ring-based, spear-shaped, and spatulate setæ are illustrated in fig. 1, I, K, and L respectively. FIRST STAGE (fig. 2, F): Length, ·54 mm. Colour as in second-stage nymph. Differs from it in its two-segmented antennæ, in absence of wing-pads, these being represented by two pairs of sclerotised plates each with a seta, and in complete absence of circumanal rings of pores and those situated around them and marginally.

2. *Psyllia alni* L.

The fifth-stage nymph of this species was described by Ferris (1925). In the following account, colour notes of this stage and descriptions of the remaining four instars are given.

Hosts.—*Alnus glutinosa* and *A. incana.*

Locality.—Royal Botanic Garden and Boghall, Edinburgh.

FIFTH STAGE: Body yellowish green, eyes pinkish, sclerotised areas on abdomen and thorax black, on head brown. Antennæ (except segments 3 and 4), wing-pads, and legs (except femora) blackish brown. Third and fourth segments of antennæ and femora brownish yellow. FOURTH STAGE (fig. 2, E): Length, 1·5 mm. Body greenish yellow, eyes red, sclerotised parts including antennæ (except segments 3 and 4), wing-pads, and legs blackish brown. Third and fourth segments of antennæ brownish yellow. Differs from fifth-stage nymph in relatively larger size of sclerotised plates on thorax, in possessing six-segmented antennæ, and in absence of tibio-tarsal articulation. THIRD STAGE: Length, ·98 mm. Body yellow, eyes red, sclerotised areas brown, antennæ (except tips), wing-pads, and legs brownish yellow, antennal tips dark. Differs from fourth-stage nymph in absence of ventral sclerotised areas at base of antennæ, in having a belt of minute pointed structures on both sides and on ventral abdomen, and in possessing four-segmented antennæ. SECOND STAGE (fig. 2, A): Length, ·83 mm. Head and thorax greenish yellow, abdomen

yellow, eyes red, sclerotised areas brown, antennæ and legs pale yellow. Differs from third-stage nymph in absence of ventral sclerotisation, in having three-segmented antennæ, and in absence of wing-pads, these being represented by

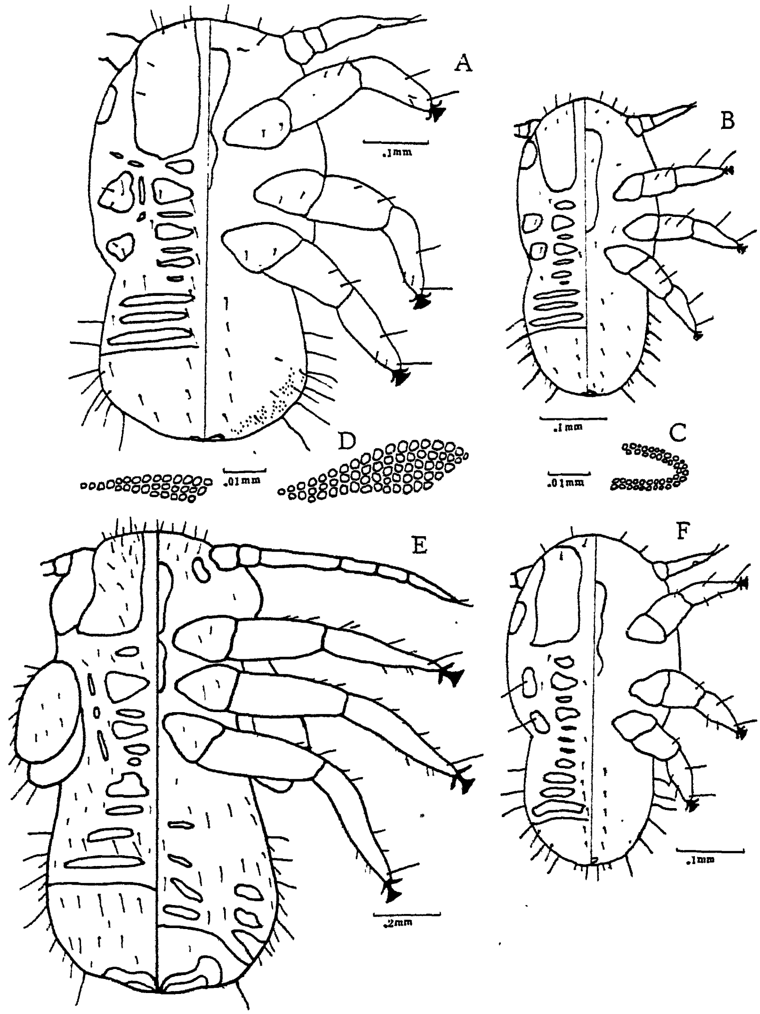

FIG. 2.—*Psyllia alni* L. A, second-stage nymph; B, first-stage nymph; C, portion of outer circumanal pore-ring of first-stage nymph; D, portion of outer circumanal pore-ring of second-stage nymph; E, fourth-stage nymph. *Psyllia försteri* Flor. F, first-stage nymph.

slight bulgings and by two pairs of sclerotised plates each with a seta. Fig. 2, D, illustrates the shape and arrangement of the pores of the outer circumanal ring. FIRST STAGE (fig. 2, B): Length, ·57 mm. Colour as in second-stage nymph. Differs from it in being relatively much smaller and in having outer circumanal pore-ring (fig. 2, C) encircling extreme apex, composed of two layers of small round pores.

3. *Psyllopsis fraxinicola* Först.

All the stages of the nymphs of this species have been described by Ferris (1923). Earlier, general descriptions of the nymphs were given by Witlaczil (1885) and by Scott (1886). In the following account, measurements and colour notes, which do not occur in Ferris's description, are given:—

Host.—*Fraxinus excelsior*.

Locality.—Dalkeith, and Boghall, Edinburgh.

Fifth Stage: Length, 1·9 mm. Body whitish green, eyes black, wing-pads pale whitish. Fourth Stage: Length, 1·0 mm. Body more yellowish than greenish, eyes deep brown, last segment of antennæ and tips of last segments of legs black. Third Stage: Length, ·70 mm. Body yellow, eyes reddish brown. Second Stage: Length, ·47 mm. Body yellow, eyes deep red. First Stage: Length, ·37 mm. Colour as in second-stage nymph.

4. *Psyllopsis discrepans* Flor.

Host.—*Fraxinus excelsior*.

Locality.—Dalkeith, and Boghall, Edinburgh.

Fifth Stage (fig. 3, A): Length, 1·8 mm. Body greenish yellow, eyes brown, sclerotised parts (including wing-pads) blackish brown, legs and antennæ (except tips) brown, antennal tips black. *Form*: Psylline. Sclerotisation on dorsum extends to a pair of ocular areas on head, certain small areas on thorax, four pairs of areas on anterior and to entire posterior half of abdomen. Ventrally two small areas on thorax, four pairs of submedian, five pairs of marginal, and two areas at apical end of abdomen are sclerotised. Dorsum, including wing-pads, beset with numerous round pores (fig. 3, J) of various sizes and small ring-based setæ (fig. 3, I), the latter assuming a more or less transverse arrangement on abdomen. *Head*: anterior margin with few setæ, antennæ (fig. 3, K) of eight segments, third longest. *Thorax* broader than head. Wing-pads oval, uniformly broad, costal margin devoid of setæ except a few at apical ends. Legs long, femora just reaching body margin, trochanter absent, tibio-tarsal articulation distinct, empodium with two claws, petiolate (fig. 3, G). *Abdomen* broad, margin surrounded by small lanceolate setæ (fig. 3, D). Anal opening a short distance from apex. Outer circumanal pore-ring (fig. 3, H) one-layered towards median line, two- to three-layered near sides, individual pores elongate and oval, inner ring consisting of round pores. Lanceolate setæ are illustrated in fig. 3, D and F. Fourth Stage: Length 1·1 mm. Colour as in fifth-stage nymph. Differs from it in having six-segmented antennæ and in absence of tibio-tarsal articulation. Third Stage: Length, ·74 mm. Colour as in nymphs of previous stages. Differs from fourth-stage nymph in possessing four-segmented antennæ, in the reduction of round pores on dorsum, and in outer circumanal pore-ring being one-layered, except for a short distance near sides. Second Stage (fig. 3, B): Length, ·66 mm. Body pinkish yellow, eyes deep red, antennæ, wing-pads, and legs pale yellow. Differs from third-stage nymph in its different

arrangement of sclerotised plates on dorsum, absent ventrally, in possessing three-segmented antennæ and very rudimentary wing-pads each with a lanceolate seta (fig. 3, D) apically. The outer circumanal pore-ring is represented in fig. 3, C.

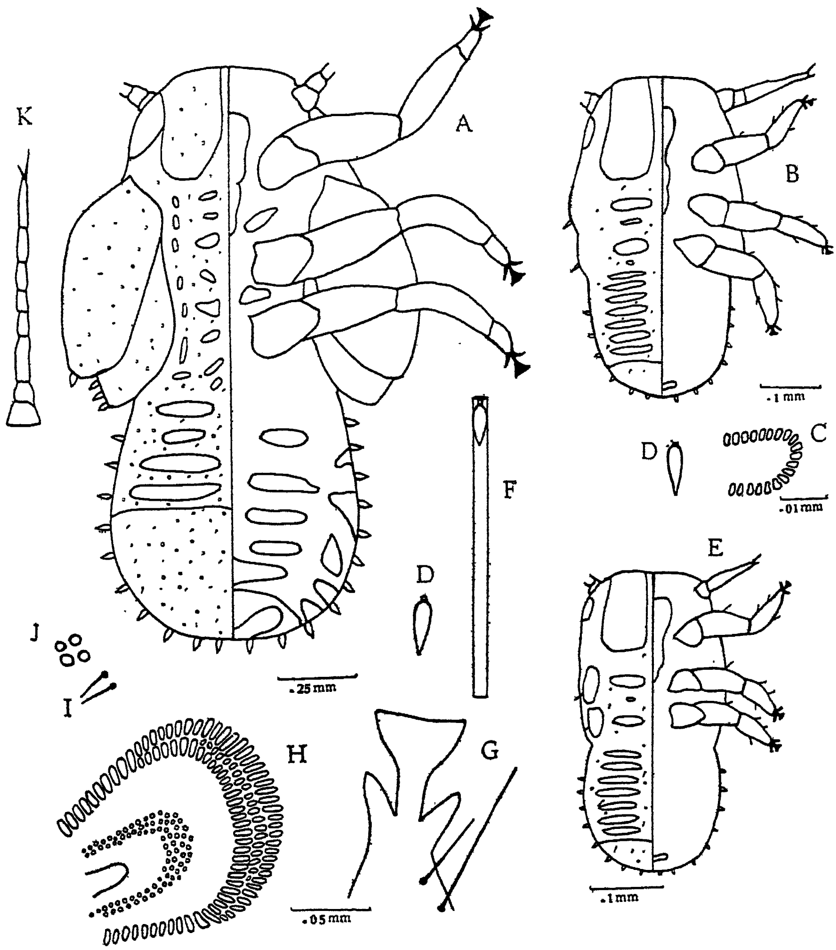

FIG. 3.—*Psyllopsis discrepans* Flor. A, fifth-stage nymph; B, second-stage nymph; C, portion of outer circumanal pore-ring of second-stage nymph; D, lanceolate setæ of second- and fifth-stage nymphs; E, first-stage nymph; F, lanceolate seta of fifth-stage nymph enclosed in a shaft of waxy substance; G, apex of tarsus of fifth-stage nymph; H, portion of circumanal pore-rings of fifth-stage nymph; I, ring-based setæ of fifth-stage nymph; J, round pores of fifth-stage nymph; K, antenna of fifth-stage nymph.

FIRST STAGE (fig. 3, E): Length, ·48 mm. Colour same as in second-stage nymph. Differs from it in possessing two-segmented antennæ and in absence of wing-pads, these being represented by two pairs of sclerotised plates each with a seta.

5. *Psyllia buxi* L.

The fifth-stage nymph of this species has been described by Ferris (1926); to this some additional notes and descriptions of the remaining four stages are now added.

Host.—Buxus sempervirens.

Locality.—Various places in and around Edinburgh.

FIFTH STAGE: Body colour deep yellow, with sclerotised parts brown, eyes deep red. Antennæ of nine segments, last three imbricate, third twice as long as others following. Inner ring of circumanal pores situated ventrally only, six-layered and individual pores round. FOURTH STAGE: Length, 1·5 mm. Differs from fifth-stage nymph in deeper brown colour of sclerotised parts and in having antennæ of five segments, third and fifth being twice as long as first and second together. THIRD STAGE (fig. 4, C): Length, ·97 mm. Body yellow, eyes red, sclerotised parts, including antennæ, wing-pads, and legs, brown. Differs from fourth-stage nymph in being more elongate, in possessing antennæ of three segments, third twice as long as first and second together, and in having smaller wing-pads with not more than three or four setæ near costal margin. Fig. 4, D, represents ring-based setæ, and fig. 4, E, a portion of the circumanal pore-rings. SECOND STAGE (fig. 4, F): Length, ·72 mm. Body colour same as in third-stage nymph. Differs from it in different arrangement of sclerotised plates on dorsum and their absence ventrally except a small apical area, in absence of wing-pads, these being represented by slight bulgings in thoracic region and by two pairs of sclerotised plates each with a seta, and in outer circumanal pore-ring (fig. 4, G) at extreme apex dorsally consisting of a single row of slit-like pores, ventrally, a short distance from apical margin, one-layered towards median line, three near sides; inner ring of pores round, one-layered. FIRST STAGE: Length, ·61 mm. Body, soon after hatching, pale yellow, eyes red, antennæ and legs pale white. Later, colour of body deepens into yellow with sclerotised parts brown. Differs from second-stage nymph in possessing two-segmented antennæ, in absence of any trace of wing-pads, and in having outer circumanal pore-ring one-layered at extreme apex of abdomen; inner ring absent.

6. *Psyllia melanoneura* Först.

Hosts.—Cratægus oxyacantha and other species of *Cratægus.*

Locality.—Various places in Edinburgh.

FIFTH STAGE (fig. 5, A): Length, 1·5 mm. Body yellowish green, eyes pinkish white, antennæ, legs, and wing-pads pale yellowish, antennal tips dark. *Form*: Psylline. Body fairly broad. Sclerotisation on dorsum includes a pair of ocular areas on head, two large and five smaller pairs on thorax, three pairs of transverse areas towards base of abdomen and its posterior two-thirds. Ventrally, a small apical part of abdomen and two pairs of submedian and three pairs of marginal areas are sclerotised. *Head* as broad as thorax, anterior margin with four setæ. Antennæ of eight segments, third and eighth longest and weakly jointed in middle. *Thorax*: Wing-pads large and oval. Costal margin devoid of setæ, apical ends of anterior pair having one and posterior pair two setæ each, surface studded with minute points. Legs somewhat slender, femora reaching body margin, trochanter absent, tibio-tarsal articulation distinct, pulvillus with two claws at base and a long spatulate seta (fig. 5, K). *Abdomen* broader than

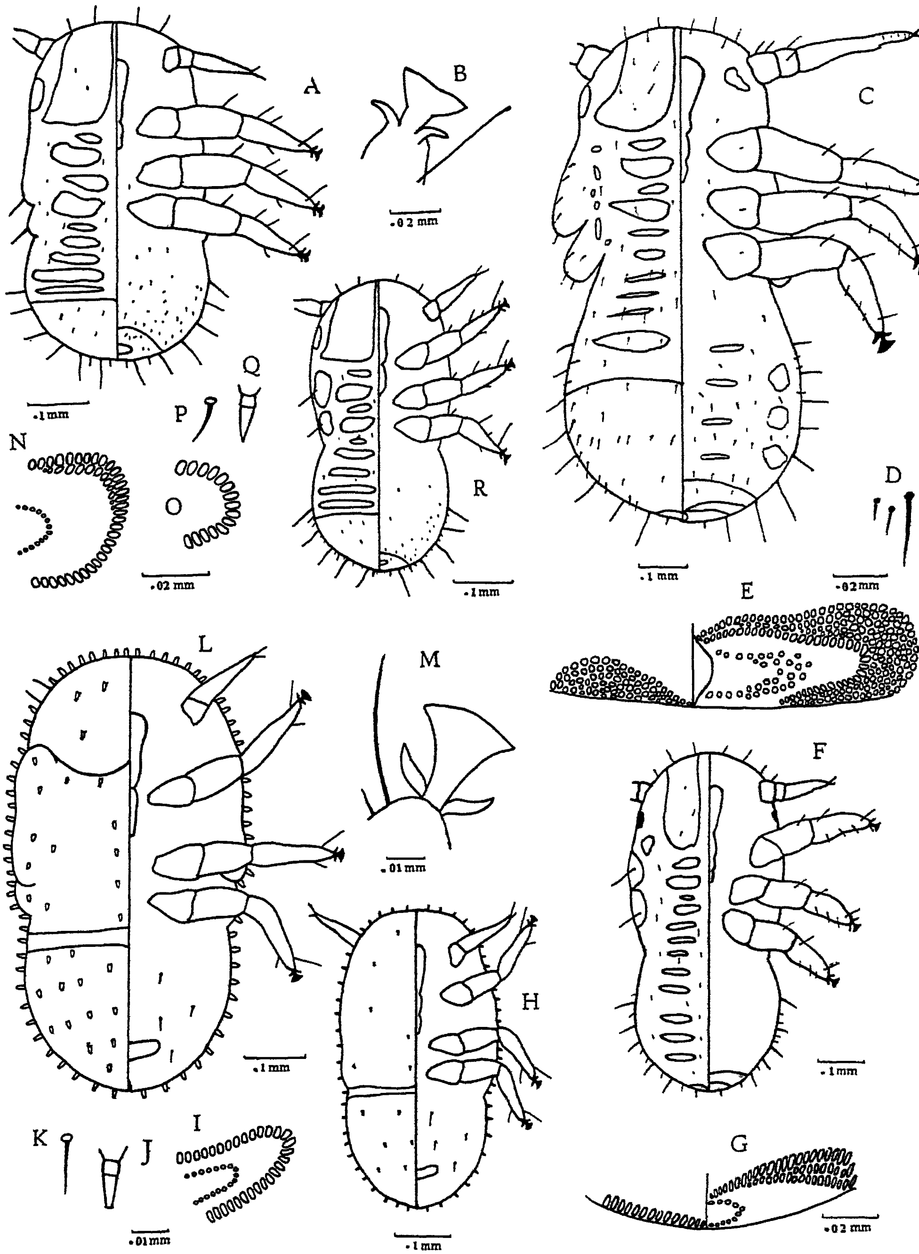

FIG. 4.—A, second-stage nymph of *Psyllia mali* Schmidb., race *mali*; B, apex of tarsus of the same; C, third-stage nymph of *P. buxi* L.; D, ring-based setæ of third-stage nymph of *P. buxi*; E, portion of circumanal pore-rings of third-stage nymph of *P. buxi*; F, second-stage nymph of *P. buxi*; G, portion of circumanal pore-rings of the same; H, first-stage nymph of *Trioza urticæ* L.; I, portion of circumanal pore-rings of second-stage nymph of *T. urticæ*; J, secta-seta of the same; K, ring-based seta of second-stage nymph of *T. urticæ*; L, second-stage nymph of *Trioza urticæ*; M, apex of tarsus of the same; N, portion of circumanal pore-rings of second-stage nymph of *Psyllia mali*, race *mali*; O, portion of outer circumanal pore-ring of first-stage nymph of *Psyllia mali*, race *mali*; P, ring-based seta of second-stage nymph of *Psyllia mali*, race *mali*; Q, secta-seta of second-stage nymph of the same; R, first-stage nymph of *Psyllia mali*, race *mali*.

head and thorax, broadest towards base, ventrally a number of bluntly pointed setæ (fig. 5, J) and numerous small pointed structures near margin; marginal setæ of two kinds, viz. six spear-shaped (fig. 5, I) and usually about ten long spatulate setæ (fig. 5, H) variously curved. Anal opening a short distance from

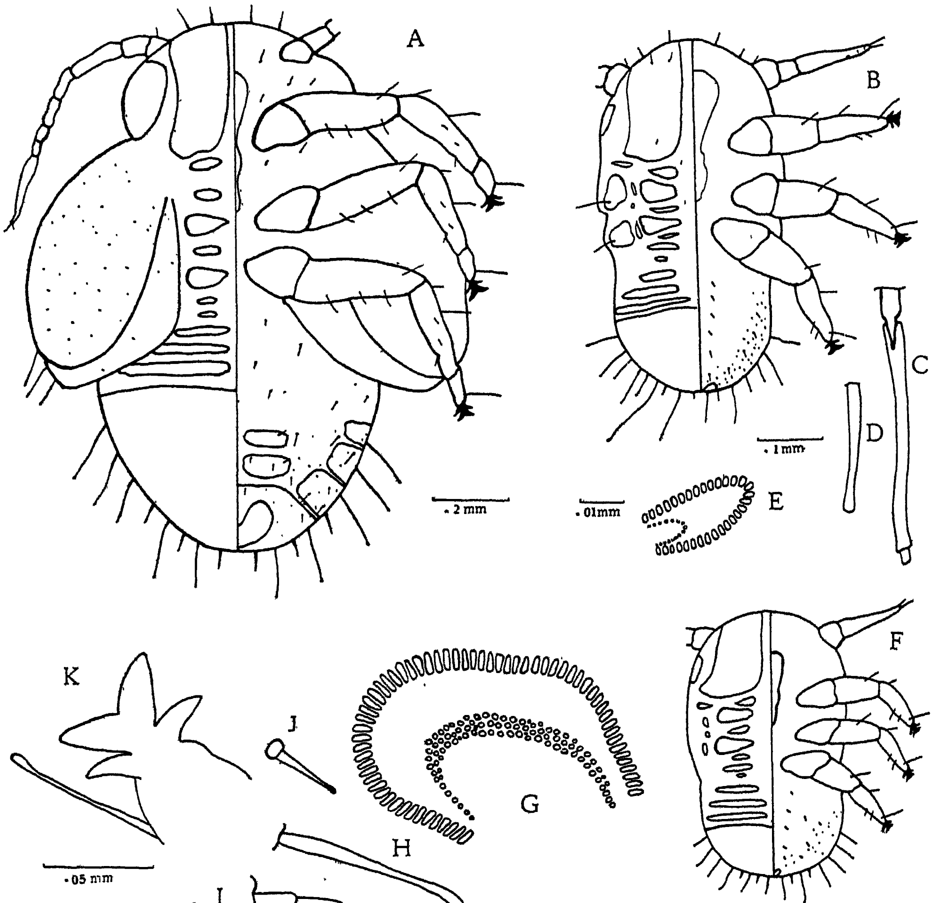

FIG. 5.—*Psyllia melanoneura* Först. A, fifth-stage nymph; B, second-stage nymph; C, spear-shaped seta of second-stage nymph enclosed in a shaft of waxy substance; D, spatulate seta of second-stage nymph; E, portion of circumanal pore-rings of second-stage nymph; F, first-stage nymph; G, portion of circumanal pore-rings of fifth-stage nymph; H, spatulate seta of fifth-stage nymph; I, spear-shaped seta of fifth-stage-nymph; J, bluntly pointed seta of fifth-stage nymph; K, apex of tarsus of fifth-stage nymph.

apex; outer circumanal pore-ring (fig. 5, G) one-layered, consisting of oval pores; inner ring mostly three- or four-layered, of round pores. FOURTH STAGE: Length, 1·1 mm. Body greenish yellow, eyes red, antennæ, wing-pads, and legs pale yellow, antennal tips dark. Differs from fifth-stage nymph in having five-segmented antennæ and in absence of tibio-tarsal articulation. THIRD STAGE: Length, ·88 mm. Body yellow, eyes red, antennæ, wing-pads, and legs pale yellow, antennal tips dark. Differs from fourth-stage nymph in its smaller

size and in having three-segmented antennæ. Second Stage (fig. 5, B): Length, ·51 mm. Colour as in third-stage nymph. Differs from it in absence of ventral sclerotisation and of wing-pads, these being represented by slight bulgings in thoracic region and two pairs of sclerotised plates each with a seta, and in inner circumanal pore-ring (fig. 5, E) being one-layered. A spear-shaped and a spatulate seta are represented in fig. 5, C and D. First Stage (fig. 5, F): Length, ·38 mm. Head and thorax yellowish white, abdomen yellow, eyes red, antennæ and legs pale white. Differs from second-stage nymph in having two-segmented antennæ, in absence of any trace of wing-pads and of inner ring of circumanal pores.

7. *Psyllia pyricola* Först.

The nymphs of this species were recently described by Klyver (1931). In the following account, therefore, only colour notes are given:—

Host.—*Pyrus communis.*

Locality.—Dalkeith; Royal Botanic Garden, Edinburgh.

Fifth Stage: Body yellow, eyes red; all sclerotised parts, including antennæ, wing-pads, and legs, brown, antennal tips black. General appearance of nymph dark brown. Fourth Stage: Head and abdomen pale yellow, thorax pale green, eyes red, antennæ, wing-pads, and legs pale whitish, antennal tips dark, sclerotised areas dark brown. General appearance of nymph brown. Third Stage: Colour as in previous-stage nymph. Second Stage: Head and thorax pale yellow, abdomen yellow, antennæ and legs pale white, antennal tips dark, eyes red, sclerotised areas not prominently pigmented. First Stage: Head and thorax pale yellow, abdomen deep yellow, eyes red, antennæ and legs whitish.

8. *Psyllia mali* Schmidb., race *mali.*

As has been shown in other papers (Lal, 1933 and 1934 *a*), *P. mali* Schmidb. has a biological race living on hawthorn, from which the apple form is distinguished as the race *mali*, appended to the specific name. Being an insect of economic importance, the nymphs of this species have been figured and described by, among others, Awati (1914), Brittain (1923), Minkiewicz (1927), and Speyer (1929). Recently the fifth-instar nymph has also been described by Klyver (1931), and the following descriptions of the remaining four stages, together with colour notes on the fifth stage, are referable to his account and figure:—

Hosts.—*Pyrus malus* and other species of Pyrus.

Locality.—Dalkeith; Royal Botanic Garden, Boghall, and garden, King's Buildings, Westmains Road, Edinburgh.

Fifth Stage: Body green, eyes pinkish white, antennæ, wing-pads, and legs pale white, antennal tips dark. Antennæ of eight segments, last longest (*cf.* Klyver). Fourth Stage: Length, 1·4 mm. Head and thorax yellowish green, abdomen green, eyes, antennæ, wing-pads, and legs pale white. Differs from fifth-stage nymph in possessing stouter, shorter antennæ of five segments, third as long as first and second together, fifth slightly longer than third, and in absence of tibio-tarsal articulation. Third Stage: Length, 1·0 mm. Body pale yellow,

eyes, sclerotised areas on head, and a small area round anal aperture brown; antennæ, wing-pads, and legs dull white, antennal tips dark. Differs from fourth-stage nymph in being more elongated in form, in having antennæ of three segments, third twice as long as first and second together, and in the spine-like structures being more concentrated towards margins of apical third of abdomen. SECOND STAGE (fig. 4, A): Length, ·77 mm. Body yellow, sclerotised parts including wing rudiments brown, eyes deep brown, legs and antennæ pale brownish. Differs from third-stage nymph in having slightly different arrangement of sclerotised areas on dorsum, absent ventrally except a small apical area on abdomen, in possessing rudimentary wing-pads with only one long seta at each apical end, and in outer ring of circumanal pores (fig. 4, N) being two- to three-layered near sides; inner ring single-layered. A ring-based seta and a secta-seta are represented in fig. 4, P and Q, and the apex of the tarsus in fig. 4, B. FIRST STAGE (fig. 4, R): Length, ·41 mm. Body yellow, eyes red, sclerotised areas brown, antennæ and legs pale yellow. Differs from second-stage nymph in complete absence of wing-pads, their places being taken by two pairs of sclerotised plates each with a seta, and in outer circumanal pore-ring (fig. 4, O) being single-layered; inner ring absent.

8*a*. *Psyllia mali* Schmidb., race *peregrina*.

This race has been known so far under the name of *Psyllia peregrina* Först. As shown in other papers (Lal, 1933 and 1934 *a*), its adults are morphologically indistinguishable from those of *P. mali* on apple. The nymphs of the two, however, differ in some particulars, and the last-instar nymph of the hawthorn race has a brown longitudinal streak on each of its wing-pads, which was also mentioned by Löw (1879) and Scott (1880) in their descriptions of the last-instar nymphs of *Psyllia peregrina* Först.

Hosts.—*Cratægus oxyacantha* and other species of *Cratægus*.

Locality.—Various places in Edinburgh.

FIFTH STAGE (fig. 6, A): Length, 2·3 mm. Body yellowish green, eyes brown, antennæ and legs pale brownish, wing-pads pale yellow with longitudinal streak deep brown. *Form*: Psylline. Sclerotisation on dorsum includes a pair of ocular areas on head, four pairs of small, each followed by a pair of smaller areas on thorax, three pairs of transverse narrow areas on anterior third of abdomen and its entire posterior two-thirds. Ventrally, posterior third of abdomen, together with three pairs of submedian and four pairs of marginal areas, are sclerotised. Dorsum with numerous round tubercles, especially abundant on abdomen. *Head*: anterior margin with a few setæ. Antennæ as long as width of head, of eight segments, last about twice as long as each of the preceding. *Thorax* narrower than abdomen. Wing-pads broadly oval with a brown longitudinal streak running down the middle of each; surface with numerous stout, minute, pointed structures; costal margin devoid of setæ except apices, which have one each. Legs with femora just reaching body margin, trochanter absent, tibio-tarsal articulation distinct, empodium fish-tail shaped with two claws at base and a long spatulate setæ (fig. 6, E). *Abdomen* with numerous stout, pointed structures in apical half and a few ring-based setæ on ventral side; apical third scalloped, extremity of each scallop with a long spatulate seta, usually sixteen in number. Anal opening a short distance from apex, outer circumanal ring (fig. 6, F) consisting of oval slit-like pores in two layers (three or four layers for

a very short distance), inner ring of small round pores, six- or seven-layered. FOURTH STAGE: Length, 1·5 mm. Body yellowish green, eyes red, antennæ, wing-pads, and legs brownish yellow, sclerotised areas brown. Differs from

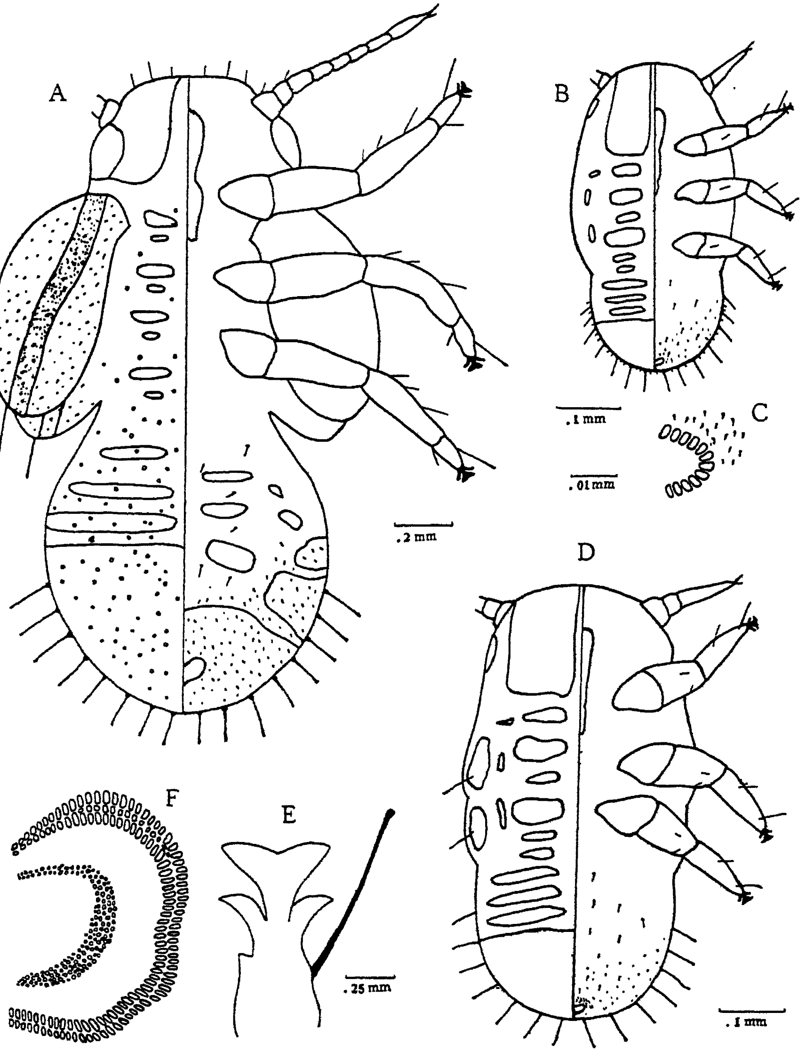

FIG. 6.—*Psyllia mali* Schmidb., race *peregrina*. A, fifth-stage nymph; B, first-stage nymph; C, portion of outer circumanal pore-ring of first-stage nymph, with stout-pointed structures surrounding it; D, second-stage nymph; E, apex of tarsus of fifth-stage nymph; F, portion of circumanal pore-rings of fifth-stage nymph.

fourth-stage nymph in having antennæ of five segments, in absence of brown streaks on wing-pads and of tibio-tarsal articulation. THIRD STAGE: Length, ·90 mm. Body greenish yellow, eyes red, antennæ, wing-pads, and legs pale yellow, antennal tips dark, sclerotised areas brown. Differs from fourth-stage nymph in having three-segmented antennæ and in outer circumanal pore-ring

being two-layered; inner ring indistinct. SECOND STAGE (fig. 6, D): Length, ·61 mm. Body yellow, eyes red, antennæ and legs brownish yellow, sclerotised areas brown. Differs from third-stage nymph in its different arrangement of sclerotised plates on dorsum, absent ventrally, in wing-pads being represented by bulgings in thoracic region and by two pairs of sclerotised plates each with a seta, in absence of round tubercles on dorsum, and in outer circumanal pore-ring being single-layered; inner ring absent. FIRST STAGE (fig. 6, B): Length, ·42 mm. Body deep yellow, eyes red, sclerotised areas brown, legs and antennæ brownish yellow. Different from second-stage nymph in having two-segmented antennæ, in complete absence of wing-pads, and in being fringed with minute, pointed structures round abdominal margin. The pores of the outer circumanal ring are illustrated in fig. 6, C.

9. *Psyllia ambigua* Först.

Hosts.—*Salix caprea* and other species of *Salix*.

Locality.—Boghall and Liberton, Edinburgh.

FIFTH STAGE (fig. 7, A): Length, 1·8 mm. Head and thorax green, eyes red, abdomen yellowish green, sclerotised areas blackish brown, antennæ, wing-pads, and legs pale yellowish, antennal tips dark. *Form*: Psylline. Body fairly broad. Sclerotisation on dorsum includes a pair of ocular areas on head, a number of smaller areas on thorax, four pairs of narrow transverse areas on anterior abdomen, and its entire posterior half except along median line. Ventrally, two pairs of submedian and two pairs of contiguous marginal areas on abdomen and its apical half are sclerotised. *Head* narrower than thorax, anterior margin with a few setæ. Antennæ of seven segments, third and seventh more than twice as long as others and weakly jointed in middle. *Thorax* narrower than abdomen. Wing-pads large, oval, studded with many stout, pointed structures (fig. 7, C), costal margin with a number of simple setæ (fig. 7, E). Legs not very long, trochanter absent, tibio-tarsal articulation distinct, empodium petiolate with two claws at base and two long spatulate setæ (fig. 7, F). *Abdomen* broadest about middle, covered with numerous small, stout, pointed structures and dorsally with long, ring-based setæ, margin surrounded by long simple setæ. Anal opening a short distance from apex, outer and inner circumanal pore-rings (fig. 7, D) one-layered, former composed of oval slit-like pores, latter of round pores. FOURTH STAGE: Length, 1·1 mm. Colour same as in fifth-stage nymph. Differs from it in possessing antennæ of five segments, third and fifth twice as long as others and weakly jointed in middle, in having fewer setæ on costal margin, and in absence of tibio-tarsal articulation. THIRD STAGE (fig. 7, I): Length, ·81 mm. Body yellow, eyes red; sclerotised parts, including antennæ, wing-pads, and legs, deep brown, tips of antennæ and legs dark. Differs from fourth-stage nymph in possessing three-segmented antennæ, in its smaller wing-pads, and in having fewer setæ on abdominal dorsum. SECOND STAGE (fig. 7, B): Length, ·55 mm. Body pale yellow, eyes red; sclerotised parts, including legs, brown; antennæ brownish yellow, tips dark. Differs from third-stage nymph in slightly different arrangement of sclerotised plates on dorsum and their absence ventrally except for a small apical area on abdomen, and in its rudimentary wing-pads each with an apical seta. Fig. 7, J, represents a ring-based seta. FIRST STAGE (fig. 7, G): Length, ·32 mm. Head and thorax pale yellow, abdomen orange, eyes red; sclerotised parts, including antennæ and legs, brown. Differs from second-stage nymph in absence of wing-rudiments, these being represented by two pairs of sclerotised plates in thoracic region, and in absence of inner circumanal pore-ring (fig. 7, H).

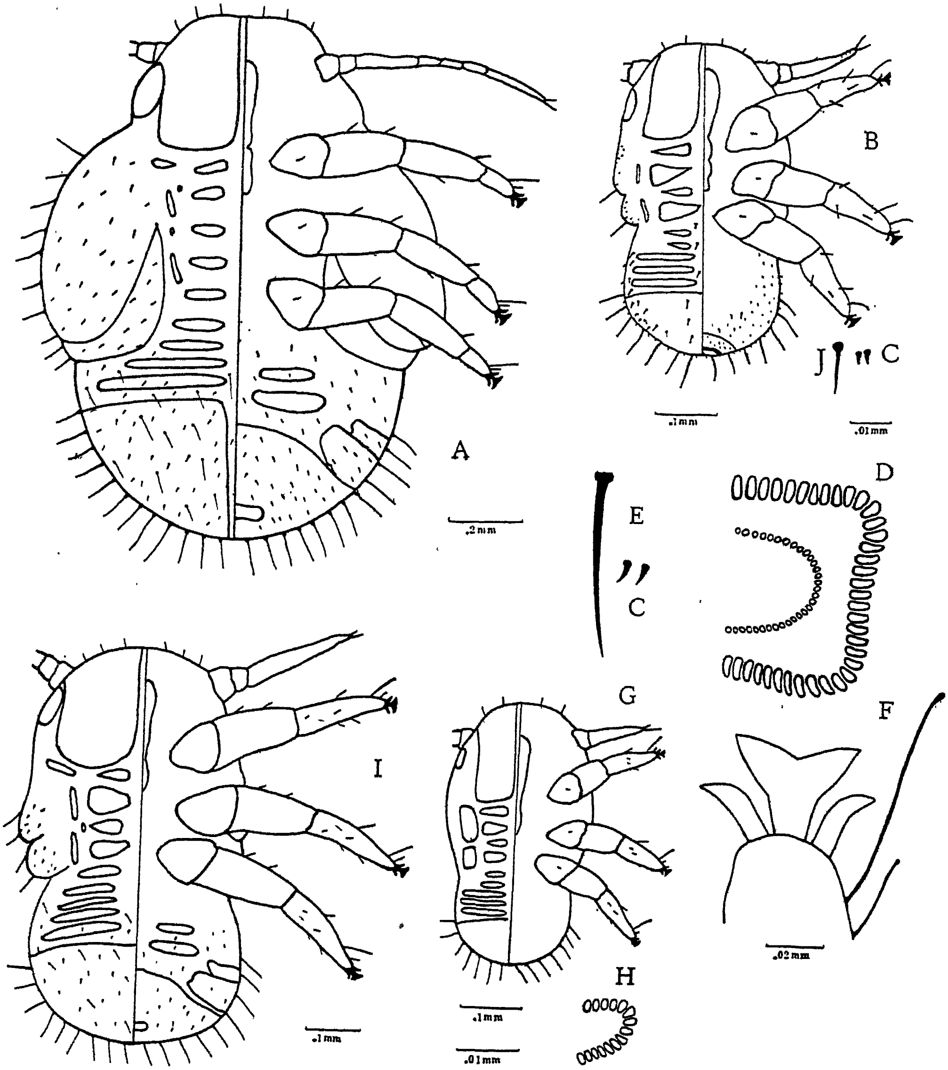

FIG. 7.—*Psyllia ambigua* Först. A, fifth-stage nymph; B, second-stage nymph; C, stout, pointed structures present on wing-pads and abdomen of second- and fifth-stage nymphs; D, portion of circumanal pore-rings of fifth-stage nymph; E, simple seta (marginal) of fifth-stage nymph; F, apex of tarsus of the same; G, first-stage nymph; H, portion of outer circumanal pore-ring of first-stage nymph; I, third-stage nymph; J, ring-based seta of second-stage nymph.

10. *Trioza urticæ* L.

The fifth-stage nymph of this species has been described by Ferris (1925). The following descriptions of the first four instars and colour notes on the fifth instar are given with reference to his account and

figures, which should be consulted. The nymphs of this species are very variable in colour.

Host.—*Urtica dioica.*

Locality.—Various places in and around Edinburgh.

Fifth Stage: Body pale to deep yellow, with grey-brown or deep brown markings on dorsum and wing-pads, especially near sides; eyes brownish. Fourth Stage: Length, 1·2 mm. Colour same as in fifth-stage nymph. Differs from it in being more elongate and in absence of tibio-tarsal articulation. Third Stage: Length, 1·0 mm. Colour as in previous stages. Differs from fourth-stage nymph in its smaller size and in possessing two-segmented antennæ. Second Stage (fig. 4, L): Length, ·74 mm. Body pale yellow, eyes red, antennæ, wing-pads, and legs pale whitish. Differs from third-stage nymph in absence of sclerotisation ventrally and in weak development of wing-pads. Other structures of the second-stage nymph are illustrated in fig. 4, I, J, K, and M. First Stage (fig. 4, H): Length, ·58 mm. Colour as in second-stage nymph. Differs from it in absence of wing-pads.

11. *Aphalara nebulosa* Zett.

Nymphs of the first and the last two instars were alone available and are here described. It is believed that the nymph of one instar—namely, the third—is missing.

Host.—*Epilobium angustifolium.*

Locality.—Dalkeith, Galashiels, Edinburgh.

Fifth Stage (fig. 8, A): Length 2·2 mm. Body yellow, eyes deep pink, antennæ, wing-pads, and legs brown, sclerotised areas dark brown. *Form*: Pauropsylline. Sclerotisation on dorsum extends to a pair of broad areas on head, two pairs of large and a number of smaller pairs of areas on thorax, three pairs of transverse areas on anterior abdomen and to its entire posterior half. Ventrally, apical fourth of abdomen, three median, three pairs of submedian and of marginal areas are sclerotised. Dorsum thickly covered with small, irregularly rounded tubercles (fig. 8, E), those on abdomen radiating from three pairs of submedian and three or four pairs of marginal areas, which in turn are composed of groups of smaller oval areas up to a maximum of seven. Pigmentation immediately around these areas darker, and sclerotisation heavier. Such areas also present on head and thorax but not so well marked. Small ring-based setæ sparsely scattered all over body. *Head* narrower than thorax with a few setæ anteriorly. Antennæ shorter than width of head, of three segments, two small basal and a long distal one. *Thorax* about as broad as abdomen. Wing-pads triangular, bluntly pointed at apex, broadest at base, projecting but little from general contour of body, costal margin with a few minute setæ. Legs with femora not reaching body margin, trochanter absent, tibio-tarsal articulation distinct, empodium (fig. 8, D) relatively small with two claws at base. *Abdomen* rounded, width uniform, posterior half with a number of small dagger-shaped setæ (fig. 8, C) marginally. Anal opening a short distance from apex, outer circumanal pore-ring (fig. 8, B) composed of two to five layers of oval pores, innermost layer elongate and slit-like, inner ring also composed of two to five layers of round, somewhat irregularly arranged pores. Fourth Stage: Length, 1·4 mm. Colour as in fifth-stage nymph. Differs from it in its two-segmented antennæ and in absence of tibio-tarsal articulation. Second Stage (fig. 8, F):

Length, ·68 mm. Body dull yellow, sclerotised parts including wing-pads brown, eyes red, antennæ and legs light brown. Differs from fourth-stage nymph in

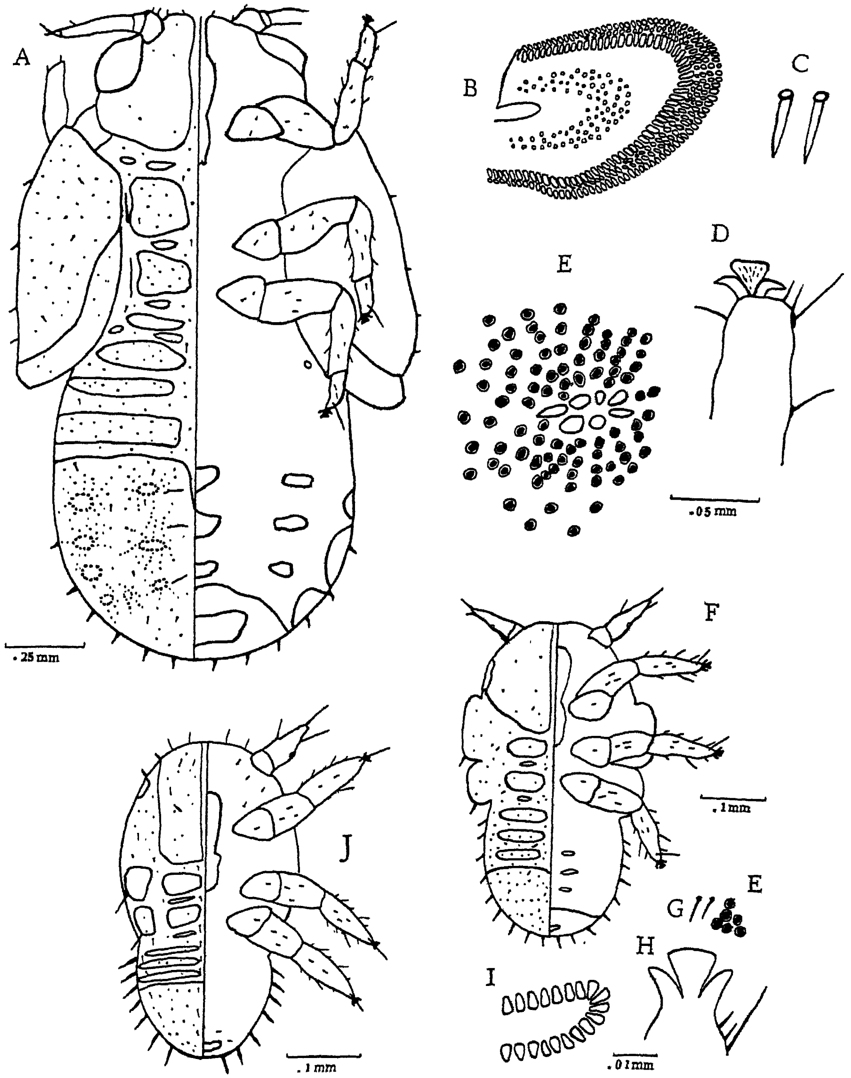

FIG. 8.—*Aphalara nebulosa* Zett. A, fifth-stage nymph; B, portion of circumanal pore-rings of fifth-stage nymph; C, dagger-shaped setæ of fifth-stage nymph; D, apex of tarsus of fifth-stage nymph; E, round tubercles radiating from a group of oval areas on dorsum; F, second-stage nymph; G, ring-based setæ of second-stage nymph; H, apex of tarsus of second-stage nymph; I, portion of outer circumanal pore-ring of second-stage nymph; J, first-stage nymph.

having few sclerotised plates on dorsum, absent ventrally except three pairs of small submedian and a small apical area on abdomen, in absence of any radiating arrangement of tubercles on dorsum, in each wing-pad possessing only one seta at apex, and in having outer circumanal pore-ring (fig. 8, I) single-layered

and inner ring absent. Fig. 8, G and H, represent ring-based setæ and apex of tarsus respectively. First Stage (fig. 8, J): Length, ·36 mm. Body yellow, eyes red, antennæ and legs pale yellow. Differs from second-stage nymph in complete absence of sclerotisation ventrally and in absence of wing-pads, these being represented by two pairs of sclerotised plates each with a seta.

12. *Eurhinocola eucalypti* Mask.

This species was first reported from New Zealand as *Rhinocola eucalypti*; later, the generic name was changed by Petty (1925) to *Eurhinocola*. Besides England it has also been recorded from Australia and South Africa. All the stages occurred throughout the year from egg to adult in England, where the insect was regarded as a pest unlike in New Zealand, where it was said by Maskell (1889) to content itself with feeding on a "white aromatic gummy matter" exuded by the leaves of the host-plant and to cause no damage. An account of the biology of the insect has been given by Fox Wilson (1924), from which the following is taken:—

"Larvæ are active and live gregariously surrounded by masses of thin cottony threads. Appearance yellowish, though they possess light purplish areas. Nymphs active, live together in colonies, surrounded by light mealy covering. The general appearance is dark on account of broad, dark purplish areas on the head."

The occurrence in England of this insect may be regarded as sporadic and localised, as it has not been heard of again since 1924. Short descriptions of the immature stages of this species were also given by Maskell (1889).

The material, which was kindly supplied by Mr G. Fox Wilson, contains nymphs of the last and three previous stages. On the assumption that this species, like other Psyllidæ, has five instars, the first-stage nymph must be taken as missing. The rest are described below.

Hosts.—*Eucalyptus globulus* and *E. cordata.*

Locality.—In England: Felixstowe, Leamington, Handcross (Sussex).

Fifth Stage (fig. 9, A): Length, 1·2 mm. *Form*: Psylline. Sclerotisation on dorsum includes a pair of ocular areas on head, two large pairs on thorax, three pairs of transverse areas on abdomen, and its entire posterior two-thirds. Ventrally, two pairs of small areas at bases of antennæ, one pair next to eyes; a number of median, marginal, and apical areas on abdomen are sclerotised. Dorsum studded with small ring-based setæ. *Head* as broad as thorax; antennæ of nine segments, third weakly jointed in middle, last longest. Wing-pads long, oval, devoid of setæ. Legs long and stout, trochanter absent, tibiæ markedly swollen towards apices, tibio-tarsal articulation distinct, tarsal extremity (fig. 9, C) with two claws and a cushion-like rudimentary empodium, with a pair of simple setæ between it and each claw. *Abdomen* spherical about middle, marginally notched inwards on both sides a short distance from apex, margin invested by

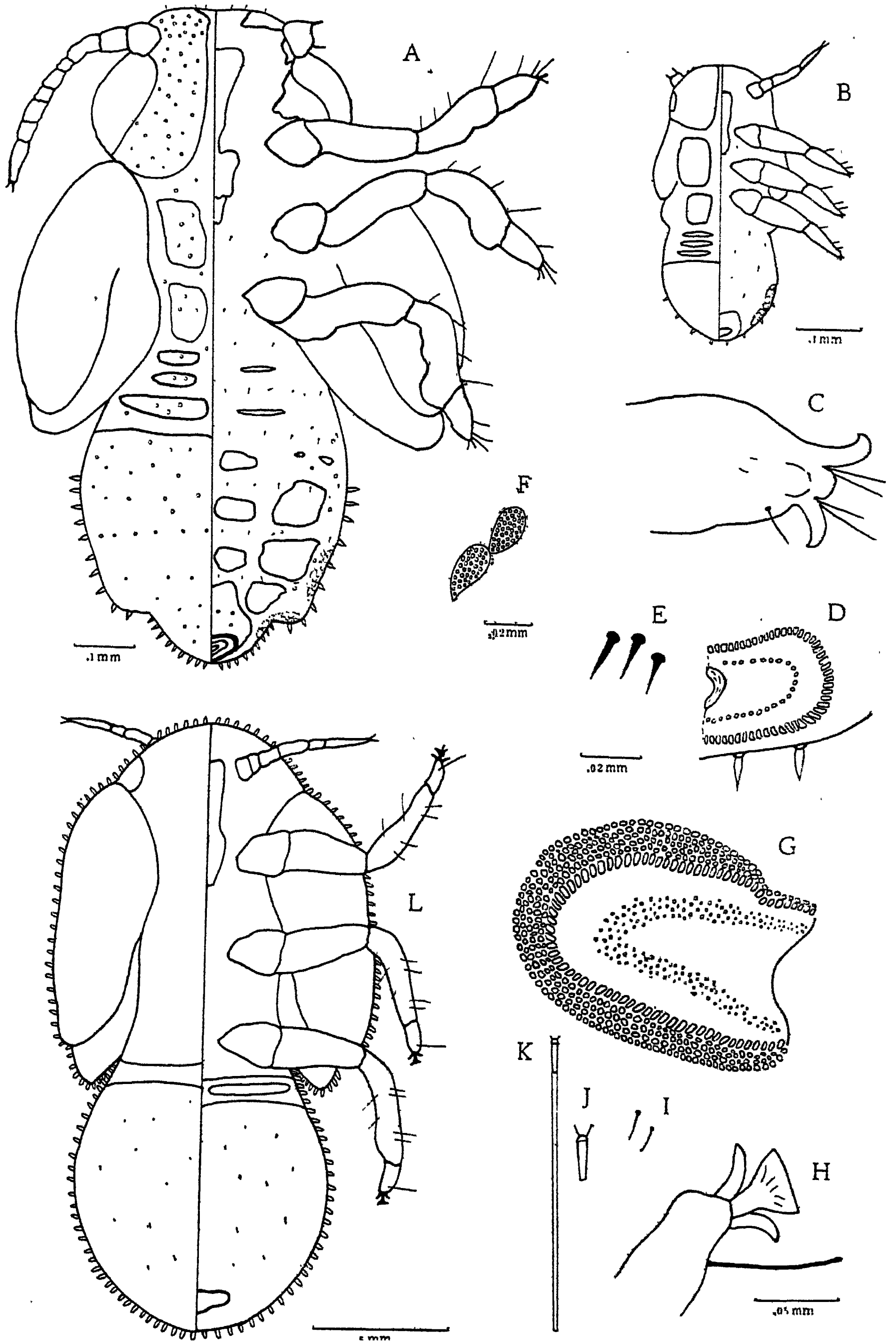

FIG. 9.—*Eurhinocola eucalypti* Mask. A, fifth-stage nymph; B, second-stage nymph; C, apex of tarsus of fifth-stage nymph; D, portion of circumanal pore-rings of fifth-stage nymph and two lanceolate setæ on abdominal margin; E, ring-based setæ of fifth-stage nymph; F, marginal longitudinal areas of groups of pores present on ventral abdomen of fifth-stage nymph. *Trichopsylla walkeri* Thoms. G, portion of circumanal pore-rings of last-stage nymph; H, apex of tarsus; I, ring-based setæ; J, secta-seta; K, secta-seta enclosed in a shaft of waxy substance; L, last-stage nymph.

small lanceolate setæ, ventrally two marginal longitudinal areas (fig. 9, F) present on each side in posterior third of abdomen, consisting of small groups of pores, about thirty in each area. Anal opening near apex, outer and inner circumanal pore-rings (fig. 9, D) one-layered, pores in former oval, in latter round. FOURTH STAGE: Length, ·90 mm. Differs from fifth-stage nymph in its shorter seven-segmented antennæ, in absence of swellings at apices of tibiæ and of tibio-tarsal articulation, and in abdominal margin towards apex being less notched. THIRD STAGE: Length, ·62 mm. Differs from fourth-stage nymph in abdomen being uniformly heart-shaped, in antennæ being four-segmented, and in smaller number of pore-groups on ventral margins, about ten in each area. SECOND STAGE (fig. 9, B): Length, ·48 mm. Differs from third-stage nymph in absence of ventral sclerotisation, except a small apical area of round tubercles on dorsum, and in smaller number of pore-groups, these being four and three in anterior and posterior areas respectively.

13. *Trichopsylla walkeri* Thom.

Material kindly supplied by Mr G. Fox Wilson, and the nymph herein described is of the last instar. Nymphs that are destined to be males differ from those which transform to females in possessing longer abdomens.

Hosts.—*Rhamnus catharticus*. Recorded by Edwards (1896) as having been found also on *Prunus spinosa* and *Euonymus europæus* in various parts of England.

Locality.—Wisley, England.

LAST STAGE (fig. 9, L): Length, 2·4 mm. Body colour in alcohol-preserved specimens dirty yellow, eyes deep brown. *Form*: Triozine. Body elongated. Sclerotisation extends to entire dorsum and venter except a narrow transverse band of abdomen. Body margin uniformly surrounded by closely spaced secta-setæ (fig. 9, J). *Head* slightly narrower than abdomen. Antennæ arising nearer median than marginal line, of six segments, last longest. Wing-pads as long as head and thorax together, produced forwards towards head, broadest in basal one-third. Legs with femora just reaching wing-margins, trochanter absent, hind tibiæ slightly swollen towards apices, tibio-tarsal articulation distinct, tarsal extremity (fig. 9, H) with two claws and petiolate empodium. Abdomen spherical, with a few minute setæ. Anal opening a short distance from apex, outer circumanal pore-ring (fig. 9, G) five-layered near sides, two-layered in anterior median, and four-layered in posterior median region; the innermost row of pores oval, rest pentagonal with rounded edges; inner ring also many-layered with individual pores round.

SUMMARY.

A sound working knowledge of the Psyllidæ or jumping plant lice is based, not only upon the characters which serve to distinguish the adults of the species, but also upon those which serve to distinguish the nymphs. Accurate descriptions of the latter are all the more important, since in those species which are harmful to their host-plants it is the nymphs, rarely the adults, which are injurious.

Descriptions of the nymphs, if they are to be serviceable, must be both accurate and detailed, since species that may be readily distinguishable as adults frequently have nymphs that closely resemble each other. Conversely, species the adults of which appear to be closely allied often display marked nymphal differences.

The characters on which the differentiation of the nymphs mainly rests are:

1. General form, of which three types, Triozine, Psylline, and Pauropsylline, are distinguished according to the shape of the wing-pads and the degree of their projection from the body margin.
2. General colour scheme.
3. Size and distribution of sclerotised areas on the surface of the body.
4. Physical and chemical character of the secretions of the wax glands.
5. Kinds of investing integumentary setæ of which eight are distinguished.
6. Arrangement and number of rows of pores of the circumanal ring, their size and shape.

In the metamorphosis of the Psyllidæ there are normally five pre-imaginal or nymphal instars. In each of ten species, including the two races *mali* and *peregrina* of the apple sucker, *Psyllia mali*, five nymphal stages have been examined, described, and illustrated. Of three other species discussed the third nymphal stage of *Aphalara nebulosa* is lacking, and likewise the first nymphal instar of *Eurhinocola eucalypti*; of *Trichopsylla walkeri* only the fifth nymphal instar is described.

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XXII.—Some Distributions associated with a Randomly Arranged Set of Numbers.

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(MS. received February 18, 1937. Read May 3, 1937.)

CONTENTS.

	PAGE
1. THE CASE OF THE LIMITED SERIES	332
1.1. Statement of Problem	332
1.2. Relationship between p_{nr} and x_{nr}	334
1.3. Distributions of "Runs" and "Gaps"	336
1.4. The Function f_{nr}	337
1.5. The Generating Function for f_{nr}.	341
1.6. The Values of $f_{n,\ n-\sigma}$, $p_{n,\ n-\sigma}$ and $x_{n,\ n-\sigma}$	346
1.7. The Generating Functions for $f_{n,\ n-\sigma}$, $p_{n,\ n-\sigma}$ and $x_{n,\ n-\sigma}$	348
2. THE CASE OF THE LIMITED SERIES	352
2.1. Statement of Problem	352
2.2. Distribution of "Runs"	352
2.3. Distribution of "Gaps"	356
3. CYCLIC ARRANGEMENTS	361
3.1. Distribution of Types of Arrangements ($\bar{P}_{mr}$) and the Associated Generating Function	361
3.2. The Generating Function for $\bar{P}_{m,\ m-\sigma}$.	364
3.3. Some Identities	366
3.4. Distribution of "Runs"	370
3.5. Distribution of "Gaps"	371
3.6. General Expression for $\bar{x}^{m}_{nr}$	374

1. THE CASE OF THE LIMITED SERIES.

1.1. *Statement of Problem.*

IN the course of a search for criteria which would assist in deciding whether a particular series of numbers was randomly arranged, it was found necessary to investigate some of the more obvious characters of a random series of numbers. If the series is an infinite one and all the numbers are unequal (the series of decimals between 0 and 1 selected in some random fashion may be considered) certain of the numbers will be maximal, that is greater than their immediate neighbours, whilst others will be minimal, that is less than their immediate neighbours. Every maximal number is succeeded by a run down, that is by a series of descending numbers ending in the succeeding minimal number. The length of a run down is the number of numbers it contains, including the maximal number which begins it and the minimal number which

ends it. The length of a run up is similarly defined. Clearly the minimum length of a run is 2.

If we take at random any sequence of r (successive) runs, these may contain $r+1$, $r+2$ or any higher number of numbers (as in the case of a single run we count both the first and last points, the points common to 2 successive runs are only counted once). By selecting m runs at random and taking each of these runs, together with its succeeding $r-1$ runs, as a random sequence of r runs we find that a certain fraction of them, q_{nr}, contain n points. If now we make m larger and larger the fractions q_{nr} will tend to statistical limits which we may denote by x_{nr}. We may speak of x_{nr} as the chance that a sequence of r runs will contain n numbers. The general problem is to find the values of x_{nr} for all possible values of n and r $(n > r > 0)$. It may be noted that x_{n1} is the chance that any run taken at random is of length n, whilst x_{n2} is the chance that any 2 successive runs should contain n points. Clearly any 2 successive runs extend either from a maximal point to its next succeeding maximal point, or from a minimal point to its next succeeding minimal point. In a previous paper (Kermack and McKendrick, 1937) we have used the term "gap" to denote a sequence of 2 runs, thus x_{n2} is the chance that a gap shall be of length n. We have also shown in the same paper that x_{n1} is given by

$$x_{n1} = \tfrac{3}{2}\Delta^2\frac{1}{n!} = \tfrac{3}{2}\left(\frac{1}{n!} - \frac{2}{(n+1)!} + \frac{1}{(n+2)!}\right) = 3\frac{n^2+n-1}{(n+2)!}. \qquad (1.11)$$

It will be found convenient to discuss the problem of evaluating x_{nr} in association with the following related problem. Let us choose at random any ν successive numbers from the infinite series of numbers randomly arranged as above. This series of ν numbers may contain ρ runs, where $\rho = \nu-1, \nu-2, \ldots 1$. By choosing m such groups of ν successive numbers we obtain a fraction of $q'_{\nu\rho}$ containing ρ runs, and as m increases $q'_{\nu\rho}$ will tend to a statistical limit $p_{\nu\rho}$. We may speak of $p_{\nu\rho}$ as the chance that ν successive numbers will contain ρ runs. The problem is to evaluate $p_{\nu\rho}$ for all possible values of ν and ρ, $(\nu > \rho > 0)$.

This latter problem is evidently very closely connected with that discussed by André in a series of papers (see also Netto (1901), pp. 106–116). André discusses the properties of certain integer numbers P_{nr}, the number of ways in which n unequal numbers (say the first n natural numbers) can be arranged so as to contain r runs. As the total number of different ways of arranging n unequal numbers is $n!$ it is easily seen that

$$p_{\nu\rho} = \frac{P_{\nu\rho}}{\sum_\rho P_{\nu\rho}} = \frac{P_{\nu\rho}}{\nu!}. \qquad (1.12)$$

André has shown, *inter alia*, that

$$\mathrm{P}_{n1}=2, \qquad \mathrm{P}_{n2}=2^n-4, \qquad (1.13)$$

whilst $\mathrm{P}_{n,\,n-1}$ is the coefficient of $\frac{z^n}{n!}$ in the expansion of

$$2\tan\left(\frac{\pi}{4}+\frac{z}{2}\right) \qquad (1.14)$$

whilst $\mathrm{P}_{n,\,n-2}$ is the coefficient of $\frac{z^n}{n!}$ in the expansion of

$$\frac{2(1-2\cos z)}{1-\sin z}. \qquad (1.15)$$

He does not seem to have obtained general expressions for P_{nr}, but proves that these numbers satisfy the difference equation

$$\mathrm{P}_{n+1,\,r}=r\mathrm{P}_{nr}+2\mathrm{P}_{n,\,r-1}+(n+1-r)\mathrm{P}_{n,\,r-2}, \qquad (1.16)$$

and that

$$\sum_{1}^{n-1} r\mathrm{P}_{nr}=\frac{2n-1}{3}n!. \qquad (1.17)$$

It follows that

$$(n+1)p_{n+1,\,r}=rp_{nr}+2p_{n,\,r-1}+(n+1-r)p_{n,\,r-2}. \qquad (1.18)$$

Reference may also be made to the related problem of Simon Newcomb discussed fully by MacMahon (1915, pp. 187–214). We shall not quote here any of the many interesting results obtained by MacMahon in his investigation of this problem, as they do not immediately concern us, and are readily accessible in his book.

1.2. *The Relationship between p_{nr} and x_{nr}.*

We shall now establish the following relationship between p_{nr} and x_{nr},

$$\underset{r}{\Delta_2} x_{nr}=\tfrac{3}{2}\underset{n}{\Delta^2} p_{nr} \qquad (1.21)$$

where

$$\underset{n}{\Delta^2} p_{nr}\equiv p_{n+2,\,r}-2p_{n+1,\,r}+p_{nr}$$

is the second *upper* difference of p_{nr} with respect to n, and

$$\underset{r}{\Delta_2} x_{nr}=x_{nr}-2x_{n,\,r-1}+x_{n,\,r-2}$$

is the second *lower* difference of x_{nr} with respect to r.

Consider the n consecutive numbers of the series $a_1, a_2, a_3, \ldots a_n$, beginning with an arbitrarily chosen number a_1. The chance that this series contains r runs is p_{nr}. Let a_{n+1} be the number which follows a_n, and let a_0 be the number which precedes a_1. We shall denote by y_{nr}

the probability that a_n is the rth maximal or minimal number following the arbitrarily chosen number a_1.

The probability that the series $a_1, a_2, \ldots a_{n+1}$ contains r runs is evidently $p_{n+1, r}$. Such a series is derived from a series $a_1, a_2, \ldots a_n$ by the addition of one number a_{n+1}, and can have r runs, only if the series $a_1, a_2, \ldots a_n$ contains either (i) $r-1$ runs or (ii) r runs.

(i) A series $a_1, a_2, \ldots a_n$ containing $r-1$ runs will give rise to a series $a_1, a_2, \ldots a_{n+1}$ containing r runs, only when the number a_n is maximal or minimal; and the chance of this is $y_{n, r-1}$. The contribution to $p_{n+1, r}$ is therefore $y_{n, r-1}$.

(ii) A series $a_1; a_2, \ldots a_n$ containing r runs gives by addition of a_{n+1} a series still containing r runs except in the case where a_n is either maximal or minimal. Thus the contribution to $p_{n+1, r}$ from (ii) is $p_{nr} - y_{nr}$.

Whence

$$p_{n+1, r} = y_{n, r-1} + (p_{nr} - y_{nr})$$

or

$$\underset{n}{\Delta^1} p_{nr} = -\underset{r}{\Delta_1} y_{nr}, \quad . \quad . \quad . \quad . \quad . \quad (1.22)$$

where $\underset{n}{\Delta^1}$ is the first *upper* difference with respect to n, and $\underset{r}{\Delta_1}$ is the first *lower* difference with respect to r.

Let c be the chance that any arbitrarily chosen number is either maximal or minimal. The chance that a_1 is maximal or minimal and that a_n is the rth succeeding maximum or minimum is clearly the chance c_1—that a_1 is maximal or minimal—multiplied by x_{nr}—the chance that the r runs immediately succeeding an arbitrarily chosen maximal or minimal number should contain in all n numbers.

Let us now consider the probability that the series $a_0, a_1, \ldots a_n$ contains n runs, a_n being maximal or minimal and ending the rth run. This probability must be $y_{n+1, r}$, as we may now think of a_0 as the arbitrarily chosen number instead of a_1. This series may contain r runs when the series $a_1, a_2, \ldots a_n$ contains either (i) $r-1$ runs or (ii) r runs.

(i) If the series $a_1, a_2, \ldots a_n$ contains $r-1$ runs, the series $a_0, a_1, \ldots a_n$ may contain r runs, only when a_1 is maximal or minimal. The contribution to $y_{n+1, r}$ from (i) is therefore $cx_{n, r-1}$.

(ii) If the series $a_1, a_2, \ldots a_n$ contains r runs, the series $a_0, a_1, \ldots a_n$ will always contain r runs except in the case where the number a_1 is maximal or minimal. Hence the contribution of (ii) to $y_{n+1, r}$ is $y_{nr} - cx_{nr}$.

Whence

$$y_{n+1, r} = cx_{n, r-1} + (y_{nr} - cx_{nr})$$

or

$$\underset{-}{\Delta^1} y_{n, r} = -c\Delta_1 x_{nr}. \quad . \quad . \quad . \quad . \quad . \quad (1.23)$$

Consequently

$$\begin{aligned}c\underset{r}{\Delta_2}x_{nr} &\equiv c\underset{r}{\Delta_1}\underset{r}{\Delta_1}x_{nr}\\ &= -\underset{r}{\Delta_1}\underset{n}{\Delta^1}y_{nr}\\ &= -\underset{n}{\Delta^1}\underset{r}{\Delta_1}y_{nr}\\ &= \underset{n}{\Delta^1}\underset{n}{\Delta^1}p_{nr}\\ &\equiv \underset{n}{\Delta^2}p_{nr}.\end{aligned}$$

It is easily proved that $c=2/3$ (Kermack and McKendrick, 1937). Whence

$$\underset{r}{\Delta_2}x_{nr}=\tfrac{3}{2}\underset{n}{\Delta^2}p_{nr}, \qquad (1.24)$$

or

$$(x_{nr}-2x_{n,\,r-1}+x_{n,\,r-2})=\tfrac{3}{2}(p_{n+2,\,r}-2p_{n+1,\,r}+p_{nr}). \qquad (1.25)$$

1.3. *Distributions of "Runs" and "Gaps."*

If in (1.25) we put $r=1$,

$$x_{n1}-2x_{n0}+x_{n,\,-1}=\tfrac{3}{2}(p_{n+2,\,1}-2p_{n+1,\,1}+p_{n1}), \qquad (1.31)$$

but

$$p_{n1}=\frac{2}{n\,!},\quad \text{and}\quad x_{n0}=x_{n,\,-1}=0, \qquad (1.32)$$

so that

$$x_{n1}=\tfrac{3}{2}\underset{n}{\Delta^2}\frac{2}{n\,!}=3\Delta^2\frac{1}{n\,!}=3\frac{n^2+n-1}{(n+2)\,!} \qquad (1.33)$$

(the result previously obtained for single runs (Kermack and McKendrick, 1937), x_{n1} being identical with x_n in that paper).

In the case where $r=2$,

$$x_{n2}-2x_{n1}=\tfrac{3}{2}\underset{n}{\Delta^2}p_{n2},$$

but

$$p_{n2}=\frac{P_{n2}}{n\,!}=\frac{2^n-4}{n\,!}, \qquad (1.34)$$

whence

$$x_{n2}=\tfrac{3}{2}\underset{n}{\Delta^2}\frac{2^n}{n\,!}=\frac{3}{2}\frac{2^n}{n\,!}\frac{n-2}{n+2}. \qquad (1.35)$$

This result was quoted but not proved in our previous paper.

It is easily seen that

$$\left.\begin{aligned}\sum_{n=3}^{\infty}x_{n2}&=\tfrac{3}{2}\sum\Delta^2\frac{2^n}{n\,!}\\ &=\frac{3}{2}\left\{-\Delta^1\frac{2^n}{n\,!}\right\}_{n=3}\\ &=1\end{aligned}\right\} \qquad (1.36)$$

and that

$$\left.\begin{aligned}\frac{\sum_{3}^{\infty} n x_{n2}}{\sum_{3}^{\infty} x_{n2}} &= \tfrac{3}{2}\sum_{3}^{\infty} {}_n\Delta^2 \frac{2^n}{n!} \\ &= \frac{3}{2}\left\{-3\underset{n=3}{\Delta^1}+3\underset{n=4}{\Delta^1}-4\underset{n=4}{\Delta^1}+4\underset{n=5}{\Delta^1}-\ldots\right\}\frac{2^n}{n!} \\ &= \frac{3}{2}\left\{-3\underset{n=3}{\Delta^1}-\sum_{4}^{\infty}\Delta^1\right\}\frac{2^n}{n!} \\ &= \frac{3}{2}\left\{3\left(\frac{2^3}{3!}-\frac{2^4}{4!}\right)+\frac{2^4}{4!}\right\} \\ &= 4\end{aligned}\right\} \quad . \quad (1.37)$$

Thus the average number of numbers in 2 consecutive runs is 4. As 2 consecutive runs constitute a gap in the sense defined above, $x_{n2} \equiv g_n$ of the previous paper.

1.4. *The Function* f_{nr}.

We define

$$f_{nr} \text{ as } \sum_{1}^{r}(r-s+1)p_{ns}, \quad . \quad . \quad . \quad . \quad (1.401)$$

whence

$$\underset{r}{\Delta_2} f_{nr} = p_{nr}. \quad . \quad . \quad . \quad . \quad . \quad (1.402)$$

Further

$$\begin{aligned}\underset{r}{\Delta_2} x_{nr} &= \tfrac{3}{2}\underset{n}{\Delta^2} p_{nr} \\ &= \tfrac{3}{2}\underset{n}{\Delta^2}\underset{r}{\Delta_2} f_{nr} \\ &= \tfrac{3}{2}\underset{r}{\Delta_2}\underset{n}{\Delta^2} f_{nr},\end{aligned}$$

whence

$$\underset{r}{\Delta_2}(x_{nr} - \tfrac{3}{2}\underset{n}{\Delta^2} f_{nr}) = 0,$$

thus

$$x_{nr} - \tfrac{3}{2}\underset{n}{\Delta^2} f_{nr} = \phi(n) + kr, \quad . \quad . \quad . \quad . \quad (1.403)$$

where ϕ is a function of n and k is a constant.

To evaluate $\phi(n)$ and k, we take $r=1$ and $r=2$, and note that

$$f_{n1} = p_{n1} = \frac{2}{n!}, \quad \text{and} \quad f_{n2} = \frac{2^n - 4}{n!} + \frac{4}{n!} = \frac{2^n}{n!}. \quad . \quad . \quad (1.404)$$

Whence, from 1·403, $\phi(n) + k = 0$ for all values of n. Similarly, from 1.403,

$$(r=2), \qquad \phi(n) + 2k = 0,$$

whence

$$\phi(n) = 0 \quad \text{and} \quad k = 0.$$

Thus

$$x_{nr} = \tfrac{3}{2}\underset{n}{\Delta^2} f_{nr}. \quad . \quad . \quad . \quad . \quad . \quad (1.405)$$

Thus both p_{nr} and x_{nr} are expressible in terms of f_{nr}.

We shall now deduce the difference equation satisfied by f_{nr}. Let

$$F_{nr}=n!f_{nr} \quad \text{so that} \quad F_{nr}=\sum_{1}^{r}(r-s+1)P_{ns}. \qquad (1.406)$$

André has shown that P_{ns} satisfies the recurrence equation

$$P_{n+1,\,s}=sP_{ns}+2P_{n,\,s-1}+(n+1-s)P_{n,\,s-2}, \qquad (1.407)$$

whence

$$P_{n+1,\,s}=(r+1)P_{n,\,s}-(r+1-s)P_{ns}+2P_{n,\,s-1}+(r-s+1)P_{n,\,s-2}+(n-r)P_{n,\,s-2}$$

and

$$\sum_{1}^{r}P_{n+1,\,s}=(r+1)\sum_{1}^{r}P_{ns}-\sum_{1}^{r}(r+1-s)P_{ns}+2\sum_{1}^{r}P_{n\,s-1}$$
$$+\sum_{1}^{r}(r-s+1)P_{n,\,s-2}+\sum_{1}^{r}(n-r)P_{n,\,s-2}.$$

But

$$\sum_{1}^{r}P_{n+1,\,s}=\underset{r}{\Delta_1}F_{n+1,\,r}, \quad \text{and} \quad \sum_{1}^{r}(r-s+1)P_{n,\,s-2}=\sum_{0}^{r-2}(r-2-s^1+1)P_{ns^1}=F_{n,\,r-2},$$

whence

$$\underset{r}{\Delta_1}F_{n+1,\,r}=(r+1)\underset{r}{\Delta_1}F_{nr}-F_{nr}+2\underset{r}{\Delta_1}F_{n,\,r-1}+F_{n,\,r-2}+(n-r)\underset{r}{\Delta_1}F_{n,\,r-2}$$

or

$$\underset{r}{\Delta_1}[F_{n+1,\,r}-rF_{nr}-(n-r-1)F_{n,\,r-2}]=0,$$

so that

$$F_{n+1,\,r}-rF_{nr}-(n-r-1)F_{n,\,r-2}=\phi(n). \qquad (1.408)$$

Putting $r=1$, we have

$$F_{n+1,\,1}-F_{n1}=\phi(n).$$

But $F_{n1}=n!f_{n1}=2$ (by 1.404), so that $\phi(n)=0$ for all values of n. Whence

$$F_{n+1,\,r}=rF_{nr}+(n-r-1)F_{n,\,r-2}. \qquad (1.409)$$

It follows at once that

$$(n+1)f_{n+1,\,r}=rf_{nr}+(n-r-1)f_{n,\,r-2}. \qquad (1.410)$$

Equations (1.409) and (1.410) are not themselves sufficient to determine F and f, it is necessary in addition to know a suitable set of values which play the part of the boundary conditions in partial differential equations. We are only interested in the values of F and of f for positive values of r, and for $n\geqslant r$. It is evident from (1.409) and (1.410) that the solutions for odd and even values of r are independent of each other. Inspection shows that the equation has a definite solution if we know the values of F_{n1}, F_{n2}, and $F_{n,\,n}$ (or f_{n1}, f_{n2} and f_{nn}) for all values of n.

By definition

$$\begin{aligned} F_{nn} &= \sum_{1}^{n} (n-s+1)P_{ns} \\ &= (n+1)\sum_{1}^{n-1} P_{ns} - \sum_{1}^{n-1} sP_{ns} \quad (\text{as } P_{nn}=0) \\ &= (n+1)n! - \frac{2n-1}{3}n! \quad (n \geqslant 2) \\ &= \frac{n+4}{3}n! \quad (n \geqslant 2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.411) \end{aligned}$$

and so

$$f_{nn} = \frac{n+4}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.412)$$

It may also be easily shown that

$$F_{n, n-1} = \frac{n+1}{3}n! \quad (n \geqslant 2) \quad . \quad . \quad . \quad . \quad (1.413)$$

and

$$f_{n, n-1} = \frac{n+1}{3} \quad (n \geqslant 2). \quad . \quad . \quad . \quad . \quad (1.414)$$

Furthermore, if in (1.409) we write $r=n$,

$$F_{n+1, n} = nF_{nn} - F_{n, n-2},$$

whence

$$\begin{aligned} F_{n, n-2} &= nF_{nn} - F_{n+1, n} \\ &= \frac{n-2}{3}n! \quad (n > 2), \quad . \quad . \quad . \quad (1.415) \end{aligned}$$

and so

$$f_{n, n-2} = \frac{n-2}{3}. \quad (n > 2). \quad . \quad . \quad . \quad . \quad (1.416)$$

Equation (1.409) may be used to calculate successively the values of F_{nr} for $r=3$, 4, 5, etc. Thus the equation

$$F_{n+1, 3} = 3F_{n3} + (n-4)F_{n1} \quad . \quad . \quad . \quad . \quad (1.417)$$

becomes, after substituting $F_{n1}=2.1^n$,

$$F_{n+1, 3} = 3F_{n3} + (n-4)2.1^n,$$

and the solution is clearly

$$F_{n3} = c_3 3^n + (\alpha n + \beta)1^n.$$

Substitution of this value in (1.417) gives $\alpha = -1$, $\beta = \frac{7}{2}$, and by substituting $n=3$, and making use of the fact that by (1.411) $F_{33}=14$, it is easily found that $c_3 = \frac{1}{2}$, so that

$$F_{n3} = \frac{3^n}{2} - \frac{2n-7}{2}1^n. \quad . \quad . \quad . \quad . \quad . \quad (1.418)$$

Similarly, we may calculate successively

$$F_{n5}=\frac{5^n}{8}-\frac{2n-9}{8}3^n+\frac{n^2-9n+19}{4}1^n, \quad . \quad . \quad . \quad (1.419)$$

$$F_{n7}=\frac{7^n}{32}-\frac{2n-11}{32}5^n+\frac{2n^2-22n+57}{32}3^n-\frac{4n^3-66n^2+344n-561}{96}1^n, \quad (1.420)$$

and for even values of r, we find by making use of $F_{n2}=2^n$ by (1.404), and $F_{44}=64$ by (1.411),

$$F_{n4}=\frac{4n}{4}-\frac{n-4}{2}2^n \quad . \quad . \quad . \quad . \quad (1.421)$$

$$F_{n6}=\frac{6^n}{16}-\frac{n-5}{8}4^n+\frac{2n^2-20n+47}{16}2^n \quad . \quad . \quad . \quad (1.422)$$

$$F_{n8}=\frac{8^n}{64}-\frac{n-6}{32}6^n+\frac{n^2-12n+34}{32}4^n-\frac{2n^3-36n^2+205n-366}{96}2^n. \quad . \quad (1.423)$$

These expressions of course at once give the values of $f_{nr}(r=3, 4, \ldots 8)$.

Before giving a general expression for F_{nr} we may use (1.409) to find $S_n=\sum_1^{n-1}F_{nr}$. We have

$$F_{n+1,\,1}=F_{n1}.$$
$$F_{n+1,\,2}=2F_{n2}.$$
$$F_{n+1,\,3}=3F_{n3}+(n-3)F_{n1}-F_{n1}.$$
$$\vdots$$
$$F_{n+1,\,r}=rF_{nr}+(n-3)F_{n,\,r-2}-(r-2)F_{n,\,r-2}.$$
$$\vdots$$
$$F_{n+1,\,n}=nF_{nn}+(n-3)F_{n,\,n-2}-(n-2)F_{n,\,n-2}.$$

Addition of these equations gives

$$\left.\begin{aligned} S_{n+1}&=\sum_1^n rF_{nr}+(n-3)\sum_1^{n-2}F_{nr}-\sum_1^{n-2}rF_{nr}\\ &=nF_{nn}+(n-1)F_{n,\,n-1}+(n-3)(S_n-F_{n,\,n-1})\end{aligned}\right\} \quad . \quad . \quad (1.424)$$

To solve this we write $S_n=(n-4)!\,S_n'$, and we find

$$\begin{aligned} S'_{n+1}&=S'_n+\frac{n(n-1)(n-2)}{3}(n^2+6n+2)\\ &=S'_n+\tfrac{1}{3}\{(n+2)(n+1)n(n-1)(n-2)+3(n+1)n(n-1)(n-2)\\ &\qquad -3n(n-1)(n-2)\},\end{aligned}$$

whence

$$S'_n=\tfrac{1}{3}\left\{\frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{6}+\frac{3(n+1)n(n-1)(n-2)(n-3)}{5}\right.$$
$$\left.-\frac{3n(n-1)(n-2)(n-3)}{4}+\text{constant}\right\},$$

from which it follows that

$$S_n = \tfrac{1}{3}\left\{\frac{(n+2)!}{6} + \frac{3(n+1)!}{5} - \frac{3n!}{4}\right\} = \frac{n!}{180}(10n^2 + 66n + 11), \ (n > 3) \quad . \quad (1.425)$$

(the constant being found to be zero, by putting $n=4$, and using the value $S_4 = F_{43} + F_{42} + F_{41} = 58$). Whence, by using (1.411),

$$\sum_1^n F_{nr} = S_n + F_{nn} = \frac{n!}{180}(10n^2 + 126n + 251) \quad (n > 3). \quad . \quad (1.426)$$

For $n=2$ and $n=3$ these equations fail, the reason being that the difference equation fails to hold when $n=1$ and $n=2$. We may use this value of S_n in order to calculate $\sum_1^n r^2 P_{nr}$. From (1.402), $P_{nr} = \underset{r}{\Delta_2} F_{nr}$, and the values of F_{nn} and $F_{n,\,n-1}$ already obtained (1.411 and 1.413) it is found without difficulty that

$$\sum_1^n r^2 P_{nr} = \frac{n!}{3}(n^2 - 3n - 1 + 2S_n),$$

whence

$$\sum_1^n r^2 P_{nr} = \frac{n!}{90}(40n^2 - 24n - 19) \quad (n > 3). \quad . \quad . \quad . \quad (1.427)$$

This value may also be obtained directly from (1.16) by multiplying by r^2 and summing. The value obtained in (1.427) has been employed in our previous paper for calculating the errors of the mean length of a run and of a gap.

1.5. *The Generating Function for f_{nr}.*

The calculation of the coefficients of the various terms for F_{nr} becomes increasingly laborious as r increases. A general expression can, however, be found by obtaining the generating function for f_{nr}.

From (1.410) it follows that, if

$$\phi(zt) = \sum_0^\infty \sum_0^\infty z^n t^r f_{nr} \quad . \quad . \quad . \quad . \quad (1.501)$$

satisfies (1.410), it also satisfies the differential equation obtained by substituting $z\frac{\partial}{\partial z}$ for n, $t\frac{\partial}{\partial t}$ for r and ϕ for f_{nr}. This partial differential equation reduces to

$$(1 - zt^2)\frac{\partial \phi}{\partial z} - t(1 - t^2)\frac{\partial \phi}{\partial t} + 3t^2\phi = 0. \quad . \quad . \quad . \quad (1.502)$$

The auxiliary equations are therefore

$$\frac{dz}{1 - zt^2} = -\frac{dt}{t(1 - t^2)} = -\frac{d\phi}{3t^2\phi}. \quad . \quad . \quad . \quad (1.503)$$

These have integrals

$$\log \phi = c_1 - \tfrac{3}{2} \log (1 - t^2) \quad . \quad . \quad . \quad . \quad (1.504)$$

and

$$c_2 = z\sqrt{1-t^2} - \log \frac{1+\sqrt{1-t^2}}{t}, \quad . \quad . \quad . \quad . \quad (1.505)$$

whence, noticing that

$$\frac{1+\sqrt{1-t^2}}{t} = \frac{t}{1-\sqrt{1-t^2}},$$

we obtain

$$\phi(zt) = (1-t^2)^{-\frac{3}{2}} F\left(\frac{1-\sqrt{1-t^2}}{t} e^{z\sqrt{1-t^2}}\right) \quad . \quad . \quad . \quad (1.506)$$

$$= \lambda^{-3} F(\tau e^{\lambda z}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.507)$$

where $\lambda = \sqrt{1-t^2}$, $\tau = \dfrac{1-\lambda}{t}$ and F is an arbitrary function so chosen that the "boundary" conditions for f_{nr} are satisfied. We note that τ is expressible in the form

$$\tau = \frac{t}{2} + \frac{t^3}{8} + \frac{t^5}{16} + \frac{5t^7}{128} + \ldots \quad . \quad . \quad . \quad . \quad (1.508)$$

As $\phi(zt)$ is expressible in positive powers of z and t, we may write

$$F(\tau e^{\lambda z}) = A_0 + A_1 \tau e^{\lambda z} + A_2 \tau^2 e^{\lambda z} + \ldots A_r \tau^r e^{+r\lambda r} \ldots,$$

where the values of A_r are determined by the "boundary" conditions of the problem. Clearly the coefficient of r^n in the coefficient of $z^n t^r$ is $A_r \dfrac{2^{-r}}{n!}$, but for $r = 1, 2, 3 \ldots 8$ the coefficients in question have been found to be $\dfrac{2^{-(r-2)}}{n!}$, so that $A_r = 4$ (for $r = 1, 2 \ldots 8$), and A_0 may also be taken as 4 as f_{00} is entirely arbitrary. So that the expression

$$\phi(zt) = \frac{4}{\lambda^3(1 - \tau e^{\lambda z})} \quad . \quad . \quad . \quad . \quad (1.509)$$

gives the generating function correctly for all values of r from 0 to 8, and for all values of n, and presumably is the correct generating function.

That this is indeed the case is proved by considering the value to which P_{nr} for any finite value of r_n tends, when n is made very large. We may in this case enumerate the ways in which the n numbers may be arranged to give r runs as follows. Let us distribute the n numbers at random in r compartments arranged in a row. If n is made very large, the number of the arrangements in which one or more of the compartments contain less than 2 numbers becomes a vanishingly small fraction of the total number. The number of such arrangements is therefore $r^n(1+\epsilon)$ where $\epsilon \rightarrow 0$ as $n \rightarrow \infty$. Now we rearrange the numbers in the 1st compartment in ascending order, those in the 2nd in descending order, in the 3rd in ascending and in the 4th in descending order, etc. As each

compartment contains at least 2 numbers, we ensure in this way that the series of numbers taken as a whole will contain r runs. We obtain another set of r runs by arranging those in the 1st compartment in descending order, in the 2nd in ascending, in the 3rd in descending order, etc. In this way we obtain $2r^n(1+\epsilon)$ arrangements.

We have now to relate this set of arrangements with the P_{nr} arrangements of n numbers containing r runs. Any of these latter arrangements, provided that it contains at least 4 numbers in every run, is necessarily given by the above enumeration. However, each of the P_{nr} arrangements corresponds to 2^{r-1} different distributions of the numbers in the r compartments, for every maximal or minimal number between 2 runs may be put into the compartment corresponding to the preceding or the succeeding run, and there are $r-1$ such maximal or minimal numbers. The number of irregular cases—that is, those containing runs with less than 4 numbers — becomes a vanishingly small fraction of the total number. Whence

$$2r^n(1+\epsilon) = P_{nr}2^{r-1}(1+\epsilon')$$

where ϵ' like $\epsilon \to 0$ as n becomes very large.

Thus

$$\lim_{n\to\infty} \frac{P_{nr}}{r^n} = 2^{-(r-2)}. \qquad . \quad . \quad . \quad . \quad (1.510)$$

But

$$P_{nr} = \underset{r}{\Delta_2} F_{nr} = F_{nr} - 2F_{n,\,r-1} + F_{n,\,r-2}.$$

Now inspection of the general form of F_{nr} shows that for large values of n,

$$F_{nr} \to \frac{A_r r^n}{2^r}$$

because for large values of n, $(r-1)^n$ is small compared with r^n. For the same reason $F_{n,\,r-1}$ and $F_{n,\,r-2}$ are negligible compared with F_{nr}, so that

$$\lim_{n\to\infty} \frac{P_{nr}}{r^n} = \frac{A_r}{2^r},$$

whence, by (1.510),

$$A_r = 4 \quad (r \geqslant 0). \quad . \quad . \quad . \quad . \quad . \quad (1.511)$$

Thus

$$\phi(zt) = \frac{4}{\lambda^3}(1 + \tau e^{\lambda z} + \tau^2 e^{2\lambda z} + \ . \ . \ .) = \frac{4}{\lambda^3(1-\tau e^{\lambda z})}. \quad . \quad . \quad (1.512)$$

As $x_{nr} = \frac{3}{2}\underset{n}{\Delta^2} f_{nr}$, it follows that the generating function for x_{nr} is

$$\xi(zt) = \frac{6(1-z)^2}{z^2\lambda^3(1-\tau e^{\lambda z})}, \quad . \quad . \quad . \quad . \quad (1.513)$$

whilst the generating function for $p_{nr} \equiv \frac{P_{nr}}{n!}$ is

$$\pi(zt) = \frac{4(1-t)^2}{\lambda^3(1-\tau e^{\lambda z})}. \qquad (1.514)$$

We may now make use of $\phi(zt)$ to obtain the general expressions for F_{nr} and f_{nr}. We may write

$$F_{nr} = \sum_s s^n K^s_{nr} \quad \text{and} \quad f_{nr} = \frac{1}{n!}\sum_s s^n K^s_{nr}. \qquad (1.515)$$

where K^s_{nr} is a function of n and r.

From the expression for ϕ, (1.512), it follows that K^s_{nr} is the coefficient of $t^r z^n$ in $\frac{4\tau^s e^{s\lambda z}}{\lambda^3}$, *i.e.* the coefficient of t^r in $4\tau^s(1-t^2)^{\frac{n-3}{2}}$. This is evidently the coefficient of t^{r+s} in the expansion of $4(1-\sqrt{1-t^2})^s(1-t^2)^{\frac{n-3}{2}}$; that is in

$$4\sum_l C_{sl}(1-t^2)^{\frac{l}{2}}(-)^l(1-t^2)^{\frac{n-3}{2}};$$

that is in

$$4\sum_l (-)^l C_{sl}(1-t^2)^{\frac{n+l-3}{2}}, \qquad (1.516)$$

where C_{sl} is the coefficient of x^l in the expansion of $(1+x)^s$. This is an even function of t, and therefore the coefficient of t^{r+s} is zero when $r+s$ is odd. Let $r+s=2\theta$. The coefficient of $t^{2\theta}$ is

$$4\sum_l (-)^l C_{sl}(-)^\theta C_{\frac{n+l-3}{2},\theta}. \qquad (1.517)$$

whence

$$K^s_{nr} = 4\sum_l (-)^{l+\theta} C_{sl} C_{\frac{n+l-3}{2},\theta} \qquad (1.518)$$

s having the values r, $r-2$, $r-4$ . . . corresponding to $\theta=r$, $r-1$, $r-2$ . . . . The lowest value of s is 2 or 1 according as to whether r is even or odd.

This expression may also be written in the form

$$K^s_{nr} = 4(-)^{\theta+s}\underset{n}{\Delta^s} C_{\frac{n-3}{2},\theta}. \qquad (1.519)$$

It is perfectly general, but is inconvenient for the numerical calculation of K^s_{nr} except for low values of s.

For values of $s=r-2$, $r-4$ the following procedure may be adopted. From the identity

$$\tau = \frac{t}{2} + \frac{t^3}{8} + \frac{t^5}{16} + \frac{5t^7}{128} \ldots$$

it may be found that

$$\tau^s = \left(\frac{t}{2}\right)^s (1 + \lambda_{s1}t^2 + \lambda_{s2}t^4 + \lambda_{s3}t^6 + \ldots). \qquad (1.520)$$

where

$$\lambda_{s1}=\frac{s}{4},\quad \lambda_{s2}=\frac{s(s+3)}{32}=\frac{s(s+3)}{2!\,2^4},\quad \lambda_{s3}=\frac{s(s+4)(s+5)}{384}=\frac{s(s+4)(s+5)}{3!\,2^6}. \quad (1.521)$$

Thus K^s_{nr} is the coefficient of t^r in

$$4\left(\frac{t}{2}\right)^s(1+\lambda_{s1}t^2+\lambda_{s2}t^2+\ldots)(1-t^2)^{-\frac{n-3}{2}}$$
$$=\frac{t^s}{2^{s-2}}\left\{1+\left(\lambda_{s1}-\frac{n-3}{2}\right)t^2+\left(\lambda_{s2}-\frac{(n-3)}{2}\lambda_{s1}+\frac{(n-3)(n-5)}{2.4}\right)t^4\right.$$
$$\left.+\left(\lambda_{s3}-\frac{n-3}{2}\lambda_{s2}+\frac{(n-3)(n-5)}{2.4}\lambda_{s1}-\frac{(n-3)(n-5)(n-7)}{2.4.6}\right)t^6+\ldots\right\} \quad (1.522)$$

This gives for $r=s+2$

$$K^{r-2}_{nr}=2^{-(s-2)}\left(\lambda_{s1}-\frac{n-3}{2}\right), \quad . \quad . \quad . \quad . \quad (1.523)$$
$$=2^{-(r-4)}\left(\frac{r-2}{4}-\frac{n-3}{2}\right)$$
$$=-2^{-(r-2)}(2n-r-4). \quad . \quad . \quad . \quad . \quad (1.524)$$

Also for $r=s+4$

$$K^{r-4}_{nr}=2^{-(s-2)}\left(\lambda_{s2}-\frac{n-3}{2}\lambda_{s1}+\frac{(n-3)(n-5)}{8}\right), \quad . \quad . \quad (1.525)$$

which, substituting $s=r-4$, we find equal to

$$2^{-(r-1)}(4n^2-4(r+4)n+r^2+7r+16). \quad . \quad . \quad . \quad (1.526)$$

Similarly, for $r=s+6$

$$K^{r-6}_{nr}=2^{-(s-2)}\left(\lambda_{s3}-\frac{n-3}{2}\lambda_{s2}+\frac{(n-3)(n-5)}{8}\lambda_{s1}-\frac{(n-3)(n-5)(n-7)}{48}\right), \quad (1.527)$$

which may be reduced if desired.

For the calculation of K^s_{nr} for values of s neither small nor equal to $r-2, r-4, r-6$ the following recurrence relationship may be made use of. Let

$$\tau\equiv\sum_1^\infty(-)^m C_{\frac{1}{2},\,m}t^{2m-1}=\sum_1^\infty\gamma_m t^{2m-1}, \quad . \quad . \quad . \quad (1.528)$$

where

$$\gamma_m=\frac{1.3.5\ldots(2m-3)}{2.4.6\ldots2m}. \quad . \quad . \quad . \quad . \quad (1.529)$$

(We may note that $\lambda_{1m}=2\gamma_{m+1}$.) Because the values of K^s_{nr} are given by the coefficient of t^r in $4\tau^s(1-t^2)^{-\frac{n-3}{2}}$, it follows that

$$\sum_{r=s+1}^\infty K^{s+1}_{nr}t^r=\sum_{r=1}^\infty K^s_{nr}\times\sum_1^\infty\gamma_\sigma t^\sigma\times t^r.$$

Equating the coefficients of t^r, we have

$$\mathrm{K}_{nr}^{s+1} = \mathrm{K}_{n,\,r-1}^{s}\gamma_1 + \mathrm{K}_{n,\,r-3}^{s}\gamma_2 + \mathrm{K}_{n,\,r-5}^{s}\gamma_3 + \ .\ .\ .\ + \mathrm{K}_{n,\,r-2l+1}^{s}\gamma_l + \ .\ .\ .\ . \qquad (1.530)$$

If, then, we know (by 1.519) the values of K_{nr}^{1} for all values of r less than and including the given value, we may proceed by steps to $s=2$, $s=3$, etc.

1.6. *The Values of* $f_{n,\,n-\sigma}$, $p_{n,\,n-\sigma}$ *and* $x_{n,\,n-\sigma}$.

The value of $x_{n,\,n-1}$ is of rather special interest as it is the chance that m consecutive runs taken at random are all of length 2. It is obviously closely related to $\mathrm{P}_{m,\,m-1}$, the number of ways in which m numbers can be arranged to give $m-1$ runs—that is, in such a way that all runs are of size 2. Such a series has been called by André an alternating series.

Let us consider n numbers $y_1, y_2, \ldots y_n$ of the infinite series which we may think of as taken at random from the whole set of decimal numbers between 0 and 1. Let y_0 be the number preceding y_1, and y_{n+1} the number succeeding y_n. Then the condition that y_1 be maximal, y_2 minimal, y_3 maximal, etc., is that

$$y_0 < y_1 > y_2 < y_3 \cdot \cdot \cdot y_{n+1}. \qquad (1.601)$$

Now each number has equal chances of occurring anywhere between 0 and 1, so that the chance that the condition (1.601) should be satisfied is

$$\mathrm{A} \equiv \int_0^1\int_{y_0}^1\int_0^{y_1}\int_{y_2}^1 \cdot \cdot \cdot dy_{n+1}dy_n dy_{n-1}dy_{n-2} \cdot \cdot \cdot dy_0. \qquad (1.602)$$

Now $x_{n,\,n-1}$ is the chance that a number chosen to be maximal is followed by $n-1$ runs each of size 2. The integral A is evidently the chance that the number y_1 is maximal combined with the chance that a maximal number is succeeded by $n-1$ runs of size 2, that is

$$\mathrm{A} = \tfrac{1}{3}x_{n,\,n-1},$$

since the chance that y_1 is maximal is $1/3$. Thus

$$x_{n,\,n-1} = 3\mathrm{A}. \qquad (1.603)$$

Now it may be shown that the integral A, which ends in

$$\cdot\cdot\cdot \int_{y_{n-1}}^1\int_0^{y_n} dy_{n+1}dy_n \cdot\cdot\cdot \quad \text{or} \quad \cdot\cdot\cdot \int_0^{y_{n-1}}\int_{y_n}^1 dy_{n+1}dy_n \cdot\cdot\cdot$$

according as to whether n is odd or even, is equal to

$$\mathrm{D}_n \equiv \begin{vmatrix} a_1, & a_3, & a_5, & a_7 \ldots a_{2p+1}, & a_{2p+3} \\ 1, & a_2, & a_4, & a_6 \ldots a_{2p}, & a_{2p+2} \\ 0, & 1, & a_2, & a_4 \ldots a_{2p-2}, & a_{2p} \\ \vdots & & & & \\ 0, & 0, & 0, & 0 \ldots a_2, & a_4 \\ 0, & 0, & 0, & 0 \ldots 1, & a_2 \end{vmatrix} \quad n \text{ odd} = 2p+1, \qquad (1.604)$$

or

$$D_n \equiv \begin{vmatrix} a_2, & a_4, & a_6, & a_8 & \ldots & a_{2\rho}, & a_{2\rho+2} \\ 1, & a_2, & a_4, & a_6 & \ldots & a_{2\rho-2}, & a_{2\rho} \\ 0, & 1, & a_2, & a_4 & \ldots & a_{2\rho-4}, & a_{2\rho-2} \\ \vdots \\ 0, & 0, & 0, & 0 & \ldots & a_2, & a_4 \\ 0, & 0, & 0, & 0 & \ldots & 1, & a_2 \end{vmatrix} \quad n \text{ even} = 2\rho, \qquad . \quad (1.605)$$

where $a_\rho = \dfrac{1}{\rho!}$.

The numerical values of the determinants D_n are

$$\left.\begin{aligned} D_{-1} &= 1 && = 1 \\ D_0 &= \frac{1}{2} && = \frac{1}{2!} \\ D_1 &= \frac{1}{3} && = \frac{2}{3!} \\ D_2 &= \frac{5}{24} && = \frac{5}{4!} \\ D_3 &= \frac{2}{15} && = \frac{16}{5!} \\ D_4 &= \frac{61}{720} && = \frac{61}{6!} \\ D_5 &= \frac{17}{315} && = \frac{272}{7!} \\ D_6 &= \frac{277}{8064} && = \frac{1385}{8!}, \quad \text{etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.606)$$

so that

$$x_{n,\,n-1} = 3D_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.607)$$

By (1.405),

$$\tfrac{2}{3} x_{n,\,n-1} = (\Delta_n^2 f_{nr})_{r=n-1}$$
$$= f_{n+2,\,n-1} - 2f_{n+1,\,n-1} + f_{n,\,n-1}.$$

But by (1.414) and (1.416),

$$f_{n,\,n-1} = \frac{n+1}{3} \quad \text{and} \quad f_{n+1,\,n-1} = \frac{n-1}{3},$$

so that

$$f_{n+2,\,n-1} = \frac{2}{3}x_{n,\,n-1} + \frac{2(n-1)}{3} - \frac{n+1}{3}$$
$$= 2D_n + \frac{n-3}{3},$$

and so

$$f_{n,\,n-3} = 2D_{n-2} + \frac{n-5}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.608)$$

We shall now consider the relation of the determinants D_n to $p_{n,\,n-1}$. Following an argument similar to that given for $x_{n,\,n-1}$, it is not difficult to show that

$$p_{n,\, n-1} = 2\mathrm{D}_{n-2}, \qquad (1.609)$$

the coefficient 2 coming from the fact that the first run may either be a run up or a run down. It follows that

$$x_{n,\, n-1} = \tfrac{3}{2} p_{n+2,\, n+1}, \qquad (1.610)$$

a result which may be obtained directly from the difference equation (1.21). Now André has shown that $p_{n,\, n-1}$ is the coefficient of z_n in the expansion of

$$f(z) = 2 \tan\left(\frac{z}{2} + \frac{\pi}{4}\right) \equiv 2\frac{1 + \sin z}{\cos z}. \qquad (1.611)$$

Thus the determinant D_{n-2} is the coefficient of z^n in the expansion of $-2\dfrac{(1 + \sin z)}{\cos z}$, a result which can of course be proved independently.

We now proceed to evaluate $f_{n,\, n-4}$, $x_{n,\, n-2}$ and $p_{n,\, n-2}$. If in (1.410) we put $r = n - 2$, we get

$$(n + 1) f_{n+1,\, n-2} = (n - 2) f_{n,\, n-2} + f_{n,\, n-4},$$

and substituting the values of $f_{n,\, n-2}$ (1.416) and $f_{n+1,\, n-2}$ (the latter being obtained from $f_{n,\, n-3}$ (1.608)), we find

$$f_{n,\, n-4} = 2(n + 1)\mathrm{D}_{n-1} + \frac{n - 8}{3}, \qquad (1.612)$$

from which it follows that

$$\begin{aligned} x_{n,\, n-2} &= \tfrac{3}{2}(f_{n+2,\, n-2} - 2f_{n+1,\, n-2} + f_{n,\, n-2}) \\ &= 3(n + 3)\mathrm{D}_{n+1} - 6\mathrm{D}_{n-1}, \qquad (1.613) \end{aligned}$$

and

$$\begin{aligned} p_{n,\, n-2} &= f_{n,\, n-2} - 2f_{n,\, n-3} + f_{n,\, n-4} \\ &= 2(n + 1)\mathrm{D}_{n+1} - 4\mathrm{D}_{n-2}, \qquad (1.614) \end{aligned}$$

whilst

$$\mathrm{P}_{n,\, n-2} = 2(n + 1)!\,\mathrm{D}_{n-1} - 4n!\,\mathrm{D}_{n-2}. \qquad (1.615)$$

This last expression can of course easily be obtained directly from (1.16).

André has shown that the generating function for $p_{n,\, n-2}$ is $\dfrac{e(1 - 2\cos z)}{1 - \sin z}$, and it is not difficult to verify that this result is consistent with equations (1.614) and (1.615) above.

1.7. *The Generating Functions for $f_{n,\, n-\sigma}$, $p_{n,\, n-\sigma}$ and $x_{n,\, n-\sigma}$.*

It is natural to inquire whether the results obtained above for $p_{n,\, n-1}$ and $p_{n,\, n-2}$ and for $f_{n,\, n-\sigma}$ ($\sigma = 1$, 2, 3 or 4) can be generalised.

If in the difference equations for f and p, (1.410) and (1.18), we write $r = n - \sigma$, we find

$$(n + 1) f_{n+1,\, n-\sigma} = (n - \sigma) f_{n,\, n-\sigma} + (\sigma - 1) f_{n,\, n-\sigma-2} \qquad (1.701)$$

and

$$(n+1)p_{n+1,\ n-\sigma} = (n-\sigma)p_{n,\ n-\sigma} + 2p_{n,\ n-\sigma-1} + (\sigma+1)p_{n,\ n-\sigma-2}. \quad (1.702)$$

These difference equations allow us to calculate $f_{n,\ n-5}$, $f_{n,\ n-6}$, etc., and $p_{n,\ n-3}$, $p_{n,\ n-4}$, etc., successively from the values already obtained. The results, however, become increasingly complicated, and it is more interesting to deduce the generating function for $f_{n,\ n-\sigma}$ and $p_{n,\ n-\sigma}$ as we then obtain the whole solution in a compact form.

If we denote $f_{n,\ n-\sigma}$ by $e_{n\sigma}$, then (1.701) can be written in the form

$$(n+1)e_{n+1,\ \sigma+1} = (n-\sigma)e_{n\sigma} + (\sigma-1)e_{n,\ \sigma+2}. \quad (1.703)$$

If $E(zs) \equiv \sum\sum e_{n\sigma} z^n s^\sigma$, then corresponding to (1.703) we have a partial differential equation which reduces to

$$\left(\frac{1}{s} - z\right)\frac{\partial E}{\partial z} - \frac{(1-s^2)}{s}\frac{\partial E}{\partial s} = -\frac{3}{s^2}E, \quad (1.704)$$

of which the general solution is

$$E(zs) = \frac{2s^3}{(1-s^2)^{3/2}}\Phi(S) \quad (s \geqslant 3), \quad (1.705)$$

where $S = (z\sqrt{1-s^2} + \sin^{-1} s)$ and Φ is an arbitrary function.

It will be noted that any analytic expansion of Φ in ascending powers of s results in E containing no power of s less than s^3—that is, the generating function represents $f_{n,\ n-\sigma}$ for $\sigma \geqslant 3$. Now $f_{n,\ n-3} = 2D_{n-2} + \frac{n-5}{3}$ (by (1.608)), and it follows from André's generating function for $\frac{p_{n,\ n-1}}{2} = D_{n-2}$ that the coefficient of s^3 includes the term $2\frac{(1+\sin z)}{\cos z}$, along with presumably other terms which give rise to the term $\frac{n-5}{3}$ in $f_{n,\ n-3}$. The part of Φ independent of s must therefore include $2\frac{(1+\sin z)}{\cos z}$, from which it is natural to assume that

$$E(zs) = \frac{2s^3}{(1-s^2)^{3/2}}\,\frac{1+\sin S}{\cos S} + \text{terms which account for the term } \frac{n-5}{3}, \quad (1.706)$$

If now we consider the coefficient of s^4 in the above expression, it is not difficult to show that it is $\frac{2(1+\sin z)}{\cos^2 z} = \frac{2}{1-\sin z}$, in which the coefficient of z^n may be shown to be $2(n+1)D_{n-1}$. Consequently the assumed form generates the correct values for $f_{n,\ n-3}$ and $f_{n,\ n-4}$ as far as the terms containing the D's are concerned. Investigation shows, however, that it is impossible to find an expression of the given form which generates

the terms $\frac{n-5}{3}$ and $\frac{n-8}{3}$. This is because this series of terms does not begin like the D terms at $\sigma=3$, but appears also in $f_{n,\,n-2}$, $f_{n,\,n-1}$ and $f_{n,\,n}$, so that no generating function beginning with s^3, nor indeed any analytic function, can properly represent these terms, and at the same time satisfy the partial differential equation completely.

The investigation of the generating function for $p_{n,\,n-\sigma}$ sheds light on this point. If we write $p_{n,\,n-\sigma}=o_{n\sigma}$, it is readily found that the following difference equation is satisfied:

$$(n+1)o_{n+1,\,\sigma+1}=(n-\sigma)o_{n\sigma}+2o_{n,\,\sigma+1}+(\sigma+1)o_{n,\,\sigma+2}, \quad . \quad . \quad (1.707)$$

so that $\Omega(zs)\equiv\Sigma\Sigma o_{n\sigma}z^n s^\sigma$ satisfies a partial differential equation which reduces to

$$\left(\frac{1}{s}-z\right)\frac{\partial\Omega}{\partial z}+\left(s-\frac{1}{s}\right)\frac{\partial\Omega}{\partial s}=\frac{2s-1}{s^2}\Omega. \quad . \quad . \quad . \quad (1.708)$$

The general solution of which is

$$\Omega(zs)=\frac{s(1-s^2)^{\frac{1}{2}}}{(1+s)^2}\Phi(\mathrm{S}). \quad . \quad . \quad . \quad . \quad (1.709)$$

Now for $\sigma=1$, $p_{n,\,n-1}=\frac{2(1+\sin z)}{\cos z}$, so that, following an argument similar to that given in the case of $e_{n\sigma}$, the following expression for Ω suggests itself:

$$\Omega(zs)=\frac{s(1-s^2)^{\frac{1}{2}}}{(1+s)^2}2\left\{\frac{1+\sin \mathrm{S}}{\cos \mathrm{S}}\right\} \quad . \quad . \quad . \quad (1.710)$$

The coefficient of s in this expression is $\frac{2(1+\sin z)}{\cos z}$, and it is not difficult to show that the coefficient of s^2 is $\frac{2(1-2\cos z)}{1-\sin z}$, which André has found to be the correct generating function for $p_{n,\,n-2}$. Thus the above generating function Ω gives the correct values for $\sigma=1$ and $\sigma=2$, and therefore must be correct for all values of σ; it is therefore the generating function for $o_{n\sigma}$.

Now, by equation (1.402) we know that

$$o_{n\sigma}\equiv p_{n,\,n-\sigma}=\underset{n-\sigma}{\Delta_2}f_{n,\,n-\sigma}=\underset{\sigma}{\Delta}e^2_{n\sigma}, \quad . \quad . \quad . \quad (1.711)$$

so that we should expect that

$$\Omega(zs)=\frac{(1-s)^2}{s^2}\mathrm{E}(zs) \quad . \quad . \quad . \quad . \quad (1.712)$$

or

$$\begin{aligned}\mathrm{E}(zs)&=\frac{s^2}{(1-s)^2}\Omega(zs)\\&=\frac{2s^3}{(1-s^2)^{3/2}}\frac{1+\sin \mathrm{S}}{\cos \mathrm{S}}. \quad . \quad . \quad . \quad . \quad (1.713)\end{aligned}$$

Again we have failed to obtain a generating function which accommodates the terms $\frac{n-5}{3}$ and $\frac{n-8}{3}$, but the explanation is now clear. It is evident from the last result that the relationship $o_{n\sigma}=\Delta^2_{\sigma} e_{n\sigma}$ must hold even when these extra terms are included, and this is certainly true as far as the series is known, for the series, starting with $\sigma=0$ is $\frac{n+4}{3}$, $\frac{n+1}{3}$, $\frac{n-2}{3}$, $\frac{n-5}{3}$, $\frac{n-8}{3}$, of which the second difference is zero. That the remainder of the series continues the arithmetical progression may be confirmed by substituting $e_{n\sigma}=\frac{n}{3}-\sigma+\frac{4}{3}$ in (1.703). It is clear therefore that the general expression for $f_{n,\,n-\sigma}$ contains, in addition to the terms containing the D's generated by (1.713), another term represented by $\frac{n+1}{3}-(\sigma+1)+2$, and it is easily seen that this is satisfactorily represented in $E(zs)$ by a term

$$\frac{1}{3(1-z)^2(1-s)}-\frac{1}{(1-z)(1-s)^2}+\frac{2}{(1-z)(1-s)}, \qquad (1.714)$$

whence

$$E(zs)=\frac{2s^3}{(1-s^2)^{3/2}}\quad\frac{1+\sin S}{S\cos}+\frac{1}{3(1-z)^2(1-s)}-\frac{1}{(1-z)(1-s)^2}+\frac{2}{(1-z)(1-s)}. \qquad (1.715)$$

The above considerations show that for negative values of σ, $f_{n,\,n-\sigma}$ has the value $\frac{n}{3}-\sigma+\frac{4}{3}$, that is

$$f_{n,\,r}=r-\tfrac{2}{3}(n-2) \quad (r>n). \qquad (1.716)$$

This result has, of course, no bearing on the practical problem which involves x and p and has a meaning only when $n>r$, but it is of theoretical interest.

As

$$\begin{aligned} x_{n,\,n-\sigma} &= \tfrac{3}{2}(f_{n+2,\,n-\sigma}-2f_{n+1,\,n-\sigma}+f_{n,\,n-\sigma}) \\ &= \tfrac{3}{2}(e_{n+2,\,\sigma+2}-2e_{n+1,\,\sigma+1}+e_{n\sigma}), \end{aligned} \qquad (1.717)$$

it follows that if

$$\eta(zs)=\sum\sum x_{n,\,n-\sigma}z^n s^\sigma, \qquad (1.718)$$

then

$$\eta(zs)=\tfrac{3}{2}E(zs)\times\left(\frac{1}{z^2s^2}-\frac{2}{zs}+1\right)=\tfrac{3}{2}\,\frac{(1-zs)^2}{z^2s^2}E(zs). \qquad (1.719)$$

However, $e_{n\sigma}=\frac{n}{3}-\sigma+\frac{4}{3}$ satisfies the equation

$$e_{n+2,\,\sigma+2}-2e_{n+1,\,\sigma+1}+e_{n\sigma}=0, \qquad (1.720)$$

and so contributes nothing to $x_{n, n-\sigma}$. Hence

$$\eta(zs) = \frac{3(1-zs)^2 s}{z^2(1-s^2)^{3/2}} \frac{1+\sin S}{\cos S}. \qquad (1.721)$$

2. The Case of the Limited Series.

2.1. *Statement of Problem.*

So far our main object has been the evaluation of x_{nr}, the chance that r consecutive runs selected at random from an infinite series of numbers should contain in all exactly n numbers. It is to be expected that the problem will be more difficult in the case where we select the r runs from a finite series of m unequal numbers. However, it is relatively easy to obtain the result in the most important cases, namely, $r=1$ and $r=2$.

The problem may be stated as follows: Let us suppose that we have m cards each bearing a number and that no two of these numbers are equal. The first m integers may be chosen. Let us choose any run at random of the $\frac{2m-1}{3}m!$ runs (*cf.* André) contained in the $m!$ arrangements of the m cards. It is required to find the chance x^m_{n1} that this run contains n cards.

2.2. *Distribution of Runs.*

We first calculate X^m_{n1}, the number of the $\frac{2m-1}{3}m!$ runs which contain n cards, the required chance x^m_{n1} being $X^m_{n1} \div \frac{2m-1}{3}m!$. It is convenient in the following to denote

$$X^m_{n1} \text{ by } Q_{mn}. \qquad (2.201)$$

We first find the difference equation satisfied by Q_{mn}. The method employed is similar to that used by André in finding the difference equation for P_{ns}. We consider the effect of adding one extra card to the pack, and note that we may without loss of generality mark this card with a higher number than that on any of the other cards, *e.g.* with the integer $n+1$. The new card may appear anywhere in any of the $m!$ arrangements, each arrangement thus giving rise to $m+1$ new arrangements, the total number being of course $(m+1)!$.

Let us consider the runs of size ν in the original set of $m!$ arrangements of the m cards, and let us investigate how they will be represented in the $(m+1)!$ arrangements of the $m+1$ cards.

The following possibilities may occur:—

(i) A run of ν may be entirely unaffected. This will happen whenever the new card falls anywhere except in the $\nu-1$ spaces within the run, or

in the case of an ascending run, immediately after the last card, or, in the case of a descending run, immediately before the first card. Thus there are $\nu Q_{m\nu}$ cases in which a run of ν will be disturbed. Now each run of ν occurring in the old $m!$ arrangements would be represented by $(m+1)Q_{m\nu}$ in the new $(m+1)!$ arrangements if no disturbance of the runs of ν takes place. It follows that the new $(m+1)!$ arrangements will actually contain in all $(m+1)\ Q_{m\nu} - \nu Q_{m\nu}$ undisturbed runs of ν; that is, $(m+1-\nu)Q_{m\nu}$ runs of ν. (The insertion of the new number between the last two numbers of an ascending run, though it produces a new run of ν from the old, is considered as disturbing the old run—see (iii) below.)

(ii) The new number may come at the end of an ascending run of ν, thus converting the run of ν into a run of $\nu+1$. In this way $Q_{m\nu}$ runs of $\nu+1$ are produced, one from each run of ν.

(iii) The new number may come between the last and second last numbers of an ascending run, or between the first and second numbers of a descending run, and convert the old run of ν into a new run of ν. This gives rise to $Q_{m\nu}$ runs of ν.

(iv) The new number may fall in any of the other $\nu-2$ spaces within the run. Each run of ν thus gives rise to 3 runs, one of size 2, one of size r, and one of size $\nu-r$, where r has the values $2, 3, \ldots \nu-1$. In this way we obtain $(\nu-2)Q_{m\nu}$ runs of 2, and $2Q_{m\nu}$ runs of sizes $2, 3, \ldots \nu-1$.

We are now in a position to evaluate $Q_{m+1, n}$ in terms of $Q_{m\nu}$, where ν may be equal to n, $n-1$ or $n-2$. Let $n \neq 2$.

By (i) we have a contribution of $(m+1-n)Q_{mn}$.

By (ii), in which case we must take $\nu=n-1$ because the original runs are increased by 1, we have a contribution of $Q_{m, n-1}$.

By (iii) we have a contribution of Q_{mn}.

By (iv) we have contributions of $2Q_{m, n+1} + 2Q_{m, n+2} + \ldots + 2Q_{mm}$, for, in this case, runs of n are produced from previously existing runs of $n+1, n+2, \ldots m$. Thus

$$\begin{aligned} Q_{m+1, n} &= (m+1-n)Q_{mn} + Q_{m, n-1} + Q_{mn} + 2Q_{m, n+1} + 2Q_{m, n+2} + \ldots 2Q_{mm} \\ &= Q_{m, n-1} + (m-n)Q_{mn} + 2(Q_{mn} + Q_{m, n+1} + Q_{m, n+2} + \ldots + Q_{mm}) \\ & \qquad (n > 2), \qquad (2.202) \end{aligned}$$

whence we obtain

$$Q_{m+1, n} - Q_{m+1, n+1} = Q_{m, n-1} + (m-n+1)Q_{mn} - (m-n-1)Q_{m, n+1} \qquad (n > 2). \qquad (2.203)$$

We have now to obtain the equation corresponding to (2.202) for the case $n=2$. Clearly here we have an additional contribution, namely, all the extra runs of 2 referred to in (iv) above. If we consider the positions in which the new number may fall so as not to give rise to such a run of 2,

it is not difficult to see that there are just as many such positions in any particular arrangement (and therefore in the whole set of arrangements) as there are runs. Now there are in all $\Sigma r P_{nr}$ runs, whilst the total number of possible positions for the new number is of course $(m+1)!$, whence the number of extra runs of 2 is

$$\begin{aligned}&(m+1)! - \sum rP_{nr}\\ &= m!\left(m+1-\frac{2m-1}{3}\right)\\ &= m!\,\frac{m+4}{3}.\end{aligned}$$

So that

$$Q_{m+1,\,2} = \frac{m+4}{3}m! + (m-2)Q_{m2} + 2(Q_{m2} + Q_{m3} + \ldots Q_{mm}). \quad . \quad (2.204)$$

But
$$Q_{m2} + Q_{m3} + \ldots Q_{mm} = \frac{2m-1}{3}m!,$$
whence
$$Q_{m+1,\,2} = \frac{5m+2}{3}m! + (m-2)Q_{m2}. \quad . \quad . \quad . \quad (2.205)$$

To solve this equation we write $Q_{m2} = (m-3)!Q'_{m2}$, whence

$$Q'_{m+1,\,2} = Q'_{m2} + m(m-1)\frac{(5m+2)}{3}. \quad . \quad . \quad . \quad (2.206)$$

It is easily shown that

$$Q'_{m2} = \frac{5}{3}\,\frac{(m+1)m(m-1)(m-2)}{4} - \frac{1}{3}m(m-1)(m-2) + c,$$

whence

$$Q_{m2} = \frac{5(m+1)! - 4m!}{12} + c(m-3)!,$$

where c is a constant.

Putting $m=4$, enumeration shows that $Q_{4,\,2}=42$, whence $c=0$, and therefore

$$Q_{m,\,2} = \frac{5m+1}{12}m!, \quad . \quad . \quad . \quad . \quad (2.207)$$

which may be written

$$Q_{m2} = \tfrac{5}{12}(m+1)! - \tfrac{1}{3}m! \quad (m \geqslant 3). \quad . \quad . \quad . \quad (2.208)$$

We may note that this formula does not hold for $m=2$, for it gives $Q_{22} = \frac{11}{12} \times 2$ instead of 2.

The value of $Q_{m,\,3}$ is now readily found. In equation (2.202) put $n=3$, and substitute the value of Q_{m2} from (2.207), and we find

$$Q_{m+1,\,3} = (m-3)Q_{m3} + \frac{11m-9}{12}m!, \quad . \quad . \quad . \quad (2.209)$$

whence it follows that

$$Q_{m3} = \frac{11m-14}{60}m! \quad . \quad . \quad . \quad . \quad (2.210)$$

(the constant which appears being easily proved to be zero), which may also be written

$$Q_{m3} = \tfrac{11}{60}(m+1)! - \tfrac{5}{12}m! \quad (m > 3). \qquad (2.211)$$

To obtain the general value of Q_{mn}, we make use of equation (2.203). If $Q'_{mn} = Q_{mn} \div m!$, then Q'_{mn} satisfies the equation

$$(m+1)\left\{Q'_{m+1,\,n} - Q'_{m+1,\,n+1}\right\} = Q'_{m,\,n-1} + (m-n+1)Q'_{mn} - (m-n-1)Q'_{m,\,n+1}. \qquad (2.212)$$

From this we can write down the differential equation satisfied by the generating function $\psi(yz) = \sum\sum Q'_{mn}y^m z^n = \sum\sum y^m z^n \frac{Q_{mn}}{m!}$, namely,

$$\left(y\frac{\partial}{\partial y}+1\right)\frac{\psi}{z} - \left(y\frac{\partial}{\partial y}+1\right)\frac{\psi}{yz} = z\psi + \left(y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} + 1\right)\psi - \left(y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - 1\right)\frac{\psi}{y}$$

or

$$(1-y)(1-\tfrac{1}{z})\frac{\partial\psi}{\partial y} + (z-1)\frac{\partial\psi}{\partial z} - (1+z)\psi = 0, \qquad (2.213)$$

of which the general solution is found by the usual methods to be

$$\psi = (1-z)^2 e^z \chi\{(1-y)z\},$$

where χ is an arbitrary function. To determine χ we make use of the value obtained for Q_{m2} (2.208), from which it follows that

$$Q'_{m2} = \tfrac{5}{12}(m+1) - \tfrac{1}{3}.$$

The coefficient of z^2 is therefore evidently

$$\frac{5}{12(1-y)^2} - \frac{1}{3(1-y)},$$

whence χ must have the form

$$\frac{A}{z^2(1-y)^2} - \frac{B}{z(1-y)}.$$

It is easily shown that $A = B = 2$, whence

$$\psi = 2(1-z)^2 e^z\left\{\frac{1}{z^2(1-y)^2} - \frac{1}{z(1-y)}\right\}. \qquad (2.214)$$

It is readily verified that this gives the correct value for Q'_{m3} $(m > 3)$.

It follows that

$$Q_{mn} = k_n(m+1)! - k_{n-1}m! \quad (m > n > 1), \qquad (2.215)$$

where

$$k_n = 2\Delta^2\frac{1}{n!} = 2\frac{(n^2+n-1)}{(n+2)!}, \qquad (2.216)$$

so that

$$\left.\begin{aligned} k_1 &= \frac{1}{3} &&= \frac{2}{3!} \\ k_2 &= \frac{5}{12} &&= \frac{10}{4!} \\ k_3 &= \frac{11}{60} &&= \frac{22}{5!} \\ k_4 &= \frac{19}{360} &&= \frac{38}{6!} \\ k_5 &= \frac{29}{2520} &&= \frac{58}{7!}, \quad \text{etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.217)$$

For $n=m$, (2.215) gives the value $Q_{mm}=2-\frac{2}{(m+1)(m+2)}$, instead of the correct value 2. The value of $Q_{m,\ m-1}$ reduces to $4(m-1)$, and it is easy to show that this is correct.

Finally

$$X^m_{n1}=k_n(m+1)!-k_{n-1}m!, \qquad (m>n>1), \quad . \quad . \quad . \quad (2.218)$$

also

$$x^m_{n1}=X^m_{n1}\div\sum X^m_{n1}=\frac{(m+1)k_n-k_{n-1}}{\frac{2m-1}{3}}, \qquad (m>n>1). \quad . \quad (2.219)$$

Evidently

$$\lim_{m\to\infty} x^m_{n1}=\tfrac{3}{2}k_n=3\Delta^2\frac{1}{n!}, \quad . \quad . \quad . \quad . \quad (2.220)$$

the result for the infinite series already obtained by a different method (1.33).

2.3. *Distribution of Gaps.*

In order to obtain x^m_{n2} the chance that a particular gap—that is, a pair of consecutive runs—taken at random out of all possible arrangements of m different numbers, contains n numbers, we first of all calculate X^m_{n2} the number of gaps (in the $m!$ different arrangements of the m numbers) which contain n numbers, including the first and the last.

We note that the total number of gaps is

$$\sum_3^m X^m_{n2}=\sum_2^{m-1}(r-1)P_{mr}=\sum_1^{m-1}(r-1)P_{mr} \quad . \quad . \quad . \quad (2.301)$$

(because an arrangement containing r runs gives rise to $r-1$ different gaps), thus

$$\begin{aligned}\sum_3^m X^m_{n2}&=\sum_1^{m-1}rP_{mr}-\sum_1^{m-1}P_{mr}\\ &=\tfrac{2}{3}(m-2)m!, \quad . \quad . \quad . \quad . \quad . \quad (2.302)\end{aligned}$$

so that

$$x_{n2}^{m} = \mathrm{X}_{n2}^{m} \div \tfrac{2}{3}(m-2)m!.$$

To save printing the suffix 2 unnecessarily it is convenient to denote

$$\mathrm{X}_{n2}^{m} \text{ by } \mathrm{G}_{mn}. \qquad (2.303)$$

It is now necessary to introduce I_{mn}, the number of arrangements which begin with a run of length n. Evidently

$$\mathrm{I}_{m,\,m} = \mathrm{P}_{m,} = 2 \qquad (2.304)$$

and

$$\sum_{2}^{m} \mathrm{I}_{mn} = m! \qquad (2.305)$$

as every arrangement begins with a run of one length or another. By symmetry I_{mn} is also equal to the number of arrangements which end with a run of length n.

We first of all obtain the difference equations for G_{mn} and I_{mn} by considering the effect of the addition of an $(m+1)$th number—greater than any of the previous numbers—to the set. Each arrangement gives rise to $m+1$ new arrangements, according to the position into which the number falls. By following a course of argument similar to that adopted in the case of single runs, it may be shown that the equation satisfied by I_{mn} is

$$\mathrm{I}_{m+1,\,n} = (m-n)\mathrm{I}_{mn} + \mathrm{I}_{m,\,n-1} + \mathrm{I}_{mn} + \mathrm{I}_{m,\,n+1} + \ldots \mathrm{I}_{mm}, \quad (n \geqslant 3), \qquad (2.306)$$

whilst for $n=2$, we obtain

$$\mathrm{I}_{m+1,\,2} = (m-2)\mathrm{I}_{m2} + \mathrm{I}_{m2} + \mathrm{I}_{m3} + \ldots + \mathrm{I}_{mm} + m!, \qquad (2.307)$$

the last term being due to the extra runs of 2 coming from the addition of a new number at the beginning of an arrangement starting with an ascending run, or at the end of an arrangement finishing with a descending run.

The equation satisfied by G_{mn} is found to be

$$\begin{aligned}\mathrm{G}_{m+1,\,n} = (m-n)\mathrm{G}_{mn} &+ 2(\mathrm{G}_{m,\,n-1} + \mathrm{G}_{mn} + \mathrm{G}_{m,\,n+1} + \ldots + \mathrm{G}_{mm}) \\ &+ 2(\mathrm{I}_{m,\,n-1} + \mathrm{I}_{mn} + \mathrm{I}_{m,\,n+1} + \ldots + \mathrm{I}_{mm}) \quad (n \geqslant 3). \qquad (2.308)\end{aligned}$$

This equation holds for

$$(n \geqslant 3), \quad \text{and } \mathrm{G}_{m2} = 0. \qquad (2.309)$$

From (2.307) we obtain

$$\mathrm{I}_{m+1,\,2} = (m-2)\mathrm{I}_{m2} + 2m!,$$

the solution of which is readily found to be

$$\mathrm{I}_{m2} = \tfrac{2}{3}m!, \qquad (2.310)$$

the constant which appears being easily evaluated as zero. From (2.306)

we readily obtain

$$I_{m+1, n} - I_{m+1, n+1} = I_{m, n-1} + (m-n)I_{mn} - (m-n-1)I_{m, n+1}, \quad . \quad (2.311)$$

whence the generating function $\eta(yz) = \sum_0^\infty \sum_0^\infty \frac{I_{mn} y^m z^n}{m!}$ satisfies a partial differential equation, which on reduction is found to be

$$(1-y)\left(\frac{z-1}{z}\right)\frac{\partial\eta}{\partial y} + (z-1)\frac{\partial\eta}{\partial z} = z\eta, \quad . \quad . \quad . \quad (2.312)$$

and it follows that

$$\eta = (z-1)e^z \Phi\{(1-y)z\},$$

where Φ is an arbitrary function.

From the value already obtained for I_{m2} (2.310) the coefficient of z^2 in this equation must be $\frac{2}{3}y^2 + \frac{2}{3}y^3 + \frac{2}{3}y^4 + \ldots$, that is (apart from terms in y^0 and y^1) it must be $\frac{2}{3(1-y)}$, so that the arbitrary function $\Phi\{(1-y)z\}$ must be equal to $\frac{A}{(1-y)z}$, thus

$$\eta = \frac{(z-1)}{z} \frac{e^z}{(1-y)} A.$$

The coefficient of z^2 is then

$$\left(\frac{1}{2!} - \frac{1}{3!}\right)\frac{A}{(1-y)} = \frac{A}{3(1-y)},$$

whence

$$\frac{A}{3} = \frac{2}{3} \quad \text{or} \quad A = 2.$$

Therefore

$$\eta = \frac{2(z-1)}{z} \frac{e^z}{(1-y)} \quad . \quad . \quad . \quad . \quad (2.313)$$

and

$$I_{mn} = \frac{2n}{(n+1)!} m! \quad (m > n \geqslant 2). \quad . \quad . \quad . \quad (2.314)$$

[This result may be obtained directly from the consideration that the chance that the initial run will be a run of n, is equal to twice the chance that, of the first $n+1$ numbers, the first n should be in ascending order, but that the last be smaller than the penultimate. It is not difficult to show that the chance that $n+1$ numbers taken at random should have this characteristic is $\frac{n}{(n+1)!}$, and, if we note that there are in all $m!$ arrangements, it follows that the number with initial runs of length n will be $2 \times \frac{n}{(n+1)!} \times m!$.]

From (2.308),

$$G_{m+1,3} = (m-3)G_{m3} + 2\sum_{3}^{m} G_{mr} + 2\sum_{2}^{m} I_{mr}$$
$$= (m-3)G_{m3} + \tfrac{4}{3}(m-2)m! + 2m!$$
$$= (m-3)G_{m3} + \tfrac{4}{3}(m+1)! - 2m!,$$

whence it follows that

$$G_{m3} = \frac{4}{3\cdot 5}(m+1)! - \frac{2}{4}m!, \quad . \quad . \quad . \quad . \quad (2.315)$$

the constant which appears having the value zero. This equation may be written in the form

$$G_{m3} = l_3(m+1)! - \tfrac{1}{2}m! \quad . \quad . \quad . \quad . \quad (2.316)$$

where

$$l_n = \Delta^2 \frac{2^n}{n!} = \frac{2^n}{n!}\,\frac{n-2}{n+2}, \quad . \quad . \quad . \quad . \quad (2.317)$$

so that

$$l_2 = 0, \quad l_3 = \tfrac{4}{15}, \quad l_4 = \tfrac{2}{9}, \quad l_5 = \tfrac{16}{37}, \quad \text{etc.} \quad . \quad . \quad . \quad (2.318)$$

Also

$$G_{m+1,4} = (m-4)G_{m4} + 2\sum_{3}^{m} G_{mr} + 2\sum_{2}^{m} I_{mr} - 2I_{m2}$$
$$= (m-4)G_{m4} + \tfrac{4}{3}(m+1)! - 2m! - \tfrac{4}{3}m!$$
$$= (m-4)G_{m4} + \tfrac{4}{3}(m+1)! - \tfrac{10}{3}m!,$$

whence

$$G_{m4} = \frac{4}{3\cdot 6}(m+1)! - \frac{10}{3\cdot 5}m!$$

or

$$G_{m4} = l_4(m+1)! - \tfrac{2}{3}m! \quad . \quad . \quad . \quad . \quad (2.319)$$

(the constant being zero as usual).

Now, from (2.308), we have

$$G_{m+1,n} - G_{m+1,n+1} = 2G_{m,n-1} + (m-n)G_{mn} - (m-n-1)G_{m,n+1} + 2I_{m,n-1}, \quad (2.320)$$

and if we write $J_{mn} = G_{mn} + 2I_{mn}$, it is readily found, using (2.311), that

$$J_{m+1,n} - J_{m+1,n+1} = 2J_{m,n-1} + (m-n)J_{mn} - (m-n-1)J_{m,n+1}. \quad . \quad (2.321)$$

Writing $\omega(yz) \equiv \sum\sum \dfrac{J_{mn}y^m z^n}{m!}$, it is found that ω satisfies the partial differential equation

$$(1-y)\left(1-\frac{1}{z}\right)\frac{\partial\omega}{\partial z} + (z-1)\frac{\partial\omega}{\partial y} = 2z\omega, \quad . \quad . \quad . \quad (2.322)$$

whence

$$\omega = (z-1)^2 e^{2z}\Phi\{(1-y)z\},$$

where Φ is an arbitrary function.

We note that

$$J_{m3} = G_{m3} + 2I_{m3} = l_3(m+1)! \quad . \quad . \quad . \quad (2.323)$$

and

$$J_{m4} = l_4(m+1)! - 2l_3 m!, \quad . \quad . \quad . \quad . \quad (2.324)$$

from which it follows that the coefficient of z^3 in ω is

$$(5y^4+6y^5+7y^6+\ldots)l_3,$$

or (disregarding y^0, y^1, y^2 and y^3) it is $\dfrac{l^3}{(1-y)^2}$. Similarly, the coefficient of z^4 is $\dfrac{l_4}{(1-y)^2}-\dfrac{2l_3}{(1-y)}$, whence it is easily found that

$$\omega(yz)=(z-1)^2e^{2z}\left(\frac{1}{z^2(1-y)^2}-\frac{2}{z(1-y)}\right). \qquad (2.325)$$

From this it follows that

$$J_{mn}=l_n(m+1)!-2l_{n-1}m!, \qquad (2.326)$$

and this evidently holds for $n=3$ as $l_2=0$. Thus

$$G_{mn}=l_n(m+1)!-2l_{n-1}m!-2I_{mn}$$

$$=l_n(m+1)!-2l_{n-1}m!-\frac{4n}{(n+1)!}m!, \quad (m>n\geqslant 3), \qquad (2.327)$$

and the generating function for G_{mn} is clearly

$$\zeta(yz)=\frac{(z-1)^2}{z^2}e^{2z}\frac{1}{(1-y)^2}-2\frac{(z-1)^2}{z}e^{2z}\frac{1}{1-y}-4\frac{(z-1)}{z}e^{z}\frac{1}{1-y}. \qquad (2.328)$$

Thus

$$x^m_{n2}=\frac{3X^m_{n2}}{2(m-2)m!}=\frac{3G_{mn}}{2(m-2)m!}, \quad (m>n\geqslant 3), \qquad (2.329)$$

which is the general expression for the chance that a gap selected at random from any arrangements of m unequal numbers should be of length n.

Evidently as $m\rightarrow\infty$, $\dfrac{G_{mn}}{(m-2)m!}\rightarrow l_n$, and so

$$\lim_{m\rightarrow\infty} x^m_{n2}=\tfrac{3}{2}ln=\tfrac{3}{2}\Delta^2\frac{2^n}{n!}, \qquad (2.330)$$

which is the value already obtained in the case of the infinite series. From the conditions of the problem it is evident that the limited case should approach the case of the infinite series as m is made very large, so that the above result is in agreement with requirements.

It is to be noted that equation (2.327) gives

$$G_{mm}=2^m-4-\frac{4}{m+1}\left(\frac{2^m}{m+2}-1\right)$$

instead of the correct value which is easily seen to be P_{m2}, that is

$$G_{mm}=2^m-4. \qquad (2.331)$$

By putting $n=m-1$ in (2.327), it is found that

$$G_{m,\,m-1}=\{(2^m-4)(m-2)-4\}=P_{m2}(m-2)-4. \qquad (2.332)$$

That this is correct may be deduced from the consideration that if we select any one of the m numbers—which we can do in m ways—and add it at the beginning or end of the other $m-1$ numbers arranged so as to give either one or two runs, then we obtain all the possible arrangements which contribute to $G_{m,\ m-1}$, together with twice the number of arrangements of the m numbers arranged either as one or as two runs. So that

$$2m(P_{m-1,\ 2}+P_{m-1,\ 1})=G_{m,\ m-1}+2P_{m2}+2P_{m1}.$$

But

$$2P_{m-1,\ 1}=2, \quad \text{and} \quad 2P_{m-1,\ 2}=P_{m2}-4,$$

whence it easily follows that

$$G_{m,\ m-1}=(m-2)P_{m2}-4.$$

This confirmation of the result not only shows that the formula (2.327) holds for the special case $n=m-1$, but it also verifies the general correctness of the theory.

Evidently the evaluation of x_{nr}^{m} becomes very complicated for $r>2$, and we have not attempted to solve the problem.

3. Cyclic Arrangements.

3.1. *Distribution of Types of Arrangements ($\bar{P}_{mr}$) and the Associated Generating Function.*

The case where the numbers are arranged in a cycle—as, for example, in the distribution of cards round a circular table—may be developed by similar methods to those employed by André, and in the preceding sections.

We will denote by $\bar{P}_{mr}$ the number of cyclic arrangements of m unequal numbers containing r runs. Clearly for all odd values of r and for even values of $r>n$, $\bar{P}_{mr}=0$. In enumerating the arrangements we shall consider as distinct only those in which the cyclic order of the numbers is different. It follows that the number of such distinct arrangements is $(m-1)!$.

We obtain the difference equation for $\bar{P}_{mr}$ by considering the effect of adding a new number, larger than any of the previous ones, to the existing set. By considerations similar to those employed by André, we obtain the equation,

$$\bar{P}_{m+1,\ r}=r\bar{P}_{mr}+(m-r+2)\bar{P}_{m,\ r-2}, \quad . \quad . \quad . \quad (3.101)$$

also it is easily seen that

$$\bar{P}_{2,\ 2}=1, \quad \bar{P}_{3,\ 2}=2. \quad \bar{P}_{4,\ 2}=4 \quad \text{and} \quad \bar{P}_{4,\ 4}=2. \quad . \quad . \quad (3.102)$$

Equation (3.101) is very similar in form to that for F_{nr}, and the generating function may be obtained in a similar way.

Let

$$\bar{\phi}(yt) = \sum\sum \bar{P}_{mr}\frac{y^m t^r}{m!},$$

then it is found by solving the partial differential equation

$$(1 - zt^2)\frac{\partial\bar{\phi}}{\partial z} - t(1 - t^2)\frac{\partial\bar{\phi}}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad (3.103)$$

satisfied by $\bar{\phi}$ that

$$\bar{\phi} = \Phi(\tau e^{\lambda y}),$$

where, as before,

$$\lambda = \sqrt{1 - t^2}, \quad \text{and} \quad \tau = \frac{1 - \lambda}{t}.$$

The initial conditions which the arbitrary function must conform to are, however, distinctly different from those obtaining in the case of F_{nr}. We may, however, evaluate the arbitrary function Φ by methods similar to those previously adopted.

Let

$$\Phi(\tau e^{\lambda y}) = A_0 + A_1\tau e^{\lambda y} + A_2\tau^2 e^{2\lambda y} + \ . \ . \ . \ + A_s\tau^s e^{s\lambda y} + \ . \ . \ .$$

Obviously, as $\bar{P}_{mr} = 0$ for all odd values of r, Φ must be an even function of t, and therefore of τ, so that $A_1 = A_3 = A_5 = \ . \ . \ . = 0$.

We now consider the limit to which $\bar{P}_{mr}$ (r even) tends when $m \to \infty$. Following the line of argument previously adopted in (1.5) we imagine r boxes (r even) arranged round a table, and m cards distributed at random in the r boxes, m being so great that the chance of any box having less than 2 cards is negligible. The possible number of such distributions is r^m, and corresponding to each distribution there will be 2 arrangements of runs obtained by arranging the cards in any one box either as an ascending or as a descending run, the cards in the next box being arranged in a descending or an ascending run, and so on, each ascending run being followed by a descending run and *vice versa*. However, each of the $\bar{P}_{mr}$ cyclic arrangements which contain more than 4 numbers in every run corresponds to $r2^r$ different distributions in the boxes, the factor 2^r appearing from the consideration that every maximal or minimal number belongs to two adjacent runs, and the factor r arising from the fact that the allocation of the runs to the boxes may be made in r ways.

Thus

$$r2^r\bar{P}_{mr} = 2r^m(1 + \epsilon),$$

where $\epsilon \to 0$ as $m \to \infty$, whence

$$\lim_{m\to\infty}\frac{\bar{P}_{mr}}{r^m} = \frac{2^{-(r-1)}}{r}. \quad . \quad . \quad . \quad . \quad (3.104)$$

As before, we find from the generating function that

$$\lim_{m\to\infty} \bar{P}_{mr} = \frac{A_r r^m}{2^r},$$

so that

$$\frac{A_r}{2^r} = \frac{2^{-(r-1)}}{r} \qquad \text{or} \qquad A_r = \frac{2}{r} \qquad (r \text{ even}), \qquad . \quad . \quad (3.105)$$

whence

$$\begin{aligned}\bar{\phi} &= \tfrac{2}{2}\tau^2 e^{2\lambda y} + \tfrac{2}{4}\tau^4 e^{4\lambda y} + \tfrac{2}{6}\tau^6 e^{6\lambda y} + \dots \\ &= -\log(1 - \tau^2 e^{2\lambda y}). \qquad . \quad . \quad . \quad . \quad . \quad (3.106)\end{aligned}$$

It is readily seen that this gives

$$\bar{P}_{m2} = \frac{2^m}{2^2}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3.107)$$

$$\bar{P}_{m4} = \frac{4^m}{4 \cdot 2^3} - \frac{(m-1)2^m}{2^3}, \qquad . \quad . \quad . \quad . \quad . \quad (3.108)$$

$$\bar{P}_{m6} = \frac{6^m}{3 \cdot 2^6} - \frac{(m-2)}{2^6}4^m + \frac{(2m^2 - 8m + 5)}{2^6}2^m. \qquad . \quad . \quad (3.109)$$

It may be verified that these expressions give the correct values for $\bar{P}_{22}$, $\bar{P}_{42}$ and $\bar{P}_{44}$.

If we write

$$\bar{P}_{mr} = \sum_s \bar{K}^s_{mr} s^m \quad (r \text{ even} = 2\rho), \qquad . \quad . \quad . \quad (3.110)$$

then clearly $\bar{K}^s_{mr}$ is the coefficient of t^r in the expansion of $\frac{2}{s}\tau^s\lambda^m$; that is, the coefficient of t^{r+s} in the expansion of $\frac{2}{s}(1-\lambda)^s\lambda^m$; that is, in $\sum_l (-)^l \frac{2}{s}\lambda^{m+l} C_{sl}$, where C_{sl} denotes the binomial coefficient.

Let $r + s = 2\theta$, so that $\theta = \rho + \sigma$, $(\sigma = \frac{1}{2}s)$, then

$$\begin{aligned}\bar{K}^s_{mr} &= \frac{2}{s}\sum_l (-)^{l+\theta} C_{sl} C_{\frac{m+l}{2},\,\theta} \\ &= \frac{1}{\sigma}\sum_l (-)^{l+\rho+\sigma} C_{2\sigma,\,l} C_{\frac{m+l}{2},\,\rho+\sigma} \qquad . \quad . \quad . \quad (3.111)\end{aligned}$$

or

$$\bar{K}^s_{mr} = \frac{1}{\sigma}(-)^{\rho+\sigma}\Delta^{2\sigma}_{m} C_{\frac{m}{2},\,\rho+\sigma} \qquad . \quad . \quad . \quad . \quad (3.112)$$

Indeed (*cf.* (1.518)).

$$\bar{K}^s_{mr} = \frac{1}{2s} K^s_{m+3,\,r}. \qquad . \quad . \quad . \quad . \quad (3.113)$$

Thus

$$\left.\begin{aligned}\bar{K}^2_{m4} &= \tfrac{1}{4}K^2_{m+3,\,4} = -\frac{m-1}{2^3}, \\ \bar{K}^4_{m6} &= \tfrac{1}{8}K^4_{m+3,\,6} = -\frac{m-2}{2^6},\end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (3.114)$$

and

$$\bar{K}^2_{m6} = \tfrac{1}{4}K^2_{m+3,\,6} = \frac{2m^2 - 8m + 5}{2_6}$$

in agreement with (3.108) and (3.109). It follows from the above that

$$\bar{P}_{mr} = \tfrac{1}{2}\sum_s K^s_{m+3,\,r}S^{m-1}. \qquad . \qquad . \qquad . \qquad . \; (3.115)$$

We may note that by writing $r = m + 1$ in (3.101) we obtain

$$\bar{P}_{m+1,\,m+1} = \bar{P}_{m,\,m-1} \quad \text{or} \quad \bar{P}_{m,\,m} = \bar{P}_{m-1,\,m-2}. \qquad . \qquad . \; (3.116)$$

But when m is odd, and we take the cyclic arrangement of the type $\bar{P}_{m,\,m-1}$, in which all the runs are of length 2 except one which is of length 3, there are 2 ways in which we may break the cycle so as to give a straight arrangement with $m - 1$ runs of 2. Whence it is easily seen that

$$\bar{P}_{m,\,m-1} = \tfrac{1}{2}P_{m,\,m-1} \quad (m \text{ odd}) \qquad . \qquad . \qquad . \; (3.117)$$

$$= D_{m-2}m! \quad (\text{by } (1.609)) \qquad . \qquad . \qquad . \; (3.118)$$

and, making use of (3.116),

$$\bar{P}_{m,\,m} = \tfrac{1}{2}P_{m-1,\,m-2} \quad (m \text{ even}) \qquad . \qquad . \qquad . \; (3.119)$$

$$= D_{m-3}(m - 1)!, \qquad . \qquad . \qquad . \qquad . \qquad . \; (3.120)$$

where D_m is the determinant, (1.604) or (1.605) Further, if in (3.101) we put $r = m$, we find

$$\bar{P}_{m+1,\,m} = m\bar{P}_{mm} + 2\bar{P}_{m,\,m-2},$$

whence, when m is even,

$$\bar{P}_{m,\,m-2} = \tfrac{1}{2}((m + 1)!\,D_{m-1} - m!\,D_{m-3}). \qquad . \qquad . \qquad . \; (3.121)$$

Again putting $r = m - 1$,

$$\bar{P}_{m+1,\,m-1} = (m - 1)\bar{P}_{m,\,m-1} + 3\bar{P}_{m,\,m-3},$$

from which it follows that, when m is odd,

$$\bar{P}_{m,\,m-3} = \tfrac{1}{6}((m + 2)!\,D_m - (3m - 1)m!\,D_{m-2}). \qquad . \qquad . \; (3.122)$$

When m is odd, $\bar{P}_{mm}$, $\bar{P}_{m,\,m-2}$, $\bar{P}_{m,\,m-4}$, and when m is even $\bar{P}_{m,\,m-1}$, $\bar{P}_{m,\,m-3}$ and $\bar{P}_{m,\,m-5}$ are of course all zero.

3.2. *The Generating Function for $\bar{P}_{m,\,m-\sigma}$.*

As in the case of the functions F, P and X (section 1.7) it is interesting to inquire in what way the above results can be extended so as to give a general expression for $\bar{P}_{m,\,m-\sigma}$. As in the previous case, the actual

expressions become highly complicated, but it is not difficult to obtain a generating function.

Let

$$\bar{o}_{m\sigma} = \bar{P}_{m,\, m-\sigma} \div m!. \qquad (3.21)$$

From (3.101) it follows that $\bar{o}_{m\sigma}$ satisfies the difference equation

$$(m+1)\bar{o}_{m+1,\, \sigma+1} = (m-\sigma)\bar{o}_{m\sigma} + (\sigma+2)\bar{o}_{m,\, \sigma+2}, \qquad (3.22)$$

whence

$$\bar{\Omega}(ys) \equiv \sum\sum \bar{o}_{m\sigma} y^m s^\sigma$$

satisfies the partial differential equation

$$\left(\frac{1}{s} - y\right)\frac{\partial\bar{\Omega}}{\partial y} + \left(s - \frac{1}{s}\right)\frac{\partial\bar{\Omega}}{\partial s} = 0. \qquad (3.23)$$

From this it follows that $\bar{\Omega}(ys)$ is of the form $\Phi(\bar{S})$, where

$$\bar{S} = y\sqrt{1-s^2} + \sin^{-1} s, \qquad (3.24)$$

and Φ is an arbitrary function.

In order to determine Φ it is convenient to rewrite (3.120), (3.118), (3.121) and (3.122) in the following form:—

$$\left.\begin{aligned}
&\bar{o}_{m0} = \frac{1}{m}D_{m-3}\ (m \text{ even}), \quad \text{and} \quad = 0\ (m \text{ odd}),\\
&\bar{o}_{m1} = 0\ (m \text{ even}), \quad \text{and} \quad = D_{m-2}\ (m \text{ odd}),\\
&\bar{o}_{m2} = \tfrac{1}{2}\{(m+1)D_{m-1} - D_{m-3}\}\ (m \text{ even}) \quad \text{and} \quad = 0\ (m \text{ odd}),\\
&\bar{o}_{m3} = 0\ (m \text{ even}), \quad \text{and} \quad = \tfrac{1}{6}\{m(m+1)D_m - (3m-1)D_{m-2}\} \quad (m \text{ odd}).
\end{aligned}\right\} \qquad (3.25)$$

If we remember that $\tan z = \dfrac{\sin z}{\cos z}$ contains all the odd terms of $\dfrac{1+\sin z}{\cos z}$ the generating function for D_{n-2}, it is evident that to $\bar{o}_{m0}$ there corresponds the generating function $\int \tan y dy = -\log\cos y$, so that $\Phi(\bar{S})$ must reduce to $-\log\cos y$ when s is zero, *i.e.* when $\bar{S} = y$. This suggests that the required generating function is

$$\bar{\Omega}(ys) = -\log\cos \bar{S}. \qquad (3.26)$$

It is evident that the coefficient of s in the expansion of $\bar{\Omega}$ is $\tan y$, which gives $\bar{o}_{m1} = 0$ (m even) and $= D_{m-2}$ (m odd) as required.

Replacing $\bar{S}$ by $y + \dfrac{s^2 y}{2} + \dfrac{s^3}{6}$, and expanding $-\log\cos\bar{S}$ by Taylor's theorem, it is not difficult to show that we obtain the correct values for $\bar{o}_{m2}$ and $\bar{o}_{m3}$, whence it follows that $\bar{\Omega}(ys)$ as given in (3.26) is the correct generating function.

In 1.7 we showed that the generating functions for $e_{n\sigma}$ and $o_{n\sigma}$ were closely related to the generating functions for f_{nr} and p_{nr}. The same type of relation exists between $\bar{\Omega}(ys)$ and $\bar{\phi}(yt)$, the generating functions

for $\bar{o}_{n\sigma}$ and F_{nr} respectively. If in (3.106), *i.e.* $\bar{\phi}(yt) = -\log(1 - \tau^2 e^{2\lambda y})$, we write $y = y's$ and $t = \frac{1}{s}$ we find $\tau = ie^{i \sin^{-1} s}$ and $\lambda y = iy'\sqrt{1-s^2}$, so that

$$\bar{\phi}(yt) = -\log(1 + e^{2i\bar{S}}) = -\log\cos\bar{S} - i\bar{S} + \log 2 = \bar{\Omega}(ys) - i\bar{S} + \log 2. \quad (3.27)$$

Thus $\bar{\phi}$ transforms into $\bar{\Omega}$ together with a constant term and a part which is purely imaginary.

3.3. *Some Identities.*

From equations (1.411), (1.413), (1.415), (1.608), (1.612) and (1.515) two expressions for $f_{n,\, n-\sigma}$ ($\sigma = 0$, 1, 2, 3 and 4) can be deduced, and so by equating these values a series of identities is obtained:

$$\sum_s K^s_{nn} s^n \equiv \frac{n+4}{3} n! \quad (n > 1) \quad . \quad . \quad . \quad . \quad . \quad (3.301)$$

$$\sum_s K^s_{n,\, n-1} s^n \equiv \frac{n+1}{3} n! \quad (n > 2) \quad . \quad . \quad . \quad . \quad (3.302)$$

$$\sum_s K^s_{n,\, n-2} s^n \equiv \frac{n-2}{3} n! \quad (n > 3) \quad . \quad . \quad . \quad . \quad (3.303)$$

$$\sum_s K^s_{n,\, n-3} s^n \equiv \left(\frac{n-5}{3} + 2D_{n-2}\right) n! \quad (n > 4) \quad . \quad . \quad . \quad (3.304)$$

$$\sum_s K^s_{n,\, n-4} s^n \equiv \left(\frac{n-8}{3} + 2(n+1)D_{n-1}\right) n! \quad (n > 5). \quad . \quad (3.305)$$

These identities may be regarded as expressing the properties of

$$K^s_{nr} = 4(-)^{\theta+s} \underset{n}{\Delta^s} C_{\frac{n-3}{2},\, \theta}. \quad . \quad . \quad . \quad . \quad (1.519)$$

In the same way we may equate the values of $\bar{P}_{m,\, m-\sigma}$ ($\sigma = 0$, 1, 2 and 3) given by equations (3.120), (3.118), (3.121), (3.122) and (3.110) (using (3.113)), and we find

$$\tfrac{1}{2}\sum_s K^s_{m+3,\, m} s^{m-1} \equiv D_{m-3}(m-1)! \quad (m \text{ even}, m > 1) \quad . \quad . \quad . \quad (3.306)$$

$$\tfrac{1}{2}\sum_s K^s_{m+3,\, m-1} s^{m-1} \equiv D_{m-2} m! \quad (m \text{ odd}, m > 2) \quad . \quad . \quad . \quad . \quad (3.307)$$

$$\sum_s K^s_{m+3,\, m-2} s^{m-1} \equiv D_{m-1}(m+1)! - D_{m-3} m! \quad (m \text{ even}, m > 3) \, . \quad . \quad (3.308)$$

$$\sum_s K^s_{m+3,\, m-3} s^{m-1} \equiv \tfrac{1}{3}(D_m(m+2)! - (3m-1)D_{m-2} m!) \quad (m \text{ odd}, m > 4), \quad (3.309)$$

and if we rewrite these equations, replacing $m+3$ by n,

$$\tfrac{1}{2}\sum_s K^s_{n,\,n-3}s^{n-4} \equiv D_{n-6}(n-4)! \quad (n \text{ odd}, n > 4) \qquad (3.310)$$

$$\tfrac{1}{2}\sum_s K^s_{n,\,n-4}s^{n-4} \equiv D_{n-5}(n-3)! \quad (n \text{ even}, n > 5) \qquad (3.311)$$

$$\sum_s K^s_{n,\,n-5}s^{n-4} \equiv D_{n-4}(n-2)! - D_{n-6}(n-3)! \quad (n \text{ odd}, n > 6) \qquad (3.312)$$

$$\sum_s K^s_{n,\,n-6}s^{n-4} \equiv \tfrac{1}{3}(D_{n-3}(n-1)! - (3n-10)D_{n-5}(n-3)!) \ (n \text{ even}, n > 7). \qquad (3.313)$$

In relation to this new set of identities the question naturally arises as to whether the limitations n odd and n even are essential. Examination of a few particular cases at once shows that it is not, and, though we have not attempted to obtain a rigorous proof, it is clear that equations (3.310) to (3.313) hold for all values of n greater than those marginally noted.

Some light is shed on this point by the following considerations. The generating function for $\dfrac{\bar{P}_{mr}}{m!}$, namely, $\bar{\phi}(yt) = -\log(1-\tau^2 e^{2\lambda y})$, which is an even function of t, may be written $-\{\log(1-\tau e^{\lambda y}) + \log(1+\tau e^{\lambda y})\}$. Now λ is an even function of t, and τ is an odd function of t, so that the original generating function $\bar{\phi}(yt)$ is really double the even terms of the series obtained by the expansion of $-\log(1-\tau e^{\lambda y})$. As this function is of the type that satisfies the partial differential equation associated with (3.101), it is to be expected that the coefficients of the odd powers of t will give values for $\tfrac{1}{2}\bar{P}_{mr}$ (r odd) corresponding to the values of $\bar{P}_{mr}$ (r even) already obtained.

In the same way the function

$$-\log\cos\bar{S} \equiv \tfrac{1}{2}(-\log\cos^2\bar{S}) \equiv -\tfrac{1}{2}\{\log(1+\sin\bar{S}) + \log(1-\sin\bar{S})\} \quad (3.314)$$

splits into two components, and the terms which are missing from $-\log\cos\bar{S}$ are included in $-\tfrac{1}{2}\log(1+\sin\bar{S})$. Furthermore, it is not difficult to show that, by the transformation $y=y's$, $t=\dfrac{1}{s}$ (*cf.* 3.2), $-\log(1-\tau e^{\lambda y})$ becomes

$$-\tfrac{1}{2}\log(1+\sin\bar{S}) - \tfrac{1}{2}\log 2 - i\tan^{-1}\left(\frac{-\cos\bar{S}}{1+\sin\bar{S}}\right), \qquad (3.315)$$

and so corresponding to the odd powers of t in $-\log(1-\tau e^{\lambda y})$ we have terms which appear in $-\tfrac{1}{2}\log(1+\sin\bar{S})$ (but not in $-\log\cos\bar{S}$), and the equivalence of these gives the complementary set of identities.

It is to be anticipated that the values of $\bar{P}_{mr}$ (r odd) are related to a problem of the same type as that which gives rise to $\bar{P}_{mr}$ (r even). Evidently we must find a peculiar form of cycle which is associated with an odd number of runs. First of all, we may remark that the simple cyclic problem discussed above has the following obvious interpretation. Let us suppose we have a strip of paper of breadth 1 unit, and that we

form it into a cylinder by gumming its ends together. Lines, m in number, each of unit length, are drawn at intervals round this cylinder parallel to its axis. m decimal numbers between 0 and 1 are selected at random, and a point is marked on each line in a position which represents the corresponding number. One end of any line would represent zero and the other end the number 1: but as the decimal numbers are taken entirely at random, it does not matter from the statistical point of view which end of the line corresponds to 0 and which to 1. In other words, we chose a point on the line entirely at random. If the m points are then joined to their immediate neighbours by straight lines we obtain a series of runs, and $\dfrac{\bar{P}_{mr}}{\sum \bar{P}_{mr}}$ (r even) is the chance that we obtain r such runs in the cycle.

Let us suppose now that instead of an ordinary cylinder we form a singly twisted Möbius surface—that is, the strip of paper is twisted through an angle of 180° before the ends are gummed together. On this surface, as before, m lines of unit length are drawn, and in this case we imagine the strip to be transparent so that the lines may be seen from either side. This is necessary because on going round the cycle once we return to the same point, but find ourselves on the other side of the paper. If we now insert the points one at random on each line as before, and join successive points by straight lines, we obtain a series of runs, but this time there is always an odd number of runs in the cycle. Though we shall not attempt to give a general proof, it seems that the chance that a cycle should contain r runs (r odd) is given by $\dfrac{\bar{P}_{mr}}{\sum \bar{P}_{mr}}$ (r odd).

In order to test this, we have taken 1000 numbers in groups of 5 from a telephone directory—the digits of the numbers being reversed in order so as to ensure randomness—and regarded them as being preceded by a decimal point. We tabulate below the results obtained by considering these groups as arranged (1) as a simple straight limited series of 5 numbers ($p_{5,r}$); (2) as an ordinary cyclic series ($\bar{p}_{5,r}$; r even); and (3) as a cyclic series on a singly twisted Möbius surface. In the latter case the theoretical results are calculated from $\bar{p}_{5,r}$ (r odd) $=\dfrac{\bar{P}_{5,r}}{\sum \bar{P}_{5,r}}=\dfrac{\bar{P}_{5,r}}{4!}$.

We find by (3.113) that

$$\bar{P}_{m1}=1, \qquad (3.316)$$

$$\bar{P}_{m3}=\frac{3^{m-1}}{2^2}-\frac{2m-1}{2^2}, \qquad (3.317)$$

$$\bar{P}_{m5}=\frac{5^{m-1}}{2^4}-\frac{2m-3}{2^4}3^{m-1}+\frac{m^2-3m+1}{2^3} \qquad (3.318)$$

(1) *Straight Arrangements.*

Number of Arrangements.	Observed.	Calculated.
With 1 run	5	3·3
,, 2 runs	47	46·7
,, 3 ,,	101	96·7
,, 4 ,,	47	53·3

($n'=4$; $P(\chi^2)=0{\cdot}6$)

(2) *Simple Cyclic Arrangements.*

Number of Arrangements.	Observed.	Calculated.
With 2 runs	69	66·7
,, 4 ,,	131	133·3

($n'=2$; $P(\chi^2)=0{\cdot}7$)

(3) *Cyclic Arrangements on Möbius Band.*

Number of Arrangements.	Observed.	Calculated.
With 1 run	7	8·3
,, 3 runs	156	150·0
,, 5 ,,	37	41·7

($n'=3$; $P(\chi^2)=0{\cdot}6$)

It will be seen that the agreement is satisfactory.

After this digression we return to the consideration of some further identities between the K^s_{nr}'s. In section 1.7 we observed (1.716) that for $r > n-3$, $f_{nr}=r-\frac{2}{3}(n-2)$, whence it follows that

$$F_{n,\, n+\sigma} \equiv \frac{n+3\sigma+4}{3}n! \quad (\sigma > -3). \qquad (3.320)$$

Hence

$$\sum_s K^s_{n,\, n+\sigma} s^n \equiv \frac{n+3\sigma+4}{3}n! \quad (n > 1,\ \sigma > -3). \qquad (3.321)$$

For $\sigma=0$, -1 and -2 this equation is identical with (3.301), (3.302) and (3.303).

From the nature of the problem of the cyclic case $\bar{P}_{m,m+\sigma}=0$ $(\sigma > 0)$, whence

$$\sum_s K^s_{m+3,\, m+\sigma} s^{m-1}=0 \quad (m > 1) \qquad (3.322)$$

or

$$\sum_s K^s_{n,\, n+\sigma'} s^{n-4}=0 \quad (n > 4,\ \sigma' > -3). \qquad (3.323)$$

These last identities may be readily verified: for instance, the expression for $\bar{P}_{m6}$ (3.109) is found to be zero for $m=2$, 3, 4 and 5.

3.4. *Distribution of Runs.*

We denote by $\bar{X}^m_{n1}$, or more conveniently by $\bar{Q}_{mn}$ the total number of single runs of length n, in all the $(m-1)!$ distinct cyclic arrangements of the m numbers. The total number of runs in the $(m-1)!$ arrangements is

$$\sum_{n=2}^{m}\bar{Q}_{mn}=\sum_{\nu=2}^{m}r\bar{P}_{mr} \quad (\bar{P}_{mr}=0 \text{ for all odd values of } r). \qquad . \quad (3.401)$$

The value of $\sum_{2}^{m} r\mathrm{P}_{mr}$ may be obtained from a consideration that the number of runs in the whole of the $(m-1)!$ arrangements is equal to twice the number of maximal points. But the chance that any point is maximal—that is, that any point taken at random is greater than its two neighbours in the complete set of arrangements, is equal to the chance that of any 3 consecutive numbers taken at random, the middle one should be the greatest, which is $1/3$. There are, however, $m\times(m-1)!$ points, these being m in each arrangement, so that the number of maximal points is $\frac{m!}{3}$, whence

$$\sum_{n=2}^{m}\bar{Q}_{mn}=\sum_{r=2}^{m}r\bar{P}_{mr}=\tfrac{2}{3}m!. \quad . \quad . \quad . \quad . \quad (3.402)$$

We now proceed to form the difference equation for $\bar{Q}_{mn}$, and by a discussion of the various possibilities similar to that in the case of Q_{mn} (section 2.2) we obtain

$$\bar{Q}_{m+1,\,n}=(m-n)\bar{Q}_{mn}+\bar{Q}_{m,\,n-1}+\bar{Q}_{mn}+2(\bar{Q}_{m,\,n+1}+\bar{Q}_{m,\,n+2}+\ .\ .\ .\ \bar{Q}_{mm}) \qquad (m\geqslant n>2) \quad (3.403)$$

and

$$\bar{Q}_{m+1,\,2}=(m-1)\bar{Q}_{m2}+2(\bar{Q}_{m3}+\bar{Q}_{m4}+\ .\ .\ .\ \bar{Q}_{mm})+\sum_{2}^{m}(m-n)\bar{P}_{mn}. \quad (3.404)$$

The last term being the number of extra runs of 2 produced when the new number falls anywhere except adjacent to a maximal point. This readily reduces to

$$\bar{Q}_{m+1,\,2}=(m-3)\bar{Q}_{m2}+\frac{5m!}{3},$$

of which the solution is

$$\bar{Q}_{m2}=\frac{5}{3\cdot 4}m!, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.405)$$

the constant being found to be zero.

From (3.403) we obtain

$$\bar{Q}_{m+1,\,n}-\bar{Q}_{m+1,\,n+1}=\bar{Q}_{m,\,n-1}+(m-n)\bar{Q}_{mn}-(m-n-2)\bar{Q}_{m,\,n+1}, \quad (3.406)$$

from which we find that

$$\bar{\psi}(yz) \equiv \sum\sum \frac{\bar{Q}_{mn} y^m z^n}{m!}$$
$$= \frac{(1-z)^2}{z} e^z \Phi\{(1-y)z\}.$$

The coefficient of z^2 must be of the form $\frac{5}{12}(y^2+y^3+\ldots)$, which, neglecting the terms y^0 and y^1, is $\frac{5}{12(1-y)}$, from which it follows that ψ is of the form $\frac{(1-z)^2}{z}e^z\frac{A}{z(1-y)}$, and comparison of the coefficient of z^2 leads to $A=2$.

Thus

$$\bar{\psi}(yz) = \frac{2(1-z)^2}{z^2}\frac{e^z}{(1-y)} \quad . \quad . \quad . \quad . \quad (3.407)$$

and

$$\bar{X}^m_{n1} = \bar{Q}_{mn} = k_n m!, \quad . \quad . \quad . \quad . \quad (3.408)$$

whence

$$\bar{x}^m_{n1} = \bar{X}^m_{n1} \div \sum \bar{X}^m_{nr} = k_n m! \div \tfrac{2}{3} m! = \tfrac{3}{2}k_n \quad (m > n+1). \quad . \quad (3.409)$$

From this it follows that the generating function for $\bar{x}^m_{n1}$ is

$$\xi_1(yz) = \frac{3(1-z)^2 e^z}{z^2(1-y)}. \quad . \quad . \quad . \quad . \quad (3.410)$$

Thus $\bar{x}^m_{n1}$ is independent of m and is equal to x_{n1} (1.33).

It is to be noted that the above formula (3.408) breaks down when $m=n$ or $m=n+1$. It is easily shown that

$$\bar{Q}_{mm} = 2 \quad \text{and} \quad \bar{Q}_{m,\,m-1} = 2(m-2) \quad . \quad . \quad . \quad (3.411)$$

instead of the values $2\frac{m^2+m-1}{(m+1)(m+2)}$ and $2\frac{m^2-m-1}{m+1}$ given by (3.408).

It may also be verified that $\bar{Q}_{m,\,m-2}$ as given by (3.408) is correct, that is

$$\bar{Q}_{m,\,m-2}(m-1)(m-2). \quad . \quad . \quad . \quad . \quad (3.412)$$

3.5. *Distribution of "Gaps."*

We denote by $\bar{X}^m_{n2}$, or more conveniently $\bar{G}_{mn}$, the number of "gaps" in the $(m-1)!$ cyclic arrangements of the m unequal numbers. The case of the gaps in the arrangements containing only 2 runs requires special consideration. These gaps may be regarded as containing m numbers, although in fact the same number must be counted twice. There are no other gaps containing m numbers, so that we shall take $\bar{G}_{mm} = 2\bar{P}_{m2}$,

there being 2 gaps of this type in each such arrangement. The total number of gaps is evidently equal to the total number of maximal and minimal numbers, and so

$$\sum_{n=3}^{m} \bar{G}_{mn} = \tfrac{2}{3} m!. \quad . \quad . \quad . \quad . \quad . \quad (3.501)$$

We proceed to form the difference equation exactly as in the case of the limited series, noting, however, that the total number of possible ways of adding the extra number is now m instead of $m+1$. As there are no end runs, the complications encountered in the limited case do not arise. We find

$$\bar{G}_{m+1,\, n} = (m-n-1)\bar{G}_{mn} + 2\sum_{1}^{m} \bar{G}_{mn}. \quad . \quad . \quad . \quad (3.502)$$

This equation holds without modification for $n=3$, $\bar{G}_{m2}$ being of course zero. We have then

$$\bar{G}_{m+1,\, 3} = (m-4)\bar{G}_{m3} + \tfrac{4}{3} m!,$$

whence

$$\bar{G}_{m3} = \frac{4}{3 \cdot 5} m! = l_3 m!, \quad . \quad . \quad . \quad . \quad . \quad (3.503)$$

the constant as usual being equal to zero.

Also putting $n=4$,

$$\bar{G}_{m+1,\, 4} = (m-5)\bar{G}_{m4} + \tfrac{4}{3} m!,$$

whence

$$\bar{G}_{m4} = \frac{4}{3 \cdot 6} m! = l_4 m!. \quad . \quad . \quad . \quad . \quad . \quad (3.504)$$

From (3.502) we find

$$\bar{G}_{m,\, n+1} - \bar{G}_{m+1,\, n+1} = 2\bar{G}_{m,\, n-1} + (m-n-1)\bar{G}_{mn} - (m-n-2)\bar{G}_{m,\, n+1}, \quad (3.505)$$

whence it is readily shown that

$$\bar{\zeta}(yz) = \sum \sum \bar{G}_{mn} \frac{y^m z^n}{m!}$$

satisfies the equation

$$(1-y)\left(1 - \frac{1}{z}\right)\frac{\partial \bar{\zeta}}{\partial y} + (z-1)\frac{\partial \bar{\zeta}}{\partial z} = \frac{2z^2 - z + 1}{z}\bar{\zeta}, \quad . \quad . \quad . \quad (3.506)$$

of which the general solution is

$$\bar{\zeta} = \frac{(z-1)^2}{z} e^{2z} \Phi\{z(1-y)\},$$

where Φ is an arbitrary function.

By making use of the values above given for $\bar{G}_{m3}$ and $\bar{G}_{m4}$, (3.503) and (3.504), it can be shown that

$$\bar{\zeta}(yz) = \frac{(z-1)^2}{z^2} \frac{e^{2z}}{(1-y)}, \quad . \quad . \quad . \quad . \quad (3.507)$$

whence

$$\bar{G}_{mn} = l_n m! \quad (m > n+1 \geqslant 4). \quad . \quad . \quad . \quad (3.508)$$

This equation does not hold for $m=n$ or $m=n+1$. In the first case it is easily seen that $\bar{G}_{mm}$ (which it will be remembered refers to the peculiar gaps occurring in arrangements containing only 2 runs) is equal to $2\bar{P}_{m2}$, thus

$$\bar{G}_{mm} = 2\bar{P}_{m2} = 2^{m-1} \quad . \quad . \quad . \quad . \quad (3.509)$$

instead of $l_m m!$ or $2^m \dfrac{(m-2)}{(m+2)}$ by (3.508).

A consideration of the ways in which a $\bar{G}_{m,\ m-1}$ or a $\bar{G}_{m,\ m-2}$ may be obtained from an arrangement of $m-1$ numbers by the addition of another number larger than any of the others leads to the difference equations

$$\bar{G}_{m,\ m-1} = 2\bar{G}_{m-1,\ m-2} + 4\bar{P}_{m2} \quad . \quad . \quad . \quad (3.510)$$

and

$$\bar{G}_{m,\ m-2} - 2\bar{G}_{m-1,\ m-3} + 2\bar{G}_{m-1,\ m-2} + 4\bar{P}_{m-1,\ 2}, \quad . \quad . \quad (3.511)$$

whence $\bar{G}_{m,\ m-1} = 2^{m-1}(m+c)$, from which, making use of $\bar{G}_{4,\ 3}=8$, we have $c=-3$; thus

$$\bar{G}_{m,\ m-1} = 2^{m-1}(m-3) \quad . \quad . \quad . \quad . \quad (3.512)$$

instead of the expected value $l_{m-1}m! = 2^{m-1}\left(m-4+\dfrac{4}{m+1}\right)$.

From (3.511) it follows that

$$\bar{G}_{m,\ m-2} = 2^{m-2}(m^2-5m+c),$$

and putting $m=5$, and noting that $\bar{G}_{5,\ 3}=32$, we find $c=4$, whence

$$\bar{G}_{m,\ m-2} = 2^{m-2}(m-1)(m-4). \quad . \quad . \quad . \quad (3.513)$$

This agrees with the result obtained by using (3.508), which confirms the general validity of that equation and shows that it holds for values of $n \leqslant m-2$.

From (3.508) it follows that if $\bar{x}^m_{n2}$ is the chance that any gap, selected at random out of a cyclic arrangement of m numbers, be of length n, then

$$\bar{x}^m_{n2} \equiv \bar{X}^m_{n2} \div \sum_3^m \bar{X}^m_{n2} = \bar{G}_{mn} \div \sum_3^m \bar{G}_{mn} = \frac{l_n m!}{\frac{2}{3}m!} = \tfrac{3}{2}l_n = \frac{3}{2}\frac{2^n}{n!}\frac{n-2}{n+2}$$

$$(m > n+1 \geqslant 4), \quad (3.514)$$

whence it follows that $\xi_2(yz)$, the generating function for $\bar{x}_{n2}^m$, is given by

$$\xi_2(yz) = \tfrac{3}{2}\frac{(z-1)^2}{z^2}\frac{e^{2z}}{(1-y)}. \qquad (3.515)$$

Thus $\bar{x}_{n2}^m$ is independent of m, and is equal to x_{n2} (1.35) the chance that, in the unlimited series, a gap should be of length n. Thus for runs and gaps the cyclic case agrees with the unlimited case, provided that we limit ourselves to runs and gaps containing not more than $m-2$ numbers.

3.6. *General Expression for $\bar{x}_{nr}^m$.*

It will be seen from the above that when $r=1$ and $r=2$, $\bar{x}_{nr}^m$ is independent of m and equal to x_{nr}. It is natural to inquire whether this result is also true when $r > 2$. Before this question can be answered it is necessary to define $\bar{x}_{nr}^m$ exactly. A difficulty arises from the fact that for $r > 2$, sequences of runs fall to be considered which involve counting certain numbers more than once. Thus when $r=4$, for example, any cyclic arrangement containing 2 runs gives rise to 2 sequences of 4 runs—one begins at the maximum and includes every point twice except the starting-point, which is included three times. To obtain the 4 runs we must in fact circulate twice round the cycle and finish at the starting-point. The other sequence of 4 runs arises similarly by beginning at the minimal point. In order that the equation $\bar{x}_{nr}^m = x_{nr}$ be generally true, it is necessary to include sequences of the above type in the enumeration. We have already seen that it is necessary to make a similar condition in the case of the peculiar gaps which appeared in the discussion of $\bar{x}_{n2}^m$. The result of this convention is that the total number of possible sequences of r runs for any value of r is equal to the number of maximal and minimal points which occur in all the $(m-1)!$ different arrangements, because any such point may be considered as a beginning of a sequence, that is

$$\sum_{n=r+1}^{\infty} \bar{x}_{nr}^m = \tfrac{2}{3} m!. \qquad (3.61)$$

Let us now consider any n successive numbers taken at random in a cyclic arrangement of m points. We shall assume $n < m-1$, so that there are at least 2 other numbers in the part of the cycle not covered by the sequence—that is, separating its end from its beginning. Let us call the n numbers $a_1, a_2, \ldots a_n$, the number preceding a_1 being a_0, and the number following a_n being a_{n+1}; a_0 and a_{n+1} are by hypothesis distinct numbers. We may now suppose that, instead of an arrangement obtained

by the random dealing of m marked cards, we obtain each series by drawing decimal numbers between 0 and 1 from a bag and arranging m of them in a cycle, each cycle being made up of a different series of m numbers. The values of $\bar{x}^m_{nr}$ must be the same in both cases, as the actual values of the numbers do not count.

This being so, we may apply the methods of section 1.2 to the series of numbers $a_1, a_2, \ldots a_n$, and find the chance that the series of numbers begins with a maximal or a minimal number and ends with a maximal or minimal number. The fact that a_0 and a_{n+1} are two distinct independent random numbers makes this possible. Thus it follows, as in section 1.2, that

$$c\underset{r}{\Delta_2}\bar{x}^m_{nr} = \underset{n}{\Delta^2}p_{nr} \quad (n < m - 1)$$

(c is the chance that a_1 is maximal or minimal), where $c\bar{x}^m_{nr}$ represents the chance that a_1 is maximal or minimal and that simultaneously a_n is the rth succeeding maximal or minimal number. Now $c = \frac{2}{3}$, so that

$$\underset{r}{\Delta_2}\bar{x}^m_{nr} = \tfrac{3}{2}\underset{n}{\Delta^2}p_{nr} \quad (n < m - 1). \qquad (3.62)$$

But, as we have seen for $r = 1$ and $r = 2$, $\bar{x}^m_{nr} = x_{nr}$. This may also be obtained by putting $r = 1$ and $r = 2$ in (3.62), and noting that $\bar{x}^m_{n0}$ and $\bar{x}^m_{n,\,-1}$ are zero. So that as in general (3.62) is true, it follows that

$$\bar{x}^m_{nr} = x_{nr} = \tfrac{3}{2}\underset{n}{\Delta^2}f_{nr} \quad (m > n + 1 \geqslant r) \qquad (3.63)$$

for all values of m.

It follows that

$$\bar{X}^m_{nr} = \tfrac{2}{3}m!\,\bar{x}^m_{nr} = m!\,\underset{n}{\Delta^2}f_{nr}. \qquad (3.64)$$

Also the generating function for $\bar{x}^m_{nr}$,

$$\begin{aligned}\bar{\xi}(y, z, t) &\equiv \sum\sum\sum y^m z^n t^r \bar{x}^m_{nr} \\ &= \frac{\phi(zt)}{(1 - y)}\,\frac{(1 - z)^2}{z^2} \\ &= \frac{6(1 - z)^2}{(1 - y)z^2\lambda^3(1 - \tau e^{\lambda z})}. \qquad (3.65)\end{aligned}$$

It is not difficult to see that the coefficients of t and of t^2 in the expansion of this function are respectively $\xi_1(yz)$ and $\xi_2(yz)$ (see equations (3.410) and (3.515)), whilst the coefficient of y^m as m tends to infinity is $\xi(zt)$, the generating function of x_{nr} given in equation (1.513).

Though the value of $\bar{x}^m_{nr}$ may be regarded as the general solution of our problem, the infinite series being considered as the limiting cyclic

case, it must be remembered that it only holds for $n < m - 1$, although in considering the chance of selecting a sequence of runs containing $n (< m)$ numbers, the sequences for which $n \geqslant m - 1$ must be taken into account as possible sequences.

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(*Issued separately October* 5, 1937.)

XXIII.—On Rotating Mirrors at High Speed. By Sir Charles V. Boys, LL.D., F.R.S.

(MS. received July 27, 1937.)

I HAVE read the observations of Dr R. A. Houstoun at the top of p. 164 of the current volume of these *Proceedings*, and I gather the impression that he has been dissuaded by general beliefs of what cannot be done from using the simplest of all possible methods. Forty-five years ago when I required a very high speed of rotation I completely ignored or defied these general beliefs with the greatest success, and it seems to me to be possible that a few words on my practice might be useful. If this is the case I should feel that as a newly appointed Hon. Fellow of the Royal Society of Edinburgh I had at least done what I could in recognition of that honour.

The occasion was when I wished to analyse the time character of electric sparks for the purpose of photographing bullets in flight. Ordinary sparks lasted far too long. A general account of the work is to be found in *Nature* of March 2 and 9, 1893. I never thought it worth while to publish details as the method was so very simple.

Ultimately it boiled down to this. I drove a dead hard steel mirror $\frac{3}{4}$ in. in diameter and $\frac{3}{8}$ in. thick with a bit of fine string tied with a common knot, normally at speeds of 512 turns a second beating against a maintained tuning-fork. I ran it up to 1024, but not more, as I did not wish to take any risk, but Dr Edwin Edser, who was my assistant, told me afterwards that he once drove it up to 1536 turns a second. This was before the days when electric motors as we now know them existed, and I had only a clumsy rotating field Ayrton direct current motor. This was incapable of high speed and was of about three cat power, so I put a big pulley on the motor, and on a steel axle mounted a big wooden pulley of about 30 in. diameter and a small one driven by the motor. The string was a piece of common string of the whip cord character.

The mirror had on each side cylindrical portions $\frac{3}{8}$ in. in diameter and $\frac{1}{8}$ in. long and beyond these cylindrical bearing surfaces $\frac{1}{8}$ in. in diameter and $\frac{1}{2}$ in. long.

We made the steel mirror blank and hardened it, and then ground the trunnions dead true and equal on dead centres. We then balanced

it by grinding the heavy side and sent it to Hilger to get one face ground and polished with a radius of curvature of 2 metres (?). It was then finally balanced. We made a steel pulley ring of about $\frac{1}{2}$ in. diameter with a sharp V groove and $\frac{1}{8}$ in. thick and pressed it on to one of the $\frac{3}{8}$-in. portions, while the other carried an aluminium piece independently balanced to pass near two electrodes and work the spark relay. By this means all the sparks to be examined by the mirror occurred when it was looking one way and all the series of sparks were seen near together on the plate. The unit of time to which I could work with certainty was $\frac{1}{100\,000\,000}$ second at 512 turns a second, and in this way I found how to make a spark used in the bullet photographs lasting $\frac{1}{13\,000\,000}$ second.

I should have said that I carried the mirror in a very heavy brass casting with short heavy pillars cast in one with the base. These were 1 in. apart and a $\frac{3}{8}$-in. hole was drilled right through both near the top. Then four holes were drilled and tapped to take holding-down screws and the tops of the pillars were sawn off through the $\frac{3}{8}$-in. holes. The inner faces were then tinned and type metal was cast *in situ*, the brass being hot, round (as far as I can remember) a smoked dummy with mica separating pieces, so that, when all was cold, the caps could be removed and all cleaned up. Then lubricating holes were drilled and we just oiled it like any other bearing. The whole thing worked perfectly and we had no belt troubles, lubrication, or any other troubles.

I believe Professor G. P. Thomson still has this mirror as a relic of my time at South Kensington.

The knot in the string was most useful. It is very easy to get an octave wrong with sounds of different character. I avoided this by letting the string, which was perhaps 10 or 15 ft. long, run lightly over my finger. I could then count the knots (like a log at sea) and a 2:1 mistake was impossible.

(*Issued separately October* 5, 1937.)

XXIV.—*Geonemertes Dendyi* Dakin, a Land Nemertean, in Wales.

By **A. R. Waterston,** B.Sc., and **H. E. Quick,** M.B., F.R.C.S., B.Sc. *Communicated by* Dr A. C. Stephen.

(MS. received July 5, 1937. Read July 5, 1937.)

Introduction.—The land nemerteans of the world comprise only twelve species and at present all are referred to the genus *Geonemertes* Semper. Most of the species occur in the Australia, New Zealand, and Malaysian regions, with a single species in Bermuda. European occurrences are rare and apparently due to introductions by man. *G. chalicophora* Graff, for example, is unknown outside glass-houses in Germany, Austria, Czechoslovakia (Stammer, 1934), and the Botanic Gardens, Dublin, Ireland (Southern, 1911). One other species, *G. dendyi* Dakin, which was originally described from a single female discovered in "a valley in the Darling Range not far from Perth," West Australia (Dakin, 1915), was later found in glass-houses in the Botanic Gardens at Breslau, Germany (Stammer, 1934). More recently this worm has been found by us to be breeding in the open in wild situations around Swansea, Wales. We believe this occurrence to be the first record of a land nemertean established in the temperate climate of Europe.

When collecting land planarians near Swansea in September 1935, one of us (H. E. Q.) noticed among a number of the planarian *Rhynchodemus terrestris* (Müll.), a creature of similar general appearance, but which bore two dorsal stripes and shot out a thin white thread-like proboscis when touched. This was seen to be a land nemertean by A. R. W., who was granted leave by the authorities of the Royal Scottish Museum, Edinburgh, to visit the Welsh localities. We went over the ground together from October 25 to 27, 1935, and some of the material obtained on these occasions was subsequently used by A. R. W. for study of the internal anatomy. The following account is based on the results of the examination of more than thirty living specimens and five examined microscopically from serial sections, cut by the ordinary paraffin-wax method, 5 μ and 8 μ thick. We have also had access to a valuable collection of microscope slides of *G. australiensis* prepared by the late Dr A. D. Darbishire and now housed in the Ashworth Laboratory of the University of Edinburgh. We are much indebted to the late Professor J. H.

Ashworth, F.R.S., for his kindness in placing this material at our disposal and for the loan of several important papers on *Geonemertes*.

External Characters.—The resting animal resembles a small yellow land planarian (such as *Rhynchodemus bilineatus* Metschn.) and is soft and slimy. When crawling, the worm is long and slender and somewhat flattened. Mature females measure from 8 to 25 mm. in length and 1 to 1·5 mm. in breadth. The head is rounded and not constricted off from the body; a narrow vertical slit, the opening of the rhynchodaeum, is borne at the anterior end. The eyes are arranged laterally in four groups, two anterior with from three to seven eye-spots, and two posterior with from two to eight eye-spots. The anterior group is usually shaped like a triangle with the vertex directed forward, while the posterior group is more nearly linear and arranged transversely. The number of eye-spots varies, and four mature females, 8 mm. in length, had 10, 11, 19, and 22 eyes respectively, and a large female, 25 mm. in length, had 29 eyes. Numerous minute brown pigment granules surround the eyes. The ground colour is pale yellow, with a pair of dorso-lateral brown stripes extending from behind the eyes to the tail. The sole is narrow, paler, and evenly coloured.

Internal Characters.—Several workers, notably Dendy (1892), Graff (1879), Coe (1904), and Hett (1924 and 1927), have studied the internal anatomy of *Geonemertes*, and the various species are separated chiefly by the number of the proboscis nerves, the number and arrangement of the accessory stylet sacs, the nephridia, and the blood-vessels.

Stammer gives the numbers of proboscis nerves as 14–15, and of accessory stylet sacs 2–4, but five specimens from Wales have only 13 proboscis nerves and two accessory stylet sacs each containing about five reserve stylets. The lateral blood-vessels form a network with the dorsal vessel as in the two closely allied species, *G. australiensis* and *G. hillii*, but the nephridia are not sufficiently well preserved to be traced. *G. dendyi* agrees with *G. hillii* in the absence of a cephalic gland. This gland is present in all other species of *Geonemertes*.

All the specimens examined by us proved to be females, many of them with well-developed ova. Like Dakin we were unable to trace the oviducts, but it is probable that these ducts are ephemeral and developed only when the eggs are about to be shed.

Habits.—The worms rest during the day under logs, fallen branches, and sometimes under damp decaying leaves. In captivity they crawl freely at night over the moss and leaves in their jar. The crawling worm is much elongated, and a specimen of 10 mm. (resting) stretches to 15 mm. and one of 12 mm. to 18 mm. When crawling, the body glides evenly

forward, but the posterior one-fifth sometimes shortens and lengthens rhythmically as the animal progresses. Like slugs and land planarians, land nemerteans crawl on a slime track; when the worm rests, the mucus hardens somewhat and on resuming activity a thin mucous tube is left behind. A specimen crawling over the edge of a glass slip usually falls off, but sometimes remains suspended for a few seconds by a thread of mucus about half an inch long. A similar observation was made by Kew (1900) for *Rhynchodemus terrestris* (Müll.), a land planarian which is common in the British Isles.

When a crawling worm is touched, the proboscis is generally shot out with astonishing rapidity to a length exceeding that of the body and then slowly retracted. A second touch usually fails to evoke a response. Occasionally the proboscis is shot out so violently that it parts from the body. The papillæ on the everted proboscis sometimes grip the substratum and then the act of inversion hauls the body forward more rapidly than the normal mode of progression. Dendy (1892) records a similar observation for *G. australiensis* and Willemoes-Suhm (1874) seems to have believed this to be the normal mode of progression for *G. agricola*, but it is more likely that this mode of progression is used in nature as a means of more rapid movement under a special stimulus.

Feeding.—No account appears to have been published of the food or method of feeding in land nemerteans. In the field, and in captivity, we have on several occasions seen *G. dendyi* feeding on small nymphs of Delphacid bugs and young Collembola. The actual capture of the prey was not witnessed, but the worms sucked the insect's juices by means of a series of peristaltic waves passing backwards from the head and involving the contour of the body wall. The proboscis was never protruded on these occasions.

Reproduction.—Knowledge of the sexual phases of *Geonemertes* is incomplete. In *G. rodericana* (Gull.), *G. australiensis* Dendy, and *G. hillii* Hett, the sexes are separate and the males are usually smaller than the females; *G. palæensis* Semper, *G. agricola* (Willemoes-Suhm), *G. chalicophora* Graff, and *G. arboricola* Punnett, are hermaphrodite and it is probable that cross-fertilization occurs; in *G. novæ-zealandiæ* Dendy, *G. dendyi* Dakin, and *G. graffi* Burger, only females are known; while *G. spirospermia* Darbishire is known from a single male. Our material of *G. dendyi* consisted of ovigerous females, but it is probable that further search will reveal the occurrence of small male specimens as in the case of its nearest allies, *G. australiensis* and *G. hillii*.

Coe (1904) has investigated the breeding habits of *G. agricola* which is normally viviparous; and Dendy (1893) has described *G. australiensis*

which is normally oviparous. In the latter species the eggs when laid are white opaque spherical bodies about 0·6 mm. in diameter. "Some thirty of these eggs are enclosed together in a sausage-shaped mass of colourless transparent jelly. The surface of the gelatinous matrix is smooth, and the jelly appears to be common to all the eggs. . . . One such mass of eggs is deposited at a time, and . . . at least three can be deposited in succession by the same animal, at intervals of several days, the animal itself remaining perfectly uninjured" (Dendy, *loc. cit.*, pp. 127–130). The embryos were ready to hatch in about a month and had one pair of eyes. Eggs of *G. australiensis* were laid in July both in the field and in captivity.

During the course of our investigation of *G. dendyi*, we found a number of egg-masses. Several worms collected in October, November, and April deposited egg-capsules in vivaria a few days after capture and we have found a single capsule in the field in March. This last capsule was in no way different from those laid in captivity. The egg-capsules of *G. dendyi* have a coat of gelatinous material similar to that of *G. australiensis* and are about 3 mm. long. The capsules vary in shape and may be pear-shaped fixed by the narrow end to a leaf, stem, or tuft of moss, or long and narrow and attached to the substratum at both ends. The capsules contain from eight to thirty closely aggregated eggs, which are creamy-white and measure $0{\cdot}32 \times 0{\cdot}26$ mm. but vary somewhat in shape. They hatch in about three weeks and the young worms are then 0·5 mm. long, pale yellowish-pink and unstriped, with two pairs of eyes and a few brown pigment granules between them; a median stylet (0·015 mm. excluding base) and two accessory stylet sacs each bearing a single accessory stylet (0·027 mm.) are already present in the proboscis. The number of eyes increases as the worm grows and specimens 4–5 mm. long have at least ten eyes.

Habitats.—All the sites lie roughly south-west of the Brynmill district of Swansea, Glamorganshire.

The site at Mayals, about two miles from the Brynmill station of the Mumbles Electric Railway, is a small shallow marshy valley on the Millstone Grit and Glacial Boulder Clay. East and south it is limited by a row of small oaks and a few hollies, which separate it from fields grazed by sheep. Beyond the shade of the trees the ground slopes to a rivulet, and supports a growth of rough grass with rushes, and here and there a bramble-bush.

The nemerteans are found sparingly beneath fallen branches and stones on the muddy leaf-strewn ground beneath the trees and under the few branches and stones to be found on the marsh beyond the shade

of the trees. Millipedes, small oligochaets, snails, slugs, and a land planarian (*Rhynchodemus terrestris* (Müll.)) are associated with the nemerteans. The following mollusca were noted: *Agriolimax agrestis* (Linn.), *Agr. lævis* (Müll.), *Arion subfuscus* (Drap.), *A. minimus* Simroth, *Retinella nitidula* (Drap.), *R. pura* (Alder), *R. radiatula* (Alder), *Vitrea crystallina* (Müll.), *Zonitoides nitidus* (Müll.), *Z. excavatus* (Bean), *Euconulus fulvus* (Müll.), *Vertigo substriata* Jeff., *Cochlicopa lubrica* (Müll.), and *Carychium minimum* Müll.

To the north-west is moorland, and to the south-west a damp slope with oaks and firs. A narrow muddy lane leads to the site from Mayals, where there are a few scattered houses and cottages, and a field path leads from it, beside Clyne Castle grounds, to a lane leading to Blackpill. The site therefore seems to be as wild a bit of waste marshy ground as one ever sees apart from mountain, moorland, cliff, or dune areas, and there is no evidence that greenhouse rubbish has been dumped there. There are greenhouses at Clyne Castle, but they are at least five minutes walk away over the fields, and the exit from the Castle grounds is into the lane leading downhill into Blackpill. There is no reason to suppose that material from the greenhouses would find its way to the nemertean site, though of course the possibility cannot be excluded. The associated mollusca testify to the wildness of the site. The nemerteans were active on a chilly October day and have been found in three successive years, so, however they originally arrived there, they now seem to be a well-established element in the fauna.

The site at Clyne Valley, one and a half miles from Brynmill station, on the Lower Coal Series, is a damp overgrown bank of a small stream, with alders. Only two specimens have been found there.

The Derwen Fawr site, over half a mile from Clyne Valley, is also the bank of another small stream with alders, and has afforded three specimens.

At the Cwm Farm site about a furlong off, a single nemertean was found under a stone at the side of a lane close to a running ditch.

The Singleton Park site is situated about one-third of a mile from Brynmill station, in a small damp copse on blown sand near a rivulet, and has afforded four specimens.

The nemerteans are therefore most abundant at what appears to be the site least interfered with by man, namely Mayals. There are greenhouses with tropical plants in Singleton Park within half a mile of the nemertean site, but search therein has so far failed to reveal any specimens.

These sites have been marked on the 6″ *Ordnance Survey Map, Glamorgan, Sheet XXIII*, 1921, which has been deposited in the National Museum of Wales, Cardiff.

SUMMARY.

In 1935, *Geonemertes dendyi* Dakin, a land nemertean, previously known from West Australia and greenhouses in Germany, was found in several localities near Swansea, Wales.

The external and internal characters are described.

The worms are active at night and crawl on a slime track like land planarians and slugs, and feed by sucking the juices of small Collembola and nymphs of Delphacid bugs.

Clear gelatinous capsules containing from eight to thirty eggs were laid in captivity in October, November and April, and in nature in March. Hatching takes place in about three weeks.

The habitats are all damp shady places near streams or trees and the nemerteans occur under stones and fallen branches. The sites are all in wild places and the worms are most frequent where there seems to be least interference by man.

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XXV.—An Histological Analysis of Eye Pigment Development in *Drosophila pseudo-obscura*. By Flora Cochrane, Ph.D., Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E. Crew, M.D., D.Sc. (With Three Coloured Plates and Two Text-figures.)

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It is generally accepted that the function of the mutant genes which affect eye colour in *Drosophila* is to produce either quantitative or qualitative changes in the normal development of eye pigment. Wright (1932) believes that these genes act by interfering with some part of a chain of reactions which give rise to eye-colour characteristic of the wild type. Mainx (1935) suggested that recessive eye-colour genes (with the exception of sepia) reduce the total amount of pigment, and that the degree of reduction is characteristic for each gene. Transplantation experiments of Beadle and Ephrussi (1935, 1936, 1937 *a* and *b*) have led them to the view that certain eye-colour mutants lack specific substances which are necessary for the development of additional pigment present in wild type eye colour. Johannson (1924) and Schultz (1935) studied the histology of the adult eye of *D. melanogaster* and classified the mutant eye colours according to the distribution of granules in their pigment cells. Schultz (1932) had previously noted the time of pigment deposition and subsequent changes in wild type and some mutants of *D. melanogaster*. In an earlier paper the author (1936) described in detail similar colour changes in the eyes of developing pupæ of *D. pseudo-obscura*. Previous work in this laboratory (1932, 1934, 1935) by Crew and Lamy make it appear that each mutant eye colour is the expression of a gene which acts to suppress normal development during a certain period of time, allowing the rest of the development to proceed as in wild type.

The object of the present work has been to make a comparative histological study of pigment, in normal and mutant eyes of *D. pseudo-obscura* at various stages of development, in order to determine when and how mutant genes affect the normal process of colour formation.

The author wishes to thank Professor F. A. E. Crew and his colleagues at the Institute of Animal Genetics for facilities for work, helpful suggestions, and encouragement during its progress.

Technique and Methods.

Pupæ of wild type and eight eye-colour mutant types: orange (or), vermilion (v), sepia (se), three allelomorphs of purple (pr), eosin (w^e), and white (w^5) were isolated. Having previously determined the times at which pigment appears in the eyes as well as the times at which colour changes take place, pupæ of the appropriate ages were dissected and fixed. Carnoy Lebrun, Eltringham's fluid, and alcoholic Bouin were used as fixatives. Carnoy Lebrun gave the best results, though Bouin was also useful. Material was fixed for not more than 10 minutes then washed in absolute alcohol (to which a few drops of iodine had been added when Carnoy Lebrun was used as a fixative). The material was never allowed to remain in the alcohol for more than 10 minutes. It was then cleared and embedded by the quick cellodin paraffin method of Peterfy (1928). Sections were cut 7μ in thickness and mounted without staining in order to study the distribution of the pigment granules. These histological preparations were compared with fresh smears of the eyes, and it was found that the colour of the pigment granules was practically unaffected by fixation and clearing. A few sections were stained with Ehrlich's hæmatoxylin and eosin for comparison with the unstained material. Microphotographs of sections and smears were used to compare the relative sizes of the pigment granules.

Description.

The eye colour of Diptera as well as other insects, as described by various investigators (Hickson, 1885; Lowne, 1895; Eltringham, 1919), is due to the presence of pigment granules in definite pigment cells which form sheaths around the ommatidia. In *Drosophila* (Johannson, 1924) each sheath is composed of a pair of primary pigment cells surrounding the pseudocone and about twelve secondary pigment cells around the retinulæ (Pl. I, fig. a). Adjacent ommatidia share the same secondary pigment cells. Johannson (1924) described a third type, called basal pigment cells, which were supposed to lie at the bases of the ommatidia. Hertwick (1931) denied the existence of these. During this study small clusters of yellow and brown granules were found in some pupæ lying at the bases of the ommatidia. These have no connection with the secondary pigment cells, nor could their connection with any other cells be determined. Aside from these clusters of granules there is no indication of a third type of pigment cells. These observations agree with those of Hertwick (1931). The secondary pigment cells are thicker at their proximal ends, and these thickenings contain more granules than do the

other portions of the cells. Possibly these thickened ends of the secondary cells are what Johannson (1924) and Casteel (1929) described as basal pigment cells. In older pupæ and adult flies some granules are present below the basal membrane. As these are identical in appearance with the granules in the secondary pigment cells, and possibly lying within processes of these cells, they are included in the description of granules in the secondary pigment cells.

The pigment cells originate as the pyramidal supporting cells described by Krafka (1924). By 96 hours after pupation the eyes are completely formed, all the pigment cells are present, and in wild type the first coloured granules appear. Aside from the deposition of pigment granules the only further changes to take place are the cupping of the pseudocones late in pupal life and the lengthening of the ommatidia. This lengthening causes the secondary pigment cells to stretch until they form thin sheaths around the retinulæ. At about 168 hours after pupation the adult fly emerges from the pupa case.

At two distinct times during the pupal period large numbers of coloured granules are laid down in the pigment cells. These times mark the beginnings of two phases of pigment development.

I. *The Early Phase of Pigment Development.*—In the eyes of 104-hour-old pupæ (about 8 hours after the onset of pigment formation) of wild type, sepia and three allelomorphs of purple, two kinds of granules (yellow and brown) are apparent in the primary and secondary pigment cells (Pl. I, fig. b). These granules are of uniform size in both types of cells. They are undoubtedly yellow when they appear, and gradually become brown during this phase. Small clusters of yellow and brown granules are present at the bases of the ommatidia, but their connection with any cells could not be determined. The eyes of eosin pupæ of this age and also a little older contain the same yellow and brown granules in similar numbers as in wild type in the primary pigment cells. In the secondary pigment cells there are fewer of the same type of granules. No yellow and brown granules were present at the base of the ommatidia (Pl. I, fig. c). In vermilion, orange, and white pupæ of this age no pigment granules of any kind are present either in the primary or secondary pigment cells (Pl. I, fig. a). Stained sections of orange and vermilion, as well as white pupæ of similar ages, show that the eyes are fully formed as in wild type, and that both types of pigment cells are present in the usual numbers. Hertwick (1931) describes a white mutant in *D. melanogaster* which lacks pigment cells entirely. This is not the case in white[5] of *D. pseudo-obscura.*

In the eyes of wild type, sepia and purple[2] pupæ between the ages of

110 and 120 hours, the number of granules is practically the same as that noted in younger pupæ (Pl. I, fig. e). The granules in the primary cells are all yellow and brown. In the secondary cells there are numerous yellow and brown granules with red ones amongst them. In older pupæ there are more red granules and correspondingly fewer yellow and brown ones. These observations indicate that during the early phase of development the yellow granules originally laid down in the secondary pigment cells undergo two colour changes by which they become brown, then red. Similar yellow granules in the primary cells become brown but never red.

In eosin pupæ (Pl. I, fig. f) of this age, although there are few granules in the secondary pigment cells, they become red as in wild type. The granules in the primary pigment cells also behave normally.

In vermilion, orange, and white[5] pupæ under 120 hours old no coloured granules of any kind are present in primary or secondary pigment cells.

In pupæ of purple[1] and purple[3] at the end of the early phase of development, the eyes contain only yellow and brown pigment granules in both types of pigment cells. No red granules are present (Pl. I, fig. d).

The histological data showing the differences in distribution of eye pigment in the various mutants during the early stages of development are summarized in Table I.

II. *The Late Phase of Pigment Development* (over 120 hours after pupation).—In the eyes of vermilion and orange pupæ over 120 hours after pupation the secondary pigment cells contain numerous orange granules (Pl. II, fig. a). In younger pupæ these granules are yellowish, in older pupæ they are more red than yellow (Pl. II, fig. d). This gradual increase in redness as well as in the size of the granules continues into adult life. No granules are present in the primary pigment cells, and no clusters of yellow and brown granules at the bases of the ommatidia.

In the eyes of wild type in the late phase of pigment development there is no change in the number or colour of the granules present in the primary pigment cells (Pl. II, fig. b). There is a marked increase in the number of granules in the secondary pigment cells. The additional granules are similar in number and colour to the granules present in the secondary pigment cells in the eyes of orange and vermilion pupæ over 120 hours old. These new wild type granules increase in size and redness as the pupæ become older (Pl. II, fig. e), and after emergence in the same manner as the similar granules in orange and vermilion eyes.

The eyes of purple[2] pupæ during this late phase of development are similar to those of wild type in that additional granules are present in the secondary pigment cells, and that the granules in the primary pigment

TABLE I.—EARLY PHASE OF PIGMENT DEVELOPMENT (96–120 HOURS AFTER PUPATION).

	96–108 Hours after Pupation.	109–120 Hours after Pupation.
Wild type (+)	P. Yellow and brown granules S. Yellow and brown granules	P. Yellow and brown granules. S. Red and brown granules present, more red than brown in older pupæ
Sepia (se)	P. Same as wild type S. Same as wild type	P. Same as wild type S. Same as wild type
Purple[1] (pr[1])	P. Same as wild type S. Same as wild type	P. Same as wild type S. Brown and yellow granules only
Purple[2] (pr[2])	P. Same as wild type S. Same as wild type	P. Same as wild type S. Same as wild type
Purple[3] (pr[3])	P. Same as wild type S. Same as wild type	P. Same as wild type S. Same as purple[1]
Eosin (w^e)	P. Same as wild type S. Fewer yellow than in wild type	P. Same as wild type S. Few granules present, mostly red at the end of this period
White (w^5)	Both types of pigment cells present, no granules in either type	No granules present
Vermilion (v) and orange (or)	Same as white	Same as white

P = primary pigment cells.
S = secondary pigment cells.

cells remain unchanged. The difference from wild type becomes obvious in older pupæ (Pl. III, fig. b). The late-phase granules in purple[2] eyes are not as red as those of wild type. It is difficult to decide if they are as large.

In the eyes of purple[3] pupæ just over 120 hours after pupation, many of the early-phase granules in the secondary pigment cells are still brown though some of them are red (Pl. II, fig. c). In older pupæ the majority of the early-phase granules in the secondary pigment cells have become red. In pupæ of 140 hours additional yellowish granules are also present in the secondary pigment cells (Pl. III, fig. c). The granules in the primary pigment cells are yellow and brown as in the early phase of development.

The eyes of purple^1 pupæ over 120 hours old do not differ from the eyes of younger pupæ. There are no additional granules in the secondary pigment cells, and the colour of the granules previously laid down in both types of pigment cells remains unchanged. Sections of purple^1 pupæ 140 hours old show some red granules scattered among the brown ones in the secondary pigment cells (Pl. III, fig. a). Thus the transition to red which takes place in the early-phase granules of the secondary cells of wild type at about 115 hours, in purple^1 begins very late in pupal life. This colour change continues into adult life. In these adults the number of red granules increases and the number of brown ones decreases proportionally, indicating that these processes are causally connected. By two weeks after emergence all of the brown granules have become red, showing that the transition is complete.

An accurate understanding of late pupal development of the purple alleles is difficult for the reason that the presence of early-phase granules obscures the granules laid down later. Therefore it was decided to make a study of orange-purple^1, orange-purple^2, vermilion-purple^2 and orange-purple^3 where no early-phase granules exist. The histological structure of the eyes of old and young adults carrying these colour combinations was compared with that of orange and vermilion adults of similar ages (Pl. III, figs. e, f, g). It was found that the granules in the eyes of orange-purple^2 and vermilion-purple^2 are much smaller and less red than those in orange and vermilion eyes of flies of the same age. The granules in orange-purple^3 are smaller and less red than those of orange-purple^2. A corresponding increase in size of granules was noted in orange, orange-purple^2, and orange-purple^3 as the flies became older. The eye of an orange-purple^1 adult shows a complete absence of pigment granules, though the cells are present in the usual number.

The deposition of pigment granules in the sepia eye follows the same course as wild type until after the 120th hour, at which time there are numerous red granules in the secondary pigment cells. During the late phase of development additional granules of a yellowish colour may be observed in the secondary pigment cells among the red ones (Pl. II, fig. g). These granules remain yellow and relatively small during pupal and adult life instead of becoming red as in wild type, orange, vermilion, and purple eyes. This is also demonstrated by the fact that in the eyes of vermilion-sepia flies where, due to the action of the vermilion gene, no early-phase pigment is present, all the granules are small and yellow in colour (Pl. III, fig. d). The early-phase granules in the secondary pigment cells of the sepia eye undergo a final change. During early adult life they gradually change back from red to brown or yellow, so that

in the sepia eye of an old adult the pigment granules are all either brown or yellow (Pl. III, fig. h). The granules in the primary pigment cells resemble those of wild type.

The eyes of eosin pupæ in the late phase of development are very little different from those of younger pupæ (Pl. II, fig. f). A few yellowish granules are present in the secondary pigment cells of pupæ 141 hours after pupation. Similar granules were observed in sections of very old vermilion-eosin adults, indicating that these granules are produced very late in the pupal period and become only very slightly red.

The histological observations show that practically all of the deviations from the normal course of development of eye pigment concern granules in the secondary pigment cells only. In all mutants where granules are present in the primary pigment cells, they are yellow when they appear and take on various shades of brown as they become older.

Variations in pigment distribution in the eyes of wild type and the mutants under consideration are summarized in Table II.

DISCUSSION.

The early-phase granules which appear in the eye after the 96th hour of pupal life are yellow when laid down. In the secondary pigment cells they become brown and finally red. In the primary cells, though they take on various shades of brown they never become red. In adults the granules in the primary pigment cells are coarse and dark brown in colour. The late-phase granules which appear after the 120th hour of pupal life are orange in colour; small in size and yellowish at first, they gradually become more red and larger as development proceeds. These granules are present chiefly in the secondary pigment cells, though in a few instances they have been observed in the primary cells as well.

Analysing the histological structure of the eye at various stages in development, it is apparent that eye-colour genes influence the development of one or the other or both types of pigment granules in three distinct ways:

(*a*) by suppressing granule formation.
(*b*) by altering the rate of pigment development.
(*c*) by changing the granules qualitatively.

(*a*) *Some Eye-colour Genes suppress Granule Deposition.*—Orange (or) and vermilion (v) prevent the formation of all early-phase granules in the primary as well as in the secondary pigment cells, but do not interfere with the deposition of the late-phase granules which appear and follow the normal course of development as in wild type. White (w^5) entirely

TABLE II.—LATE PHASE OF PIGMENT DEVELOPMENT (OVER 120 HOURS AFTER PUPATION).

	120–132 Hours after Pupation.	132 Hours after Pupation to Emergence.	Post Pupal Development.
Wild type (+)	P. Yellow and brown granules S. Numerous additional yellowish orange granules	P. Granules darker in colour S. Late-phase granules increase in size and redness	P. Granules very dark and coarse S. Late-phase granules continue to increase in size and redness
Sepia (se)	P. Same as wild type S. Same as wild type	P. Same as wild type S. Late-phase granules remain small and yellow	P. Same as wild type S. All granules in secondary cells eventually yellow and brown
Purple[1] (pr[1])	P. Same as wild type S. No additional granules. Early-phase granules *brown* and yellow	P. Same as wild type S. Some early - phase granules become red toward the end of this period	P. Same as wild type S. All early-phase granules eventually red. No late-phase granules
Purple[2] (pr[2])	P. Same as wild type S. Same as wild type	P. Same as wild type S. Late-phase granules less red and smaller than wild type	P. Same as wild type S. Similar to wild type but late-phase granules never become as large or as red
Purple[3] (pr[3])	P. Same as wild type S. No late-phase granules. Some red among early-phase granules	P. Same as wild type S. Early-phase granules red. A few small yellowish late-phase granules appear at this time	P. Same as wild type S. Late-phase granules increase in redness but not as much as in purple[2]
Eosin (w^e)	P. Same as wild type S. No late-phase granules present	P. Same as wild type S. A few yellowish late-phase granules present	P. Same as wild type S. Late-phase granules coarser but few in number and light in colour
White (w^5)	No granules in either type of pigment cell	No granules in either type of pigment cell	No granules in either type of pigment cell
Vermilion (v) and Orange (or)	P. No granules present S. Numerous yellowish orange granules similar to late-phase granules in wild type	P. No granules present S. Orange granules increase in size and redness	P. Occasional granules like the ones in the secondary pigment cells S. Orange granules continue to increase in size and redness

P = primary pigment cells. S = secondary pigment cells.

suppresses the deposition of both early- and late-phase granules. Eosin (w^e), an allele of white, does not seem to affect the granules of the primary pigment cells, but prevents the appearance of part of the early-phase granules normally present in the secondary pigment cells. The few granules laid down during the early phase of development in the secondary pigment cells turn red, as do similar pigment granules in wild type. During the late phase of development very few additional granules appear. These do not become red in the late pupal period and in the adult their change of colour is very slight. The influence of the eosin (w^e) gene is very slight at the onset of pigment development (in pupæ over 96 hours old), allowing the formation of all of the yellow granules in the primary pigment cells and a few of the granules in the secondary cells. This gene, however, prevents the formation of the late-phase granules until the very end of the pupal period, when a few of these appear but fail to develop completely.

(*b*) *Some Eye-colour Genes influence the Rate of Pigment Development.*—The allelomorphs of purple influence the rate of production of red pigment in the secondary pigment cells. They have no effect upon the granules in the primary pigment cells. Purple^2 (pr^2) has no visible effect on the early-phase granules and does not postpone the deposition of the late-phase granules, but it does retard their change from yellowish orange to red and their growth in size. Purple^3 (pr^3) prevents the early-phase granules from becoming red until after the 120th hour of pupal life. The change to red is very slow and is not complete until nearly the end of the pupal period. The late-phase granules, when they appear toward the end of the pupal period, remain smaller than the corresponding granules in purple^2 (pr^2) pupæ and their rate of reddening is very much slower. Purple^1 (pr^1) prevents the transition from brown to red in the early-phase granules until the very end of pupal life, and retards it to such an extent that only in the old adult is the change to red complete. No late-phase granules ever appear in purple^1 eyes. Text-fig. 1 shows graphically how these three allelomorphs of purple affect pigment development as compared with wild type, and how their different rates of development influence the final colour of the eyes.

It is apparent that the three allelomorphs of purple influence pigment production in the same manner but in different degrees. The differences are what Goldschmidt (1927) has called quantitative differences of the allelomorphs acting on one and the same process. Purple^1 (pr^1) may be classified as the strongest allele because it retards the process of pigment development most. Purple^3 (pr^3) has much less effect, and purple^2 (pr^2) has so little effect as to make the eye barely distinguishable from wild type. Ford and Huxley (1928) and Wolsky and Huxley (1934) have described

a similar situation in *Gammarus chevreuxi*, where an allelomorphic series of genes has varying effects on eye pigment due to varying the rates and extent of pigment deposition.

(*c*) *Some Eye-colour Genes affect the Pigment Qualitatively.*—Sepia eyes are identical with wild type until the end of the early phase of pigment

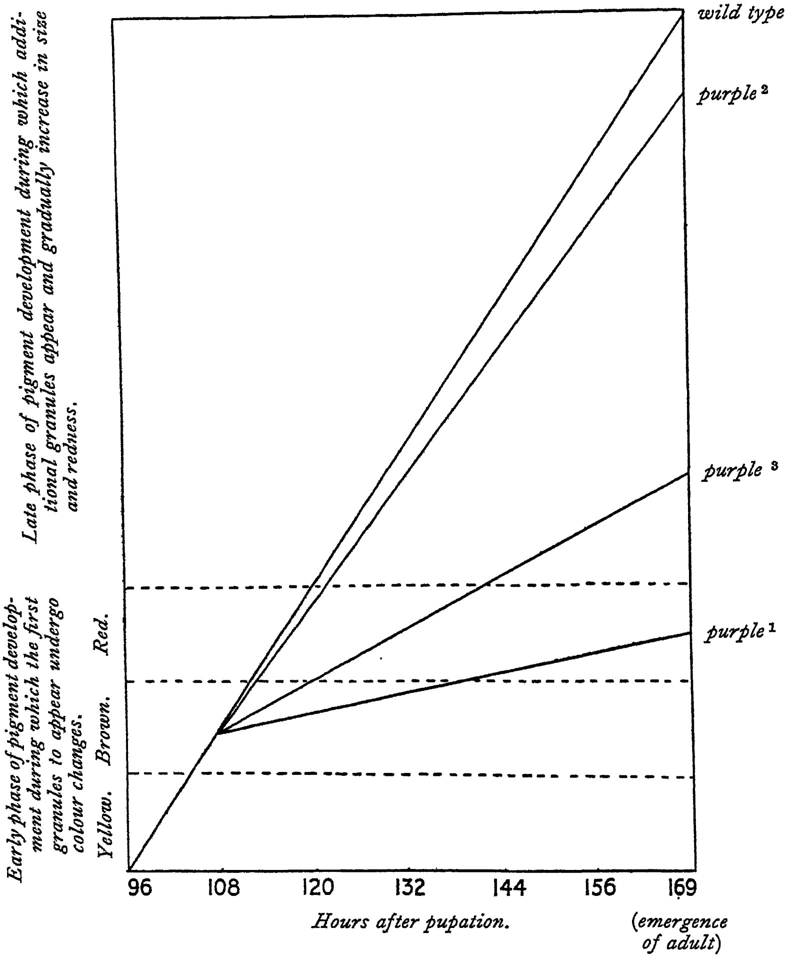

FIG. 1.—Diagram showing the different rates of pigment development in the presence of three purple allelomorphs and wild type of *D. pseudo-obscura.*

development. At this time the primary pigment cells contain only yellow and brown granules and the secondary cells only red ones. The usual number of late-phase granules is formed, but instead of becoming red and large in size these remain yellow and small. During adult life the early-phase granules in the secondary cells which were previously red undergo

a reversed chemical reaction and become yellow. Text-fig. 2 illustrates how the proportions of red to yellow and brown pigment granules in the secondary pigment cells of wild type and sepia (se) eyes differ at different times during development.

According to Schultz (1935) yellow pigment may be an oxidation product of red. It is probable, as suggested by Mainx (1935), that the sepia

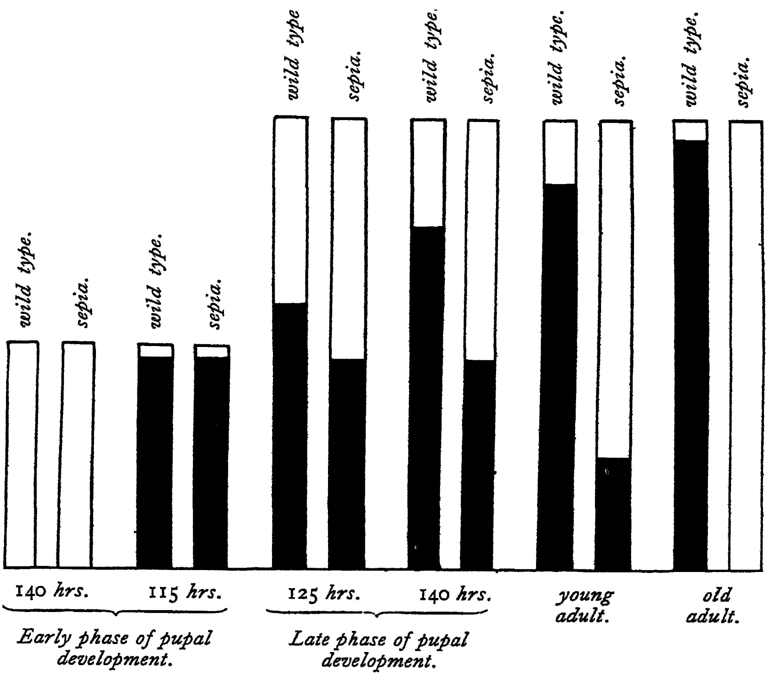

FIG. 2.—Diagram showing the proportion of red to total pigment in the eyes of wild type and sepia flies at various stages of development. Shaded portions of lines indicate the amount of red pigment.

(se) gene eliminates the inhibition of oxidation of red pigment which is characteristic of the wild type allele of sepia. As a result of this elimination all the granules in the secondary cells of the sepia eye may be oxidized to become yellow. As yellow pigment readily changes to brown, it is natural that brown granules should appear among the yellow ones in the secondary pigment cells. Bridges (1919) in *D. melanogaster*, and Crew and Lamy (1935) in *D. pseudo-obscura*, have noted that old sepia flies have much darker eyes than wild type. This very dark colour is not due to the presence of more pigment granules; it is obvious that the numerous coarse red granules in the secondary pigment cells of the wild type eye interfere with the distinct observation of the brown colour of the granules in the primary pigment cells, while the yellow and brown granules in the

secondary cells of the sepia eye intensify the effect of the brown granules in the primary cells of the sepia eye.

The detailed analysis of the histological structure of the eye during the various stages of its development clearly indicates that the maximum amount of pigment is present in the eyes of wild type flies. It also shows how mutant genes alter the process of pigment formation and behaviour. Each gene comes into operation at a definite stage and remains effective during a limited period of time. The effect is specific for each gene.

SUMMARY.

A histological study of wild type and seven eye-colour mutants of *Drosophila pseudo-obscura* at various stages of development has been made. The genes concerned may be classified according to their action, thus:

1. Genes which suppress pigment formation: (*a*) Vermilion (v) and orange (or) which suppress the early phase of development entirely, but allow the late phase to proceed as in wild type. (*b*) White (w^5) and eosin (w^e) which suppress granule formation. The suppression is complete in the case of white (w^5), partial in eosin (w^e). Eosin (w^e) does not affect the subsequent development of the granules which do appear.

2. Genes which alter the rate of pigment production. The three allelomorphs of purple (pr) which affect the rates of production of red pigment in varying degrees. Purple^2 (pr^2) retards the development of red so little as to make the eye almost indistinguishable from wild type. Purple^3 (pr^3) retards the development of red considerably, and purple^1 (pr^1) to such an extent that few red granules are present at emergence though many appear in older flies.

3. Genes which have a qualitative effect upon pigment. Sepia (se) influences the development of pigment during the later stages of pupal life and early adult life. All of the eye pigment of sepia flies eventually becomes yellow and brown, which indicates that the influence of sepia (se) is qualitative and that the normal amount of pigment is probably unchanged.

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DESCRIPTION OF PLATES.

Semi-diagrammatic longitudinal sections of ommatidia of the eyes of *D. pseudo-obscura*, showing the distribution of pigment granules at various stages of development.

EXPLANATION OF LETTERING.

c. cornea.
bm. basement membrane.
pcc. pseudocone cell.
ppc. primary pigment cell.
r. retinula.
rh. rhabdom.
spc. secondary pigment cell.

PLATE I.

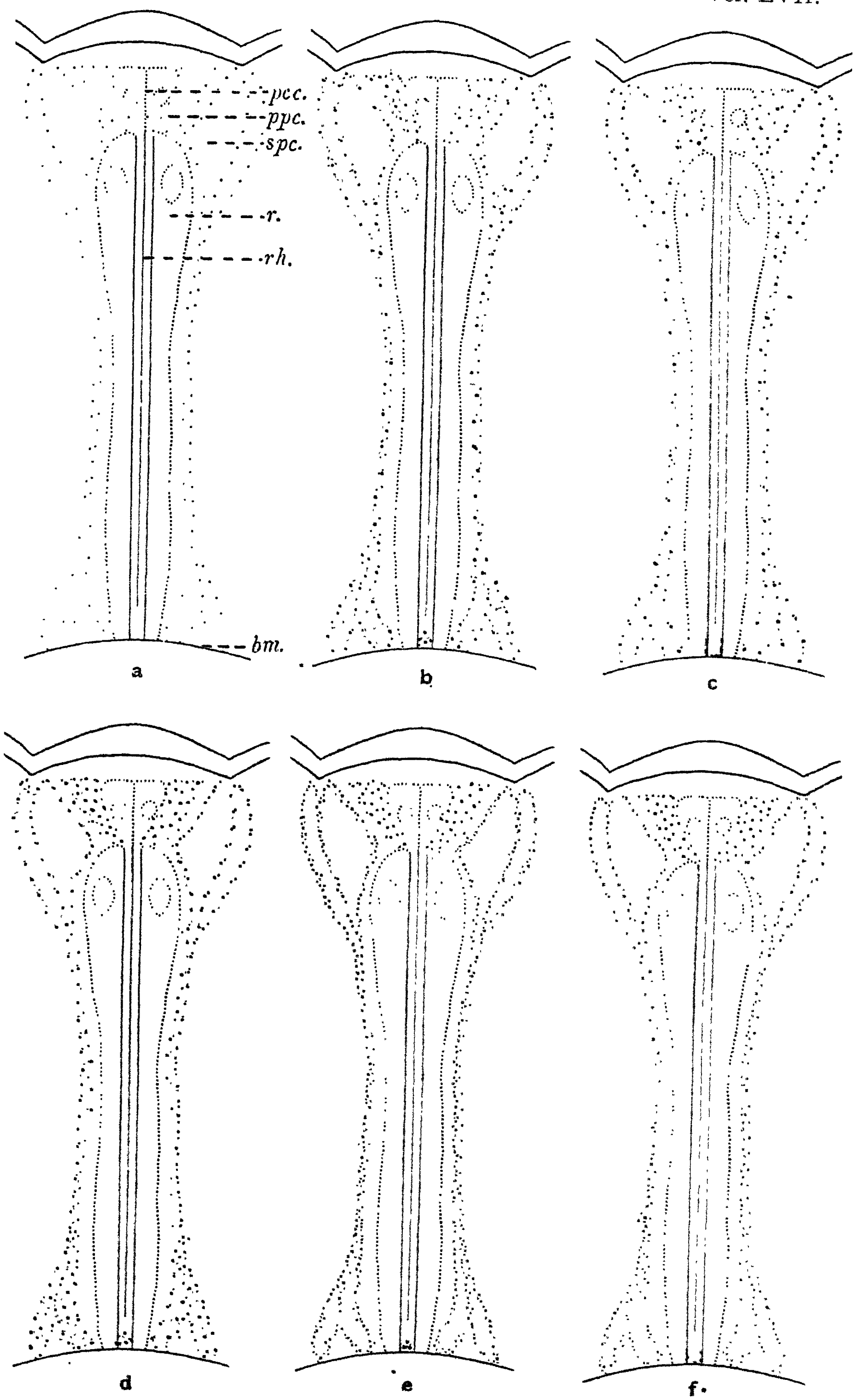
FIG.

a. vermilion (v), orange (or) or white (w^5), pupa about 104 hours old.
b. wild type, sepia (se), purple (1, 2 or 3), about 104 hours old.
c. eosin (w^e), pupa about 104 hours old.
d. purple[1] (pr^1) or purple[3] (pr^3), pupa 115–120 hours old.
e. wild type, sepia (se) or purple[2] (pr^2), pupa 115–120 hours old.
f. eosin (w^e), pupa 115–120 hours old.

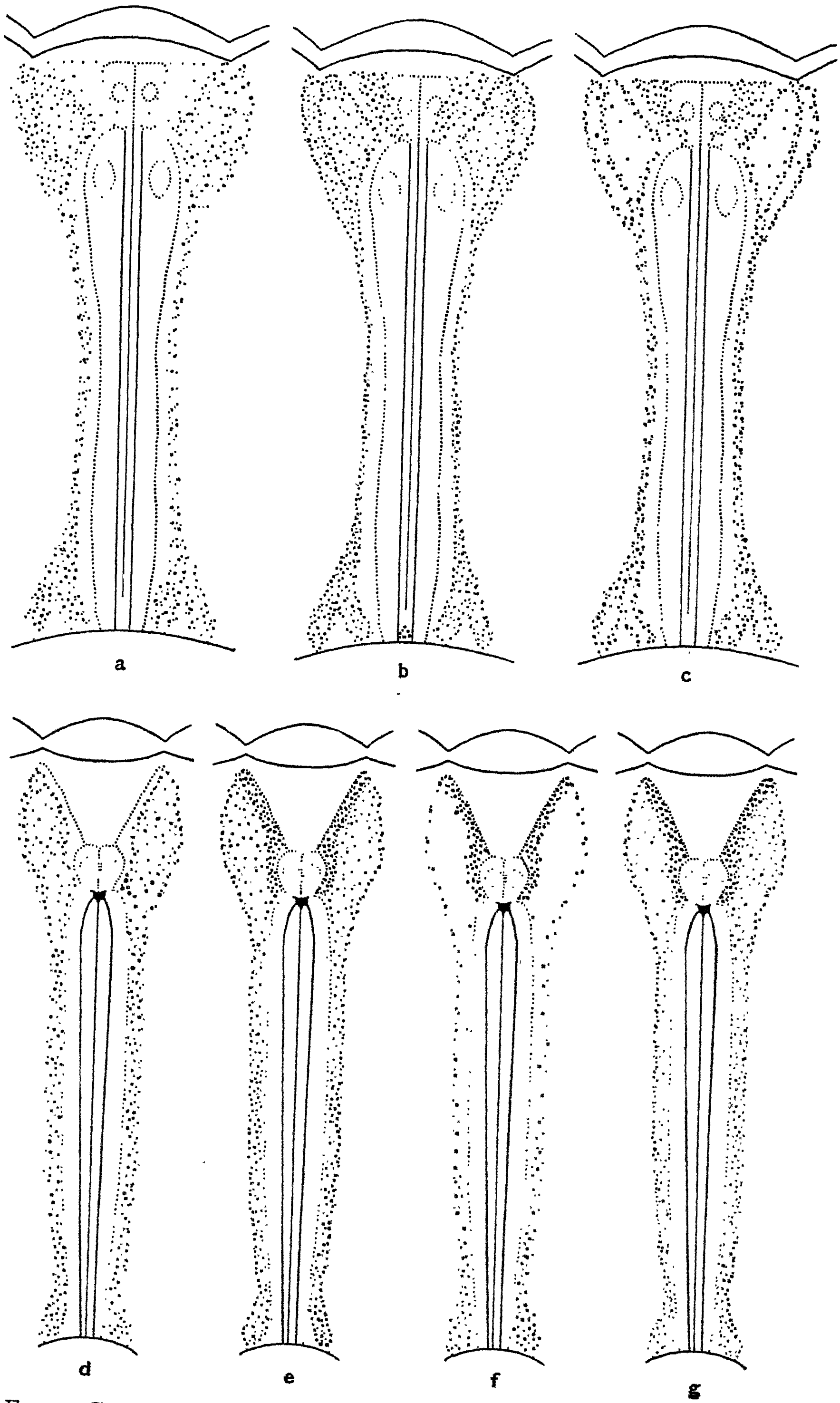
PLATE II.

FIG.

a. orange (or) or vermilion (v), pupa 120–132 hours old.
b. wild type, pupa 120–132 hours old.
c. purple[3] (pr^3), pupa 120–132 hours old.
d. vermilion (v) or orange (or), pupa about 140 hours old.
e. wild type, pupa about 140 hours old.
f. eosin (w^e), pupa about 140 hours old.
g. sepia (se), pupa about 140 hours old.



pcc.
ppc.
spc.
r.
rh.
bm.
a
b
c
d
e
f.



a b c d e f g

FLORA COCHRANE.

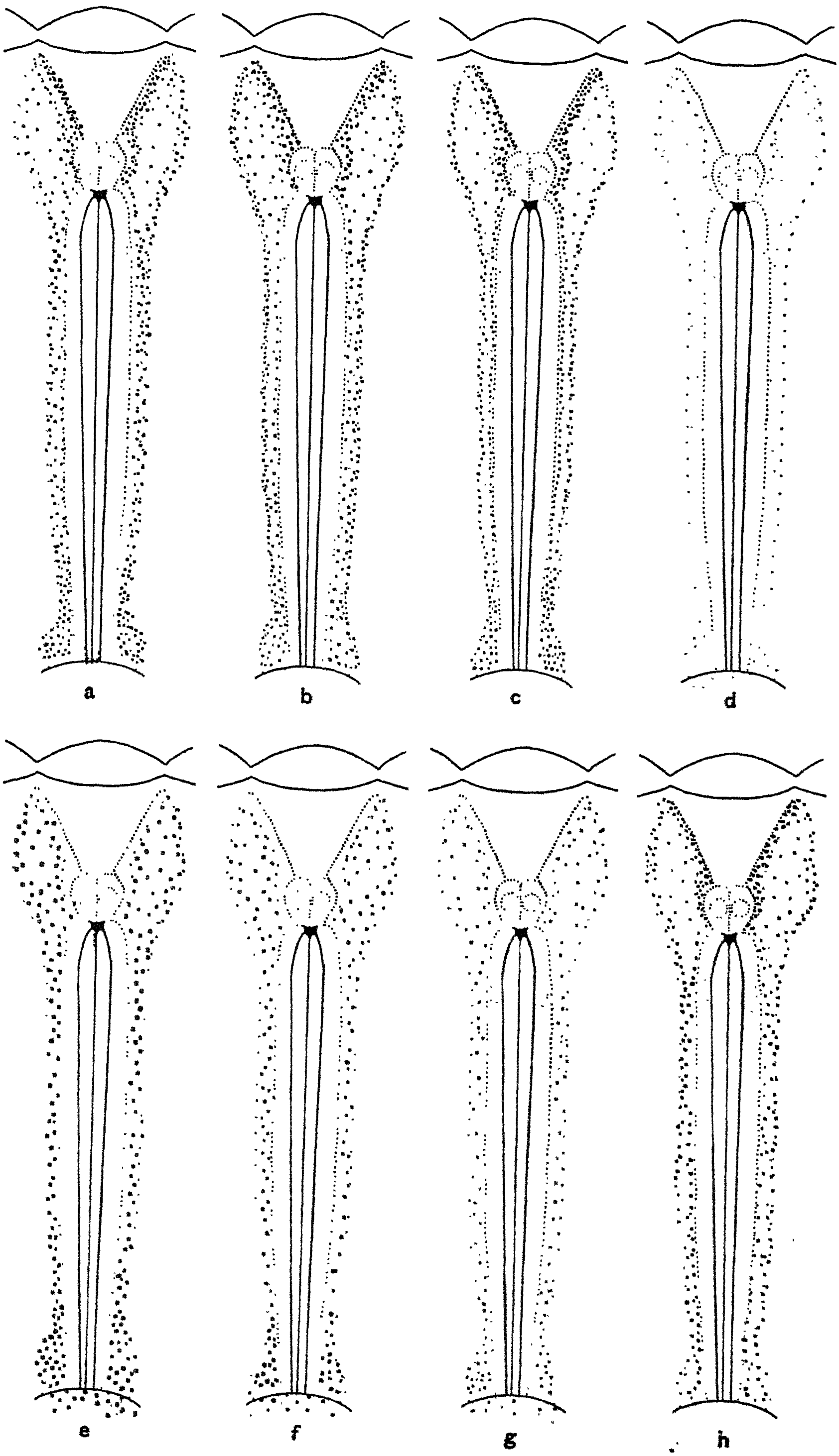

a
b
c
d
e
f
g
h

PLATE III.

FIG.

a. purple^1 (pr^1), pupa about 140 hours old.
b. purple^2 (pr^2), pupa about 140 hours old.
c. purple^3 (pr^3), pupa about 140 hours old.
d. vermilion sepia (v se), adult.
e. orange (or) or vermilion (v), adult about two weeks old.
f. orange purple^2 ($or\text{-}pr^2$), adult about two weeks old.
g. orange purple^3 ($or\text{-}pr^3$), adult about two weeks old.
h. sepia (se), adult about two weeks old.

(*Issued separately November* 13, 1937.)

OBITUARY NOTICES.

Richard Anschütz.

One of the foremost organic chemists of the latter part of the nineteenth and the earlier years of the twentieth centuries, Richard Anschütz, Emeritus Professor of Chemistry at Bonn, an Honorary Fellow of the Society, died in his eighty-fifth year on January 8, 1937, in his native town of Darmstadt, where he had resided from the date of his retiral from his Chair in 1921. After studying the natural sciences, and chemistry in particular, at Darmstadt, Heidelberg, and Tübingen, Anschütz obtained his doctor's degree in 1874, and in the following year he was appointed assistant to Kekulé, then professor at Bonn. His academic career was thereafter connected continuously with the University of Bonn, where he became successively dozent, extraordinary professor, and ordinary professor, and succeeded in 1898 to the Chair rendered vacant by the death of Kekulé. He was the author of a large number of papers recording the results of researches that extended over a wide range of subjects in organic chemistry and were carried out either by himself or in collaboration with colleagues or pupils. These papers appeared from about 1884 onwards, chiefly in the publications of the German Chemical Society and in Liebig's *Annalen*. In addition to carrying on his original investigations and fulfilling the ordinary duties of his Chair, he devoted much of his time to literary work, including the production of more than one edition of "Little Beilstein"—a masterly revision, amounting to rewriting, of v. Richter's *Text-Book of Organic Chemistry*—and the writing of a useful monograph on distillation under reduced pressure. He also made various valuable contributions to the historical literature of chemistry. One of the chief of these is his biography with papers, letters, addresses, etc., of August Kekulé, which engaged his attention principally during the years after his retirement. The result of his labour upon this biography was the appearance of two stately volumes that were published in 1929/31, and formed a dignified and worthy tribute to the preeminence of Kekulé as an investigator and a teacher. Other publications included a reproduction, in Ostwald's *Klassiker der exakten Wissenschaften*, of Lohschmidt's *Constitutional Formulæ of Organic Chemistry*;

a memorial notice of W. Körner; an account of the Chemical Institute of the University of Bonn; and a publication on the importance of chemistry in the World War. One conspicuous service that Anschütz rendered to the history of chemistry resulted from his interest in the work and publications of Archibald Scott Couper. He first met with Couper's name, about the year 1893, in connection with certain investigations on salicylic acid that he was himself engaged upon, and again, in a more arresting manner, about twenty years later when dealing with Kekulé's position in relation to the recognition of the quadrivalence of carbon and the linkage of carbon atoms. As a result of these contacts, he became intensely interested in endeavouring to gain a knowledge of the origin and personality of this evidently gifted chemist, who had written a number of meritorious chemical papers in the later fifties of the nineteenth century and had then disappeared entirely from further productive scientific activity. By means of inquiries made through chemical friends in Great Britain, Anschütz was put into communication with the late Professor Crum Brown, and, after somewhat prolonged investigations, the efforts of the latter were eventually successful in completely clearing up the obscurity that had for many years surrounded the name and nationality of Couper. An account of this noteworthy inquiry, along with an appreciation of Couper by Anschütz, was published in the Society's *Proceedings* (vol. xxix, 1909, pp. 193–260).

Anschütz was elected to Honorary Fellowship of this Society in 1935.

L. D.

de Burgh Birch, C.B., M.D., C.M.

THE son of Dr de Burgh Birch of the Madras Medical Service, de Burgh Birch was born on May 18, 1852. He was educated in Switzerland, then at a school in Clifton and later at the Bristol Medical School. In 1874 he entered the University of Edinburgh, graduating M.B., C.M. in 1877, and three years later as M.D., being awarded the gold medal. After spending three years as assistant to the professor of the Institutes of Medicine in Edinburgh, Birch was elected to the Chair of Physiology at Leeds. It was the first whole-time Chair in the School of Medicine, and a new epoch was just beginning, for the School had just been amalgamated with the Yorkshire College of Science, a constituent of the Victoria University.

Birch's first efforts were directed to building up a department worthy of his subject. There was a minimum of equipment and very little money. Nevertheless he gradually evolved simple but efficient apparatus for teaching experimental physiology and practical histology. In this he received considerable help from a local instrument-maker named Kershaw, a man of great ingenuity and technical skill. Ten years after Birch went to Leeds the present Medical School in Thoresby Place was opened. The physiology department which it contained was a tribute to his skill and inventive genius, and was probably at that time the best of its kind in the Provinces. It is perhaps not surprising that, having to devote so much of his time to teaching and the provision of adequate apparatus for it, Birch's early promise as a research worker did not materialise. On the other hand, in the invention and construction of new and improved apparatus and in the organisation of teaching he made a notable contribution to the advancement of physiology in the days when it was just becoming an experimental science.

In addition to the duties of his Chair he filled the office of Dean of the Medical School for many years with distinction. Birch retired in 1917 on reaching the age limit and was given the title of Emeritus Professor. He was elected a Fellow of the Royal Society of Edinburgh in 1880.

Mention must also be made of Birch's military hobby, in which his powers as an organiser were fully displayed. He raised a Medical Staff Corps which was later converted into a full unit in the Territorial Scheme. He became A.D.M.S. of this unit, but had retired when War broke out.

On receiving an invitation to resume his command he did so, and in 1915, when over sixty years of age, he accompanied it to France. For his public services he was awarded the C.B.

The years after his retirement were spent peacefully in the South of England, and he died at Bournemouth on September 18, 1937, aged eighty-five years.

H. S. R.

Albert William Borthwick, O.B.E., D.Sc.

By the death of Professor Borthwick, at the age of sixty-four, on April 19, 1937, British Forestry has prematurely lost one of its best-known and most highly respected Foresters.

Borthwick was a member of a very old Scottish family, being the third son of the late W. H. Borthwick, 29th Laird of Crookston and Borthwick Castle. He studied Science at St Andrews University and graduated B.Sc. in 1895. Thereafter he became interested in Forestry and studied that subject in Germany for three years under Professors Tubeuf, Hartig and Mayr. He was therefore one of the very first in this country to have received a thorough training in scientific forestry.

In 1899 he was appointed an Assistant to the late Sir Isaac Bayley Balfour and lectured on Plant Physiology, and was awarded the Degree of D.Sc. for a Thesis entitled "Production of Adventitious Roots and their Relation to Bird's-Eye Formation in the Wood of Various Trees." In 1905 he was appointed Lecturer in Forestry in the Edinburgh and East of Scotland College of Agriculture, and from 1908 to 1914 was Lecturer in Forest Botany in the University of Edinburgh. In 1914 he was appointed Chief Advisory Officer to the Board of Agriculture for Scotland, and after the War, when the Forestry Commission was at last appointed, he became its Chief Research and Education Officer, which post he held till he was appointed to the Chair of Forestry in Aberdeen in 1926.

For many years he was a very active member of the Royal Scottish Arboricultural Society. From 1911 to 1913 he was Vice-President, and from 1913 to 1929 he was Honorary Editor of the Society's Journal. In 1930 and again in 1931 he was elected President. From 1902 until his death he was Honorary Consulting Cryptogamist.

Besides being a forester he was also a botanist—and he loved trees. Consequently the subject of forest botany was his chief hobby, and it was his delight to wander in an arboretum—and it was a joy to be with him there. This aspect was suitably recognised when he was elected President of the Botany Section of the British Association for the Advancement of Science in 1934.

Borthwick was of a retiring nature, ever willing to give sound advice or help, with that peculiar quiet smile of his. Yes, Borthwick was just one of Nature's own gentle-men.

The present progress and development of Forestry in this country owes much to him.

He was elected a Fellow of this Society in 1925.

J. L.-P.

Alfred Daniell, M.A., LL.B., D.Sc.

DR DANIELL passed away at his rooms in Viewforth Gardens, Edinburgh, on January 12, 1937, in his eighty-fourth year. He had a notable record. Born at Llanelly, of purely Welsh stock, he passed his earlier years in his parents' house there, receiving, partly at home, partly at the local schools, the teaching suited to his years. His own efforts were not lacking, and the indications and results were such that, at a very early age he was sent, or came, as a student of Arts to Edinburgh University. As a student, he had a career of outstanding success, not only in Arts, but in Medicine and Science, and in due time received from his Alma Mater the degrees of Master of Arts and Doctor of Science. In 1884 he published the first edition of his *Text Book of the Principles of Physics* —a publication which enjoyed widespread popularity, being translated into various European languages, and even into Japanese. It also became the text-book of the U.S. Army. Another of his publications was *Physics for Students of Medicine*. He was for a time lecturer on Physics in the School of Medicine, and his erudition was called upon in other spheres of activity. His scientific attainments and his versatility as a scholar were recognised by his admission to the Fellowship of the Royal Society of Edinburgh in 1885.

In the early eighties of last century, Dr Daniell felt attracted to the Bar, and as a student of law he maintained his already acquired reputation and in due course became LL.B. Doubtless he had in view the value of scientific knowledge in many lawsuits, and his practice at the Scottish Bar, to which he was called in 1886, and later at the wider sphere in London (for he became a Barrister of the Inner Temple in 1894), was chiefly concerned with matters which were affected by scientific considerations.

But Science always had the first place in Dr Daniell's heart. About twenty years ago he commenced to write a new book, *Problems in Physics*, which occupied the greater part of his time during the remainder of his life. He was able to complete the manuscript before his activities were progressively more and more curtailed during the last year or two by the condition of his eyesight. We understand that the main purpose of Dr Daniell's testamentary writings is to secure publication of his *Problems in Physics*. In his later years he also revised, and partly rewrote, his *Principles of Physics*.

An incident which occasioned not a little natural gratification to

Dr Daniell in his advancing years was his call, in the year 1925, to his native Llanelly, in order that there might be conferred on him the freedom of the Borough.

True to his race, Dr Daniell was, amongst other things, an accomplished musician. He was never married.

J. P.

John Edwards, LL.D.

JOHN EDWARDS belonged to a small group of business men in Glasgow who some twenty-five or thirty years ago devoted their leisure to scholarship in History and Literature. Born in 1846, the son of Rev. John Edwards, he spent his whole life in the city of Glasgow. Educated at the High School, he was entered at the Old College as a student of Law. He began life as a legal practitioner but deserted the Law after ten years, and entered business as a partner in Alexander Harvey and Son, Dyers. All his spare hours during thirty years of business life were devoted to the pursuit of scholarship for its own sake. A humanist from his youth, he acquired sufficient Latinity to extract the ore from medieval documents, and became expert in the deciphering of ancient manuscripts. In his fifty-sixth year he retired from business and gave up his whole time to the study of the original sources of information concerning the medieval Religious Orders the Knights Templars and Knights Hospitallers in Scotland and elsewhere. He was an expert palæographer, and his scholarship was marked by acumen, accuracy and fairmindedness. Among his many contributions may be mentioned: "History of the Chronicle of the Brute" (1903), "The Religious Orders under our Early Kings" (1906), "The Knights Hospitallers and the Conquest of Rhodes" (1920), published in the *Proc. R. Phil. Soc. Glasgow*; "The Temple Barony of Maryculter" (1900), "The Order of Sempringham and its connexion with West Scotland" (1903), which appeared in the *Trans. Glasgow Archæol. Soc.*; and "The Templars in Scotland in the Thirteenth Century," and "Hospitallers in Scotland in the Fifteenth Century," which will be found in the *Scot. Hist. Rev.*, vols. v and ix.

The value of the work and the excellence of their scholarship were recognised by the Senate of the University of Glasgow in the conferment on him of the Honorary Degree of LL.D.

He was elected a Fellow of this Society in 1904 and died on February 3, 1937.

T. H. B.

David Ellis, D.Sc.

DR DAVID ELLIS, Professor of Bacteriology in the Royal Technical College, Glasgow, died suddenly in his home at Bearsden on January 16, 1937, aged sixty-two years. Less than three months previously he had suffered severe bereavement through the death of his wife. He was a native of South Wales and was educated at the University College of Wales, Aberystwyth, and graduated in 1896 at the University of London. For some time he taught as a science master at secondary schools, and then proceeded to the University of Marburg, where he gained the degree of Ph.D. in 1902 for researches in bacteriology. After his return to this country he studied at the Lister Institute, London, and later took up science teaching at Dollar Academy. In 1904 he was appointed lecturer in bacteriology and botany in the Glasgow and West of Scotland Technical College, now the Royal Technical College, Glasgow.

Ellis obtained the degree of D.Sc. of the University of London in 1905 for his researches in mycology. His further work on fungi was concerned principally with fossil species.

In 1915 he published a paper dealing with the occurrence of fossil fungi and bacteria in slides prepared from ironstones and ferruginous limestones from various districts in Great Britain, and in 1917 he described certain phycomycetous fungi found in tissues of *Lepidodendron* and *Lyginodendron* which, it is suggested, are related to the modern Peronosporaceæ; both these papers appeared in the *Proceedings of the Royal Society of Edinburgh*.

From 1907 to 1919 he was a lecturer in nature study to the Glasgow Provincial Committee for the Training of Teachers, and was for some time lecturer in technical mycology in the University of Glasgow. He took a great interest in adult education, and his popular lectures in botany attracted large audiences. His publications include *A Guide to the Common Wild Flowers of the West of Scotland*, *A Guide to the Common Wild Flowers in Wales*, *A Guide to the Study of Nature* (1912), and *Medicinal Herbs and Poisonous Plants* (1918). He was a frequent attendant at meetings of the British Association, where on several occasions he read papers on bacteriological subjects. For many years he acted as chief examiner in botany for the Central Welsh Board.

Since 1920 Professor Ellis also held the post of Superintendent of the School of Pharmacy and the Scottish School of Bakery at the Royal

Technical College, and in the latter capacity he showed a marked gift for the expression of scientific subjects in simple language. His work covered a wide field, and during this period he published (with D. Campbell) *The Science and Practice of Confectionery* (1928) and *A Science Course for Bakers* (1923). Bacteriology was, however, his chief study, and his publications include books on the *Outlines of Bacteriology*, *Practical Bacteriology for Chemical Students* (1923), and several contributions to the knowledge of *Cladothrix* and *Leptothrix*.

He is widely known for his work on the iron and the sulphur bacteria, on each of which subjects he published a monograph. The iron bacteria form an important group, which is one of the makers of geological history, for many of the bog iron ores owe their formation largely to the activities of these organisms. They are of importance to the water engineer in relation to water reservoirs, the corrosion of pipes, and the general appearance and clarity of waters. Six species were fully described in the monograph which appeared in 1919. In the monograph on the sulphur bacteria published in 1932 Professor Ellis paid much attention to the classification of the group and adopted an original scheme based on a division into colourless forms and coloured forms, the rhodo-theio-bacteria; a very complete account was given of the colouring matter of these latter organisms. The sulphur metabolism of the group was described, and the important part played by these organisms in nature fully discussed.

Professor Ellis was an expert on problems relating to the disposal of sewage, and several of his publications such as *The Sulphur Bacteria as Aids in the Study of Polluted Waters* and *The Blackening of the Sand of the Clyde Estuary* deal with this subject. He frequently acted as consultant in law cases when water pollution led to litigation. He was also consulted by Government departments, including the Department of Health, on this and kindred subjects.

He was keenly interested in the work of the Royal Philosophical Society of Glasgow, and at the time of his death was president of the Biological Section. His leisure time was divided between golf, cycling, and the care of his garden, and it was while employed in his garden that he died.

He was elected a fellow of the Society in 1906.

M. W.

Mungo McCallum Fairgrieve, M.A.

MUNGO MCCALLUM FAIRGRIEVE was born at Saltcoats, Ayrshire, on October 30, 1872, and died in Edinburgh on August 4, 1937. He entered Glasgow University in 1890, and spent five years under Kelvin, of whom he always spoke with affectionate appreciation. In 1895 he proceeded to Cambridge as Scholar of Peterhouse, being classed Senior Optime in 1898 and First Class in the Natural Sciences Tripos in 1899. For three years he held a teaching post at Eastbourne New College, which, as he used to remark, must have been too much for this school, as it closed down shortly afterwards! In 1903 he joined the staff of The Edinburgh Academy, where that exceptional schoolmaster, the late J. Tudor Cundall, was then raising the standard of science teaching to a level worthy of the school which had nurtured men like Maxwell, J. S. Haldane and D'Arcy Thompson. In 1913 Fairgrieve succeeded Cundall as Head of the Science Department, and in this capacity he remained until compelled to relinquish the post through ill health in 1935. Under his guidance the science buildings grew during the years following the War to an extent which has occasioned a shudder of disgust in more than one Academical *laudator temporis acti*!

In the limited time which he allowed himself away from his beloved school he was a keen student of meteorology. He published several papers in the *Journ. Scot. Meteorol. Soc.*, of which he was Vice-President, and later, after the Society's amalgamation with the Royal Meteorological Society, Scottish Secretary. At Glasgow he had been a member of the University Mountaineering Club; and afterwards, until his last accident robbed him of the means, he would return constantly to the Cairngorms in search of refreshment of spirit. His regard for his old chief is witnessed by the care with which he prepared one of Cundall's papers for publication (*Trans. Chem. Soc.*, vol. cv, 1914, p. 60) and constantly revised the text-books of physics and chemistry used at The Academy.

The last ten years of Mr Fairgrieve's life were shadowed by the effects of two severe accidents. Though crippled by the second he held to his post with uncomplaining tenacity. While always watchful for an opportunity to assist a colleague during a period of ill health, he refused to recognise the fact of his own disablement. He fought a gallant losing battle for several years; so a kindly, stalwart spirit has left us.

He married Miss Helen Gifford, without whose unruffled devotion even his determination must have wavered.

He was elected a Fellow of this Society in 1910.

W. P. D. W.

David Fraser Fraser-Harris, D.Sc., M.D.

DAVID FRASER FRASER-HARRIS was born in Edinburgh in 1857, the son of David Harris, F.R.S.E., and Elizabeth Fraser, daughter of Sheriff Fraser, of Kamsky. From Dr Bryce's Collegiate School he went to Edinburgh and Glasgow Universities, and did post-graduate work in Berne, Jena and Zurich. In Glasgow he was President of the S.R.C., and he returned to that University in 1893 as senior assistant to Professor J. G. McKendrick, and Muirhead Demonstrator in Physiology.

In 1898 Fraser-Harris came to St Andrews University, then in a difficult period of transition, and helped for ten years to establish for the first time a teaching department of Physiology. While there he married Eleanor, youngest daughter of Lieut.-Col. F. M. Hunter, C.B., C.S.I., by whom he had one son, now a Sub-lieutenant in the Royal Navy. After working for a time under Professor Carlier in Birmingham, Fraser-Harris became, in 1911, Professor of Physiology in Dalhousie University, Halifax, Nova Scotia; and here he had once again the difficult task of creating a new physiological school and laboratory. Here, among many other publications, he wrote the official account, on the medical side, of the great disaster caused by the explosion of an ammunition ship, which destroyed half the town with appalling loss of life, in 1917.

In 1924 Fraser-Harris suffered a breakdown in health, which, though he afterwards made a good recovery, seemed at the time so serious that he retired on a "disability pension" from his Chair. His resignation was regretted by all his colleagues; he was an extremely competent teacher, he was known as a lecturer all over Canada, and simplicity, unselfishness and a turn of kindly humour endeared him to his friends.

Returning to London with his health more or less restored, Fraser-Harris took up literary work with much courage and assiduity in scientific and medical journalism and especially in connection with the history of medicine, a subject to which he had given years of study. He wrote among other things a *History of Welsh Medicine*, and in 1933 he won a prize from Glasgow University for an Essay on "Antiseptics before Lister." He was a well-known member of the Athenæum in his latter years.

Lack of robust health, the drudgery of teaching, and constant lack of such resources as a modern laboratory affords and modern investiga-

tions require, are sufficient reasons for a somewhat scanty output of research. But his friends know that his mind was ever active, his scientific curiosity never abated, and many troubles never embittered his cheerful disposition.

He was elected a Fellow of this Society in 1896, and died on January 3, 1937.

The following are a few of his various publications:—

"On the Stereophotochromoscope, a new optical instrument," *Proc. Phil. Soc. Glasgow*, 1895; *Journ. Anat. Physiol.*, 1896.

"On the chemistry and coagulation of Milk," *Journ. Anat. Physiol.*, 1895; *Proc. Roy. Soc. Edin.*, 1897.

"On Hæmatoporphyrinuria," *Journ. Anat. Physiol.*, 1897; *Brit. Med. Journ.*, 1898.

"On Maté, or Paraguay Tea" (with J. G. McKendrick), *Pharmaceut. Journ.*, 1898.

"On post-tetanic tremor," *Journ. Physiol.*, 1903; *ibid.* (with J. Moodie), 1906.

"On the use of soluble prussian blue for investigating the reducing power of animal bioplasm" (with J. C. Irvine), *Biochem. Journ.*, 1906; and (with W. Moodie) *Journ. Physiol.*, 1906.

"On the tremors of striated muscle," various papers in *Journ. Physiol.*, 1906–1909.

"Observations on Frogs at temperatures below zero," *Journ. Physiol.*, 1910.

"Harvey *v.* Caesalpinus," *17th Int. Congress Med. London*, 1913.

"Spectroscopic investigation of the reduction of hæmoglobin by reductase" (with H. J. M. Creighton), *Journ. Biol. Chem.*, 1915.

"The reality of nerve-energy," *Brit. Journ. Med. Psychol.*, 1927.

D. W. T.

Alfred William Gibb, M.A., D.Sc.

FOR just over forty years Alfred William Gibb was associated with the teaching of geology in the University of Aberdeen, and the development of geology as an independent subject in that University, with a Department of its own, lay entirely in his hands. The Department, with its model museum, is a tribute to his comprehension of the various branches of his science and to his appreciation of educational methods.

Professor Gibb was educated at the Gymnasium in Old Aberdeen and at the University of Aberdeen, where he graduated M.A. in 1884. After spending several years partly in school-teaching and partly in business, he returned to the University on the institution of a Degree in Science and in 1897 graduated B.Sc. with Honours. In the meantime Professor H. Alleyne Nicholson, recognising his ability, had appointed Gibb (1895) to act as University assistant with the status of Lecturer in Geology, a subject which was then taught as part of Natural History. Further devolution followed upon the appointment to the Natural History Chair of Sir J. Arthur Thomson, and in 1900 Gibb was made Lecturer with sole charge of the teaching of geology, his experience having been added to by short courses of study with Rosenbusch at Heidelberg and with Judd at the Royal College of Science.

On the creation of the Kilgour Chair of Geology in 1922 the University Court made him, at the age of fifty-eight, the first professor, a recognition of his value as a teacher and his services in creating a Department of Geology in the new buildings of Marischal College opened in 1906.

Although his chief interest lay in the expounding of his subject, Professor Gibb carried on a series of investigations on varied subjects of local geology, such as the occurrence of pebbles of white chalk in Aberdeenshire clay, the structure of a felsite sill near Aberdeen, and the relation of the Don to the Avon at Inchrory, Banffshire. The results were communicated for the most part to the Geological Society of Edinburgh. But his most important work—a research upon which he concentrated for many years and for which he obtained the degree of D.Sc.—was a minute petrological analysis of *The Basic Igneous Rocks of Belhelvie, Aberdeenshire: a Study in Magmatic Differentiation.* It is regrettable that characteristic diffidence about his own thorough work prevented him from publishing this excellent thesis.

As a teacher he was outstanding; his lectures were models of clear

and direct exposition enlivened by gentle humour, and his unfailing interest in their progress gained him the devotion of a long succession of students of arts and science.

He was elected a Fellow of this Society in 1916, and died on July 12, 1937, aged seventy-three years.

J. R.

John Anderson Gilruth, M.R.C.V.S., D.V.Sc.(Melb.).

BORN near Arbroath, John Anderson Gilruth died at Melbourne on March 4, 1937, at the age of sixty-six. His intimate association in early life with farm-stock in general and sheep in particular provided him with practical experience that was to place him at great advantage when subsequently confronted with the major problems of his professional work. He spent some time in a solicitor's office, but soon determined to forsake law for veterinary science. He received his professional training in Glasgow and London, and obtained the diploma of M.R.C.V.S. in 1892. About a year later, while still in his early twenties, he was appointed to the Government Veterinary Service of New Zealand. After a period at the Pasteur Institute at Paris, he assumed the office of Pathologist to the New Zealand Health Department, an appointment he combined with research and administrative duties in connection with the Live Stock Division of the Department of Agriculture.

In 1908 Gilruth left New Zealand on his selection as Professor of Veterinary Pathology and Director of the Research Institute of the University of Melbourne. In 1911 he became a member of the Federal Scientific Mission, and in the following year was appointed Administrator of the Northern Territory when this was transferred from South Australia to the Commonwealth. He held the post of Administrator until 1920, when he returned to Melbourne to be entrusted shortly afterwards with the duties of Chief of the Division of Animal Health of the Commonwealth Council for Scientific and Industrial Research.

To Gilruth New Zealand owes much, for he was largely concerned in the framing of the Act of Parliament of 1900 which led to reform in the production of meat for export, upon which the prosperity of the country so largely depends.

A man of remarkable personality and great forcefulness, Gilruth made many contributions to Government publications, and for some years was Corresponding Member of the Société de Pathologie Exotique of Paris.

He was elected a Fellow of this Society in 1907.

O. C. B.

Sir Patrick Hehir, C.B., C.M.G., K.C.I.E., M.D., D.T.M., F.R.C.P.E., F.R.C.S.E., Major-General, I.M.S., Knight of Grace of the Order of St John of Jerusalem.

By the death of Patrick Hehir on May 1, 1937, the Indian Medical Service and the medical profession generally have lost a very remarkable figure. From being a student in the Calcutta Medical School, and a hospital apprentice in the Bengal Submedical Department, he rose to be a Major-General in the Indian Medical Service, a Companion of the Bath, a Companion of the Order of St Michael and St George, a Companion and later a Knight Commander of the Indian Empire, a Knight of Grace of the Order of St John of Jerusalem, a Fellow of the Royal College of Surgeons of Edinburgh, a Fellow of the Royal College of Physicians of Edinburgh, and in 1893 Fellow of the Royal Society of Edinburgh. He saw service in many campaigns: Burma, 1886–87, medal with clasp; North-West Frontier, Tirah, 1897–98, medal with clasp; War of 1914–18; Iraq, as A.D.M.S. of Colonel Townsend's force in Kut; advance on Kut, battle of Ctesiphon, and defence of Kut; taken prisoner at Kut but released soon after; the second Waziristan campaign and Afghanistan, 1919. He wrote many books covering a wide field. He contributed to the Report of the First Hyderabad Chloroform Commission in 1892; he wrote *Malaria in India* (1927), a subject in which he was specially interested, and a number of other volumes. He was a man of many gifts, kindly in his outlook; a very hard worker, with a keen sense of duty; he spared no effort to equip himself for the work which he was called upon to undertake.

A. G. M'K.

William Hunter, C.B., M.D., LL.D.

William Hunter was born at Ballantrae, Ayrshire, in 1861, and died in London on January 13, 1937. After graduating as M.B., C.M., at Edinburgh in 1883, when he was awarded the Ettles Scholarship as the first student of his year, he was elected a President of the Royal Medical Society and acted as a Resident House Physician in the Royal Infirmary. He studied in the Pathological Laboratory at Leipzig in 1884, and in 1886 received a Gold Medal for his M.D. thesis upon the Physiology and Pathology of Blood Transfusion and the fate of extravasated blood. Later he held a Research Scholarship of the Grocers Society, and from 1887 to 1890 the John Lucas Walker whole-time Research Scholarship at Cambridge. It was then that he carried out his experimental work on pernicious anæmia, and at that time it looked as if he might devote his life to experimental pathology. But this was not to be, for in 1889, after delivering the Arris and Gale lectures before the Royal College of Surgeons of England, he passed the examination for the Membership of the Royal College of Physicians of London, was soon after appointed an assistant physician to the London Fever Hospital and to Charing Cross Hospital, and commenced practice as a consulting physician. In 1896 he was elected to the Fellowship of the London College.

Hunter's name is especially associated with pernicious anæmia, in the etiology of which he believed that oral and gastric sepsis play an important part. His researches were a notable contribution to the clinical and gross pathological aspects of this disease.

As President of the Advisory Committee to the Eastern Mediterranean and Mesopotamia Expeditionary Forces from 1915–17, and later as consultant to the Eastern Command, he did important work for which he was awarded the C.B. and a high Serbian decoration—that of Grand Officer in the Order of St Sava. His Alma Mater conferred on him the honorary degree of LL.D. in 1927.

Devoted to the Medical School of his adoption, where he enjoyed exceptional popularity, alike with colleagues and students, his History of the Charing Cross Hospital and Medical School was a valuable addition to the annals of his profession. He was elected a Fellow of this Society in 1887.

E. B.

Rev. G. A. Frank Knight, M.A., D.D.

A SON of the Free Church of Scotland on both sides of the house, the Rev. G. A. Frank Knight was born at Dollar in 1869. He was educated at the academy there, and later he studied first at Aberdeen and subsequently at Glasgow University, where he graduated. After ordination and a journey, as secretary to his grandfather the Rev. D. A. N. Somerville, to the foreign mission fields, he held charges first at Auchterarder, then at St Leonards U.F. Church, Perth, and in 1914 he was inducted as colleague and successor to Dr Reith in the Kelvingrove U.F. Church in Glasgow. In 1928 he was offered the position of General Secretary to the National Bible Society of Scotland, for which his early experience, personal interest and energetic personality eminently suited him. This post he held with acceptance until his sudden death on May 2, 1937, in his sixty-ninth year.

Dr Knight had wide interests outside his ecclesiastical work, and was a voluminous writer on many topics. His chief absorption was in archæology, but he had a training in Natural History, and was a competent conchologist. In 1901 he published a paper in *Trans. Nat. Hist. Soc. Glasgow* on "The Marine Mollusca of Port Stewart, North Ireland, especially in Relationship with the Clyde Fauna." His larger publications included *The Nile and Jordan: being Archæological and Historical Inter-relation between Egypt and Canaan to the Fall of Jerusalem,* A.D. 70 (1921). This was a scholarly work showing a great deal of research. A second large work in two volumes published in 1933 is entitled *Archæological Light on the Early Christianizing of Scotland.* It was the product of many years of study of the records of lives of the early Saints in the Celtic Calendar, and the distribution of Dedications to the Saints throughout Scotland, as well as the study in the field of surviving remains of early monastic settlements in the Western Highlands and elsewhere.

He was elected a Fellow of this Society in 1901.

T. H. B.

Canon Albert Ernest Laurie, M.C., D.D.

CANON ALBERT ERNEST LAURIE, M.C., D.D., was born in Edinburgh in 1866, educated here, and spent his whole working life in the service of Old St Paul's Episcopal Church, Carrubber's Close, Edinburgh. He was appointed Curate in 1891, and became Rector in 1897. From then until his death his charge there was only broken by four years' service as Military Chaplain in France during the Great War. In 1906 he established a Child Garden in Chessel's Court, almost the first of its kind in Edinburgh and Scotland, and this under his inspiration and guidance has become a model of the complete education of body, mind, and spirit in the early years of childhood. While on war service he received the Military Cross in 1916, and later a Bar. On his return to Old St Paul's he built the War Memorial Chapel, which, with the adjoining Calvary and Stairway, forms one of the most beautiful war memorials in Scotland. In 1923 the University of Edinburgh bestowed on him the honorary degree of D.D. He was made a Canon of the Cathedral in 1917 and Chancellor in 1925. He was the centre and leader of his own large congregation, but found time to serve on many other public institutions and committees. He died suddenly on Sunday, April 25, 1937, in full work to the last.

He was elected a Fellow of this Society in 1921: he had a lifelong interest in science, deeply appreciated the privilege of its Fellowship, and frequently attended the meetings of the Society.

C. M.

Magnus Maclean, M.A., D.Sc., LL.D., M.Inst.C.E., M.I.E.E.

PROFESSOR MAGNUS MACLEAN was born in 1858. He died on September 2, 1937, having lived in Glasgow since his retirement in 1923. To do justice to his memory, a biographer should understand the highlander as well as the scientist. Possibly only a fellow-clansman could properly appreciate how this son of a Skye crofter reached an unrivalled position in the esteem and affection of his fellow-countrymen. This position Maclean won both by his sterling worth and by the active contribution he made to his native Gaelic, for he was not only a Lecturer in Celtic Literature and Language at Glasgow University, but also an author of these subjects—facts which were recalled when his Alma Mater conferred on him the LL.D. degree in 1919. On his side, too, Maclean loved the prominent place he had won, whether among highlanders, freemasons, electrical engineers or academic colleagues. Adulation did not destroy Maclean's strong character. He seemed to expect it and to enjoy it.

Passing to his professional side, "this gifted son of Skye," as he was more than once called, worked himself through the school-teacher stage to the University of Glasgow, like so many of his fellow-Scots, and it was not long before he attracted the notice of Kelvin (then Sir William Thomson), whose assistant he became until appointed to the Chair of Electrical Engineering at the Royal Technical College, Glasgow, in 1899. Kelvin's attraction for him marks him out as a mathematical physicist, and this he was in a high degree. During his tenures as assistant, lecturer and professor he contributed many papers to the Royal Societies of London and of Edinburgh and to the Institution of Electrical Engineers; he published compendious works on modern electric practice and he produced text-books for his students. At Glasgow University, particularly, medical and engineering students who attended his lectures on physics spoke feelingly of his sympathy with their limitations. On the other hand, his professorial outlook was probably as autocratic as that of his master, and rather strange to engineers who had lived under the softening influence of works' life. In Maclean, academic tradition of departmental dignity and independence found a firm upholder. Personally Maclean was a kindly man, always willing to help. His humorous and his wise remarks remain long in the memory.

Maclean's best work at the Royal Technical College is seen in the

fine electrical engineering department he carved out in the new building in George Street. Three large laboratories, a large drawing-office, a large lecture theatre, lecture- and class-rooms, are almost unequalled in any other place.

As a practical engineer, the limitations imposed on Maclean by his surroundings should be mentioned. Glasgow is not a centre of electrical manufactures, and most of the local electrical engineers are agents, factors or salesmen. A less inspiring place for a professor needing the constant spur of creation and production it would be hard to imagine. These local conditions, however, did not make Dr Maclean less beloved by his electrical colleagues than by his fellow-clansmen.

For his scientific work Maclean held the D.Sc. degree of Glasgow University, and he was a member of many learned societies, among which may be mentioned the Royal Society of Edinburgh (elected 1888) and the Institutions of Civil and of Electrical Engineers.

Among his sorrows is to be mentioned the loss of his wife and of two sons. He is survived by his eldest son and two daughters. He was an elder in the Westbourne Gardens Church, Glasgow. Maclean loved open-air recreation—golf, bowls and angling may be particularly mentioned.

As a man Maclean was unique—we shall not see his like again.

S. P. S.

Henry Moir, F.F.A., F.I.A., F.A.S.

By the death of Henry Moir in Rochester, in the State of New York, on June 8, 1937, the assurance world lost one of its most eminent personalities. Born in 1871, he entered the service of the Scottish Life Assurance Company, Limited, in Edinburgh in 1886. Fifteen years later he accepted the post of actuary of a life assurance office in New York. Although he spent the rest of his life in the United States, by frequent visits to this country and by constant correspondence he never lost touch with his many friends here. In 1923 he became President of the United States Life Insurance Company, and in October 1936, after fifty full and active years of life assurance work, he gave up this position for the less onerous one of Chairman of the Finance Committee. His wife, whom he married in 1899, was a daughter of the late A. T. Niven, a well-known chartered accountant of Edinburgh. She survives him, along with their son and two daughters.

It is impossible in a brief sketch to do justice to Henry Moir's strenuous career—his international reputation as an actuary, his powerful influence on the life assurance business in the United States, his public services, his wide private philanthropies, his educational work, and his warm-hearted interest in the younger members of his profession. By dint of concentration and method he was able to get through a surprising amount of work. He worked hard but he also played hard. His methods of relaxation also reflected his versatility. In his younger days he was a keen volunteer in the Queen's Edinburgh Rifles. He was a capital golfer, an enthusiastic gardener, a collector of rare books and first editions, a lover of Scottish song and story, a great devotee of Robert Burns, and no mean poet himself. No memoir of him would be complete unless it laid stress on his sterling integrity of character. He could be embarrassingly frank in his criticisms, and he was stern in his denunciation of anything that could not stand the acid test of honesty and fair dealing. Perhaps his most prominent characteristic, however, was his singularly attractive personality. His charm of manner, his genius for friendship and service, his wide culture, his almost boyish enjoyment of a game, were qualities which, joined to his solid worth, endeared him to his friends in a measure to which few men can attain. He possessed the deep religious convictions and the kindly humour of the Scot. "In quietness and confidence shall be your strength"; no phrase could more fittingly epitomise a singularly useful and highly honourable career.

He was elected a Fellow of this Society in 1929.

H. W. B.

William John Owen.

THE late William John Owen was elected a Fellow of the Royal Society of Edinburgh in 1927. He died suddenly at Canberra, Australia, on July 25, 1937.

Born in May 1887, he was educated in Melbourne, Victoria, and commenced his scientific career by studying pharmacy. He later found his special field lay with anatomy and histology, and he became associated with the Medical School of the Melbourne University.

In 1912 he joined Sir Colin MacKenzie in his work on comparative anatomy and the Australian fauna, work which at that time was undertaken in a private capacity. This activity of Sir Colin's later developed into the Australian Institute of Anatomy, situated in Australia's national capital, Canberra.

It was in 1930 that the Institute was established in Canberra, and Mr Owen, having resumed his connection with the work after being abroad in Europe and the United States of America, undertook the organisation of the Histological Department. His flair for his subject and unrivalled technique resulted in a splendid collection of thousands of preparations, for microscopic study, of the organs of the Australian Monotremes and Marsupials. It is probably correct to say that as far as the Australian fauna is concerned, no other museum houses such a complete collection of macro- and microscopical specimens, both as regards quantity and quality. On the histological side the loss of Mr Owen's services will be felt very severely.

W. C. M.

Arthur George Perkin, D.Sc., F.R.S.

DURING the last quarter of the nineteenth and the first quarter of the twentieth century no name has been more prominent in the literature of Organic Chemistry than that of Perkin. William Henry Perkin, sen. (later Sir), had two sons by his first marriage—William Henry, jun., and Arthur George—and one son by his second marriage—Frederick Mollwo. All three sons followed in their father's footsteps. "W. H., jun." was best known for his syntheses of not only "ring," but of "bridged ring" compounds. "A. G.", as we shall see, became an authority on "Tinctorial Chemistry"; while "F. M." interested himself specially in electrolytic oxidation and in low temperature carbonisation.

Arthur George Perkin was born on December 13, 1861. He attended the City of London School and then the Royal College of Chemistry. At the latter institution he came under Frankland and Guthrie, and in 1880 there appeared in the *Journ. Chem. Soc.* his first "Contribution"—"The Action of Nitric Acid on Diparatolylguanidin" (vol. xxxvii, 1880, p. 696). From London he went to Glasgow, where he spent a year with Professor E. J. Mills at Anderson's College. His next move was to Yorkshire College, Leeds, where began what, after an interval of ten years, became his permanent association with the great "Dyeing School" of the College, later known as the University of Leeds. After a year with Professor J. J. Hummel, in which they investigated "Hæmatin and Brazilien" (*Journ. Chem. Soc.*, vol. xli, 1882, p. 367), he was appointed chemist in the Alizarine Factory of Hardman and Holdens, Manchester, becoming manager in 1888. Business, however, did not offer attractions to him, and a couple of years later the Heriot-Watt College, to which "W. H., jun." had been called, gave accommodation to him and also to the youngest brother. The three Perkins, along with F. Stanley Kipping, formed a powerful team, and the Heriot-Watt College became a centre of chemical activity. The attraction of the "Dyeing School" at Leeds, however, prevailed, and from 1892 until his death on May 30, 1937, A. G. Perkin occupied successively the positions of Lecturer, Professor of Tinctorial Chemistry (from 1915), and, after retirement, Professor Emeritus.

In Leeds also he found his bride, Annie Florence Bedford, who survives him.

A. G. Perkin was an assiduous worker in the laboratory. The

Journal of the Chemical Society contains nearly sixty contributions in his own name and more than seventy in conjunction with others. These contributions are all concerned directly or indirectly with colouring matters. Many investigations were carried out on Indigo; on the Yellow Colouring Principles of Tannin Matters; on Colouring Matters of Cotton Flowers; on various colouring principles such as Apiin, Chrysin, Gentisin, Hesperitin, Maclurin, Morin, Myricetin, Quercetrin, Robinin; on derivatives of Anthracene, Anthraquinone, Oxanilide.

A measure of the chemical activity of the Perkin family may be got from the Collective Index of the Chemical Society, where in the period 1893–1902 three pages (double column), and in 1903–1912 four pages, are occupied by the family! During the Great War, "A. G." rendered important service in connection with the dye-stuff industry, and after it he continued his investigations, though not with the same energy as before, his last paper appearing in 1931.

In conjunction with A. E. Everest, he wrote a book, *Natural Organic Colouring Matters*, in which much of his own work is summarised. He also contributed articles to the *Encyclopædia Britannica* and to *Thorpe's Dictionary of Applied Chemistry*.

A modest, unassuming nature endeared "A. G." to those admitted to his friendship. Both "W. H." and "A. G." had musical attainments, and the writer has pleasant recollections of the performances of "A. G." on the flute to the pianoforte accompaniment of his brother.

He was elected a Fellow of the Royal Society of Edinburgh in 1893 and ten years later a Fellow of the Royal Society. The Davy Medal of the Royal Society was awarded to him in 1925, and on his retirement from the Chair of Tinctorial Chemistry the University of Leeds conferred upon him the Honorary Degree of D.Sc.

See also *Obituary Notices of Fellows of the Royal Society*, vol. ii, No. 6, 1938.

J. E. M.

Salvatore Pincherle, Hon. F.R.S.E.

SALVATORE PINCHERLE was born March 11, 1853, at Trieste, and graduated in 1874 at Pisa with an inaugural dissertation on Capillarity, which was published in the same year in *Nuovo Cimento*. In the following year he obtained an appointment as a teacher of mathematics in the Lyceum at Pavia, and made some advances in the theory of surfaces of minimum area, a subject evidently suggested by his interest in capillarity. From this he passed to Analysis, which was soon established as the chief interest of his scientific career.

In 1880 he obtained a University Chair of Mathematics at Palermo, and in the following year was translated to the University of Bologna, where the rest of his long life was spent.

Pincherle's name is associated chiefly with the theory of Functional Operations. The treatise which he wrote in conjunction with Dr Ugo Amaldi in 1901, *Le Operazioni Distributive e le loro Applicazioni all, Analisi*, and the lengthy contribution to the *Encyklopädie d. math. Wiss.* in 1906 on "Funktionaloperationen und Gleichungen," are to a great extent connected accounts of his own discoveries, which had been published originally in the *Mem. R. Accad. Bologna*, the *R.C. Ist. Lombardo*, *R.C. Accad. Lincei*, *R.C. Circ. Mat. Palermo*, *Ann. Mat. pura appl.*, *Acta math. Stockh.*, *Math. Ann.*, and other journals.

He was elected to the Accademia dei Lincei in 1901, and to Honorary Fellowship of our Society in 1921. He died July 11, 1936.

E. T. W.

Rt. Hon. Lord Rutherford of Nelson, O.M., F.R.S.

ERNEST RUTHERFORD was born at Nelson, New Zealand, on August 30, 1871. He was the son of a farmer, James Rutherford, of Taranaki. His paternal grandfather had emigrated from the south of Scotland. He was educated at Nelson College and at Canterbury College, Christchurch. He took the M.A. degree of New Zealand University with first-class honours in Mathematics and Physics in 1893. He was awarded an 1851 Exhibition Scholarship in 1894.

Rutherford came to Cambridge in October 1895 to work in the Cavendish Laboratory under J. J. Thomson. The discovery of argon had been published at the beginning of that year, that of X-rays in the autumn; the electron and radio-activity were to be discovered during the next three years. No time could have been more favourable for a young physicist to begin research in the Cavendish Laboratory.

Rutherford had already before leaving New Zealand begun experiments on wireless telegraphy, and he continued them in Cambridge. He very soon, however, joined Thomson in his investigations on the conduction of electricity through gases exposed to X-rays. The first systematic measurements on the mobilities of the ions produced in gases by these rays were made by Rutherford in 1897, and he had made important contributions to this subject before his appointment as Professor of Physics in McGill University, Montreal, in 1898.

Rutherford remained in Montreal till 1907. In that year he was made Langworthy Professor of Physics in the University of Manchester; in 1919 he became Cavendish Professor in Cambridge.

His discovery, shortly after going to Montreal, of the radio-active "emanation" from thorium and of the characteristic law of decay of its activity was the starting-point of the researches which led Rutherford and Soddy to put forward in 1902–3 their transformation theory of the radio-active elements. All the subsequent work on radio-activity served to confirm this view, which at first seemed so startling, that the radio-elements are undergoing spontaneous transmutations.

It is largely to Rutherford that our knowledge of the nature of the alpha particle is due. He may be said to have discovered the alpha ray while a research student at Cambridge, when he proved by absorption experiments that the rays from uranium were of two types which he named α- and β-rays. He later showed that the alpha ray is a positively

charged particle, measured its charge and mass, and proved that it is a doubly charged helium atom, *i.e.* a helium nucleus. This work on the nature of the α-particle was begun by Rutherford in Montreal; it was completed in Manchester by decisive experiments made in collaboration with Geiger and with Royds.

The alpha particle, with the scintillation method of detecting it, became in Rutherford's hands a most effective weapon for investigating the structure of the atom. Experiments on the "scattering" of alpha particles, made by himself and his pupils in Manchester, led him to the view, which later experiments have only served to confirm, that the atom is a very open electronic structure containing at its centre a minute positively charged nucleus in which most of the mass of the atom is concentrated. For many years Rutherford and his collaborators in Manchester and Cambridge continued to use the alpha particle as a projectile for gaining information about the interior of the atom.

It was by using α-particles as projectiles to bombard the nucleus itself that what is perhaps Rutherford's most famous achievement was accomplished—the first artificial transmutation of an element, nitrogen. This was in 1919; a few years later in collaboration with Chadwick he succeeded in disintegrating twelve light elements—in each case with ejection of a proton.

The rapid advances in nuclear physics and chemistry made in recent years, largely due to the use of more effective projectiles than the alpha particles—neutrons, fast protons, and deuterons—owe much directly and indirectly to Rutherford. A very important share of the work has been done in the Cavendish Laboratory.

The great part which Rutherford's work has played in the development of modern Physics and Chemistry has been universally recognised, and it would be impossible to enumerate here all the honours which came to him. He bore them easily.

He was elected a Fellow of the Royal Society in 1903 and was awarded the Rumford medal in 1905, the Copley medal in 1922; he received the Nobel prize for Chemistry in 1908, the Franklin medal in 1924. A knighthood was conferred on him in 1914, the O.M. in 1925; he became Baron of Nelson and Cambridge in 1931.

Rutherford was a man of immense vigour; in spite of his great output of scientific research he had superabundant energy to expend in other ways. The training and supervision of the large number of research students working in his laboratory was always one of his main interests. In this work he had latterly able assistance from younger colleagues, most of whom had received their training in research under him. He

took his full share in University business. He was Professor of Natural Philosophy at the Royal Institution, and also found time to give lectures to many scientific and popular audiences which were always original and stimulating. He was President of the British Association in 1923, of the Royal Society from 1925 to 1930. For the last seven years he was Chairman of the Advisory Council of the Department of Scientific and Industrial Research. On any committee on which he served he was an efficient and forceful member. He did a great service to Science by his three books relating to radio-active substances and their radiations, the latest of them being written in collaboration with Ellis and Chadwick.

There was little evidence of any falling off in Rutherford's powers with advancing years. He died after only a few days' illness on October 19, 1937; his ashes were buried in Westminster Abbey near the graves of Newton, Kelvin, and Darwin.

Rutherford married in 1900 Mary G. Newton, only daughter of Arthur and Mary de Renzy Newton of Christchurch, New Zealand. They had one daughter, who became the wife of Professor R. H. Fowler, F.R.S. She died in 1930, leaving two sons and two daughters.

Rutherford was elected an Honorary Fellow of this Society in 1921.

See also *Obituary Notices of Fellows of the Royal Society*, vol. ii, No. 6, 1938.

C. T. R. W.

Grafton Elliot Smith, Kt., M.A.(Cantab), M.D., Ch.M.(Sydney), Litt.D.(Manc.), Hon. M.D.(Adelaide, Egypt), Hon. D.Sc.(Liv., Belf., Brist.).

Sir Grafton Elliot Smith was elected an Honorary Fellow of the Society in 1934, in recognition of his valuable contributions to Biological Science, and of his outstanding position as an anatomist and anthropologist.

He was born in 1871 at Grafton, New South Wales, and received his early scientific training at the University of Sydney. At a very early period in his career he was drawn to the study of the brain, and he was fortunate in having the opportunity to obtain fresh monotreme and marsupial material, and to examine it in Professor J. T. Wilson's laboratory. This work led to new interpretations of the hippocampal formations and the cerebral commissures, and upon this foundation he raised a notable superstructure of fact and theory which revolutionised our knowledge of cerebral morphology.

Among many other things he showed how a small primordium in the reptilian brain, situated between the pyramidal lobe and the hippocampal region of the archencephalon, in which the sense of smell is alone represented, expanded in the mammals into the grey cortex or the neopallium as he termed it. In this the other senses came to be represented and were definitely located, while the motor function was taken over from the primitive corpus striatum by another region which became the psychomotor area. He traced the history of the areas through the mammals, and showed how in the Primates there was a great expansion of the visual area, as the sense of sight, and the power of stereoscopic vision came to dominate the instinctive life of the animal instead of the sense of smell as in lower mammals.

He migrated to Cambridge from Australia in 1895, where the fundamental importance of his researches was at once recognised, and the official seal set upon them by his election to the Fellowship of the Royal Society in 1907. In the year 1900 he went to Cairo as Professor of Anatomy. There and during vacations in England his morphological researches were ardently pursued, but he also, with characteristic insight and energy, exploited the great mass of anatomical material yielded by the exploration of the tombs. Further, his imagination was caught and held by what Egypt had to tell of the history of mankind in the remote past.

He returned to England in 1909 as Professor of Anatomy at Man-

chester. Here his interests expanded to embrace some of the root problems of cultural anthropology. The problem of the origin and spread of culture over the globe was added to those of pure morphology, and he began the series of studies which occupied much of his time and energy in later years. He became the chief supporter of the "Diffusion" theory in this country. For the anatomist his work on the brain will form his enduring monument—and it held him all his life. He had great fertility of ideas, and new facts wherever culled, wrought on by his suggestive mind, were fitted with sure instinct into the framework of his own researches. His writings and discourses on cultural anthropology were perhaps better known to the general public, but both professional and lay readers much appreciated his contributions to the story of human evolution, and his fascination as lecturer and writer received wide recognition. His presentation of the subject was firmly based, of course, on his studies on the comparative anatomy of the brain, and at the human level came in his work on fossil human crania. The cast of the interior of the brain-case to the discerning eye yields evidences of the relative development of areas of the brain known to subserve special activities. Elliot Smith was able to assess, not only the morphological points of these rare specimens, which indicate the place of each in the human phylum, but he was also able to give hints as to the mental life of the early progenitors of *Homo sapiens*. It may be noted that during the Great War Elliot Smith gave valuable services to the victims of the strife by bringing his knowledge of neurology and psychology to the care and cure of soldiers suffering from head injuries or shell-shock.

He received many honours, among them the award of a Royal Medal by the Council of the Royal Society in 1912. He was Vice-President of the Society in 1913–14. He received the honour of Knighthood in 1934

He died on January 1, 1937.

See also *Obituary Notices of Fellows of the Royal Society*, vol. ii, No. 6, 1938.

T. H. B.

William Morton Wheeler, Hon.F.R.S.E.

In the passing of William Morton Wheeler, Emeritus Professor of Entomology at the University of Harvard, the world of science in general, and entomology in particular, has lost one of her greatest men, a man honoured in his own country, and well known and admired for the excellence of his work far beyond the confines of the United States Wheeler was born at Milwaukee, State of Wisconsin, on March 19, 1865, and died suddenly at Cambridge, Massachusetts, on April 19, 1937.

Wheeler's early schooling was obtained at the German-American Normal School, Milwaukee, and later at Engelmann's German Academy. As a boy he showed an interest in natural history, in collections in museums—later he was a great collector himself—in nature study and open-air sights and sounds.

For a short time he was engaged in routine museum work at Rochester, U.S.A., arranging and cataloguing the collections, invertebrate and vertebrate, but resigned this work on his appointment in 1865 as Teacher of German and Physiology at Milwaukee High School. Here the Principal was Dr G. W. Peckham, whose wasp studies are well known. Peckham was certainly an influence in giving Wheeler a bias towards the study of insects. Near Milwaukee High School was the Allis Laboratory under the directorship of Professor C. O. Whitman, who was another influence in Wheeler's love for natural history. One of Whitman's assistants, Dr William Patten, gave Wheeler instruction in research methods and turned him on to work at insect embryology. This work was to bear fruit later on when Wheeler, working with a Scholarship at Clark University, graduated Ph.D. in 1892, with a thesis on The Embryology of Insects.

Before this, however, from 1887 to 1890, Wheeler had held the post of Curator of the Milwaukee Public Museum.

In 1893–94 Wheeler visited Europe to work at Wurzburg under Boveri, an inspiring teacher, famous at the time for his work on the cell, then at Liège, and then to occupy one of the tables at the Naples Zoological Station. A friend has described to me his attitude of wonder when Wheeler demonstrated to him under the microscope the fertilisation of the eggs of an echinoderm.

Wheeler returned to the United States as Assistant-Professor of

Embryology in the University of Chicago, leaving this post in 1899 on his appointment to the University of Texas.

In 1903 Wheeler was appointed Curator of Invertebrate Zoology at the American Museum of Natural History, New York, and in 1908 he joined the staff of Harvard University as Professor of Entomology, later also acting as Curator of Insects at the Museum of Comparative Zoology. During Professor Wheeler's tenure of the Chair at Harvard University he acted from 1915 to 1929 as Dean of the Bussey Institution, a Graduate Research School of the University in Applied Zoology.

While at Harvard, Wheeler, by invitation, acted as Exchange Professor in the University of Paris, his proficiency in French and German making the task easy and grateful to him.

Professor Wheeler's research work—there is a list of over 460 published papers—covers a wide field in zoology (taxonomic, morphological, embryological), while he wrote with pronounced opinions on evolution, but his latest work was devoted to insects and especially ants. His interest in ants, originally stimulated while at the University of Texas, grew with the years and took him in their study not only over North and South America but all over the world. I have had the pleasure of enjoying Wheeler's leadership in a field excursion, and an intimate friend of Wheeler who was his companion in many an ant foray in widely separated parts of the world wrote to me describing Wheeler's insatiable curiosity and constant enthusiasm. Companions had need of enthusiasm too, for the memory of wounds from Eciton ants lingered long, or, as my friend put it: "I never knew what ants could be until Wheeler led us into a tangle of inhabited (*i.e.* with their bodyguard of ants in position) Bull Horn Acacias." Wheeler was always finding things. Once the guest of a close friend in South Florida, Wheeler's host looked in to greet him on the first morning after his arrival. Wheeler had already discovered the nest of some mushroom-growing ants.

Wheeler's field-work and observations on ants in every phase of their often extraordinary life-history and habits, and his interest in other social insects, led naturally to studies in ecology and sociology and comparative psychology. Hence the authority of his scientific writings with their wealth of illustration and width of interpretation.

Among Wheeler's published works may be named *Ants, their Structure, Development and Behaviour* (1910) in the Columbia University Biological Series; *Social Insects: their Origin and Evolution*; *Foibles of Insects and Men*; *Demons of the Dust: A Study of Insect Behaviour*; *Colony Founding among Ants*. One of the chapters in *Foibles of Insects and Men* consists of an address (also published in *Science*, vol. lvii,

January 19, 1923) entitled "The Dry Rot of our Academic Biology." Apart from its relation to the teaching of biology—and everybody is interested in biology just now—the address is worth reading as an example of Wheeler's power with the pen. In this address, Wheeler defines his own attitude to research and to teaching, with a great preference for research. But his reference to his oversight of his students in training is too modest, for many graduates in prominent posts to-day bear willing testimony to what they owe to Wheeler for his sympathetic bearing and help in his laboratory.

A profound scholar, a capable linguist, steeped in the literature and philosophy of West and East, a world traveller with a wonderful record of work, Wheeler had every temptation to "wear the nodding plumes of intellectual conceit," but he remained a modest, human man, practising at home an easy natural hospitality, enjoying the humour and the stories of his friends and ready to tell a good story himself. Wheeler's work will continue to live, but his friends and colleagues will miss him much.

Professor Wheeler received an Honorary Doctorate from the Universities of Chicago (1916), California (1928), Harvard (1930), Columbia (1933). Among other honours he was made an Officer in the Legion of Honour. The Royal Entomological Society of London, the Entomological Society of France, and the Entomological Society of Belgium elected him as an Honorary Member.

He was elected an Honorary Fellow of the Royal Society of Edinburgh in 1936.

R. S. M.

WILLIAM BRODIE BRODIE, M.D.(Glas.), graduated at the University of Glasgow, and later held the following appointments: Muirhead Demonstrator in Physiology and Senior Assistant to the Professor of Physiology at the University of Glasgow; House Surgeon, General Infirmary, Worcester; and House Physician and House Surgeon, Western Infirmary, Glasgow. Dr Brodie was the author of two papers in the Society's *Proceedings* (1900, 1902), and other publications.

He was elected a Fellow of this Society in 1901, and died on August 29, 1937, at Bletchingley, Surrey, aged sixty-six years.

ROBERT CRAIG COWAN, J.P., B.A.(Cantab.), Paper Manufacturer, Penicuik, was born in 1865. He was educated at Loretto School and Pembroke College, Cambridge, where he took his B.A. in chemistry. Mr Cowan was greatly interested in botany, geology and natural history generally, and was a keen shot and angler. He was a director of the Lothian Coal Co., Ltd., the Life Association of Scotland, and served on the Board of Management of the Edinburgh Hospital for Sick Children, the Lord Roberts' Memorial Workshops, the Royal Edinburgh Hospital for Mental and Nervous Disorders, and other institutions. He was a Governor of Loretto School, and latterly a Vice-President of the Royal Physical Society, Edinburgh.

He was elected a Fellow of this Society in 1936, and died on March 22, 1937.

JAMES KNIGHT, J.P., M.A., D.Sc.(Glas.), F.G.S., formerly head of Queen's Park Higher Grade School, Glasgow, was born at Glasgow in 1860. He prepared for the teaching profession and had a distinguished career as a student at the University of Glasgow. Dr Knight was a prominent member of the Royal Philosophical Society of Glasgow, and from 1907 was Honorary Librarian. In recent years he was President of its Geographical Section and served as a Vice-President. He was greatly interested in astronomy, and was the author of books and pamphlets on various subjects.

He was elected a Fellow of this Society in 1907 and died on December 1, 1936.

JAMES ALEXANDER MACDONALD, M.A., B.Sc., formerly H.M. Chief Inspector of Schools for the Highland Division, graduated at the University of Edinburgh. He was the first Rector of Leith Academy, having been appointed in 1897.

He was elected a Fellow of this Society in 1897 and was the author of a mathematical paper in its *Transactions* (1897). He died on June 7, 1937, in his 71st year.

JOHN SMITH PURDY, D.S.O., M.D., C.M.(Aberd.), D.P.H.(Cantab.), Medical Officer of Health, Combined Metropolitan Sanitary Districts and City of Sydney, was educated at the University of Aberdeen and St Bartholomew's Hospital. Dr Purdy was the author of a text-book on *Australian Hygiene and Public Health* and other publications.

He was elected a Fellow of this Society in 1911, and died on July 26, 1936.

WILLIAM RAMSAY SMITH, J.P., M.D.(Edin.), C.M., D.Sc.(Adelaide), Permanent Head of the Department of Public Health, South Australia (1899–1929), and City Coroner of Adelaide, was born at King Edward, Aberdeenshire, in 1859. He was educated at Moray House Training College and at the University of Edinburgh. Before proceeding to Australia he was Assistant Professor of Natural History and Senior Demonstrator in Zoology in the University of Edinburgh; Demonstrator of Anatomy, Edinburgh School of Medicine; and Examiner for the Royal College of Physicians, Edinburgh. He rendered valuable service in several important appointments in South Australia—among others he was Chairman of the Advisory Committee on Food and Drugs (1909–29) and Principal Medical Officer to the Commonwealth Military Forces in South Australia (1903). Dr Ramsay Smith also had distinguished medical records in the South African War and in the Great War. For many years he was engaged as medico-legal expert for the Crown in almost every important criminal trial in South Australia. He was an Honorary Member of the Association of Military Surgeons of the United States and a Member of the Royal Commission on Lunacy (1909). He was the author of *A Manual for Coroners* (1904), *Medical Jurisprudence from the Judicial Standpoint* (1913), *Myths and Legends of the Australian Aboriginals* (1930), and other works.

He was elected a Fellow of this Society in 1907, and wrote two papers to its *Proceedings* (1907, 1908). He died on September 28, 1937.

APPENDIX.

CONTENTS.

	PAGE
PROCEEDINGS OF THE STATUTORY GENERAL MEETING, OCTOBER 1936	439
PROCEEDINGS OF THE ORDINARY MEETINGS, SESSION 1936–1937	442
PROCEEDINGS OF THE STATUTORY GENERAL MEETING, OCTOBER 1937	447
THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING VICTORIA JUBILEE, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND	451
AWARDS	455
ACCOUNTS OF THE SOCIETY, SESSION 1936–1937	463
VOLUNTARY CONTRIBUTORS UNDER LAW VI (END OF PARA. 3)	471
THE COUNCIL OF THE SOCIETY AT OCTOBER 25, 1937	472
FELLOWS OF THE SOCIETY AT OCTOBER 25, 1937	473
HONORARY FELLOWS OF THE SOCIETY AT OCTOBER 25, 1937	501
CHANGES IN FELLOWSHIP DURING SESSION 1936–1937	503
FELLOWS OF THE SOCIETY ELECTED DURING SESSION 1936–1937	503
LAWS OF THE SOCIETY	504
ADDITIONS TO THE LIBRARY—PRESENTATIONS, ETC.—1936–1937	512
INDEX	517
INDEX, UNDER AUTHORS' NAMES, OF PAPERS PUBLISHED IN "TRANSACTIONS"	520

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Beginning the 154th Session, 1936–1937.

At the Statutory General Meeting, held in the Society's Rooms, 24 George Street, on Monday, October 26, 1936, at 4.30 P.M.

Principal O. Charnock Bradley, M.D., D.Sc., Vice-President, in the Chair.

The Minutes of the Statutory Meeting held on October 28, 1935, were read, approved, and signed.

Professor THOMAS ALTY, the Rt. Hon. THOMAS MACKAY COOPER, and the Very Rev. Dr CHARLES LAING WARR signed the Roll and were admitted to the Society.

The President nominated as Scrutineers, for the election of Office-Bearers and Council, Dr E. A. BAKER and Mr T. H. GILLESPIE.

The Ballot was then taken.

The General Secretary submitted the following report:—

GENERAL SECRETARY'S REPORT, OCTOBER 26, 1936.

The following Address of condolence and loyalty was forwarded through the Secretary of State for Scotland to His Majesty King Edward VIII:—

TO THE
KING'S MOST EXCELLENT MAJESTY

May it please Your Majesty,

We, President, Council, and Fellows of the Royal Society of Edinburgh, beg leave to tell Your Majesty of our deep and heartfelt sorrow, such as all peoples of these realms and all good people of the world do feel for the death of our late beloved Sovereign King George V, of happy and most blessed memory.

We lay our humble and loyal duty before Your Majesty. We beg Your Majesty's favouring countenance and protection in all the work we have to do. We wish Your Majesty health and strength and length of days and world-wide peace and lifelong happiness. And we pray God send his richest blessings on our King.

(*Signed*) D'ARCY W. THOMPSON, *President.*
JAMES KENDALL, *Secretary.*

February 10, 1936.

The following letter was received from the Secretary of State for Scotland:—

"SCOTTISH OFFICE,
WHITEHALL,
4th March 1936.

"Sir,

I have had the honour to lay before The King the loyal and dutiful Address of the Royal Society of Edinburgh on the occasion of the lamented death of His Late Majesty King George V, and have received the King's Commands to convey to you His Majesty's grateful thanks for the assurances of sympathy and devotion to which it gives expression.

I am, Sir,
Your obedient Servant,
(*Signed*) GODFREY P. COLLINS."

The Society was represented by eight office-bearers and members of Council at the Intercessory and Memorial Services for His Late Majesty King George V, in St Giles' Cathedral, on January 22 and 28, 1936, respectively.

His Majesty King Edward VIII has graciously granted His Patronage to the Society in succession to His Late Majesty King George V.

During the session four Addresses have been given to the Society by request of the Council: on October 28, 1935, by Professor J. H. ASHWORTH, on "Charles Darwin as a Student in Edinburgh, 1825 to 1827"; on November 4, 1935, by Professor A. D. PEACOCK, on "Some Aspects of Animal Parthenogenesis"; on February 3, 1936, by Professor H. S. JENNINGS, Johns Hopkins University, Baltimore, on "Inheritance in Protozoa"; and on May 4, 1936, by Professor STEFAN JELLINEK, University of Vienna, on "The Theory of Electrical Traces." Professor ASHWORTH's address was published in the *Proceedings* of the Society, vol. 55, 1935, pp. 97–113.

21 papers were read, as compared with 22 in the previous session. The papers were in the following subjects:—Mathematics, 3; Physics, 3; Geology, 4; Palæobotany, 1; Botany, 1; Zoology, 3; Animal Genetics, 4; Social Biology, 1; and Physiology, 1. 6 papers have been published in the *Transactions* and 15 in the *Proceedings*. 8 papers have been withdrawn, and several papers are still under the consideration of the Council.

The Society has lost by death 21 Fellows and 6 Honorary Fellows. 33 Fellows and 6 Foreign and 3 British Honorary Fellows were elected.

Since the death of Professor J. H. ASHWORTH, on February 4, 1936, the duties of General Secretary have been carried on jointly by Professor F. A. E. CREW and Professor JAMES KENDALL.

Invitations were received and the Society was represented as follows on the occasions mentioned:—

1. Jubilé Scientifique de Louis Lumiere. Asnières, November 6, 1935. Letter sent.
2. 15ème Anniversaire de l'existence de L'Institut Biologique de Peterhof, July 12, 1935. Letter sent.
3. Jubilé Scientifique de M. Jacques Hadamard, Paris, January 7, 1936. Letter sent.
4. Lord High Commissioner's Levee in the Palace of Holyroodhouse, May 20, 1936. President and six members of Council.
5. 300th Anniversary of the Royal University of Utrecht, June 22–24, 1936. The President attended and presented an Address.
6. Botanical Society of Edinburgh. Centenary, July 1, 1936. Principal O. C. BRADLEY.
7. 22nd Annual Conference of the National Association for the Prevention of Tuberculosis. London, July 16–18, 1936. Sir ROBERT PHILIP.
8. International Congress of Mathematicians. Oslo, July 13–18, 1936. Professor E. T. COPSON and Dr H. S. RUSE.
9. Congrès Internationale de Linguistes. Copenhagen, August 26 to September 1, 1936. Letter sent.
10. Tercentenary of Harvard University, Cambridge, Mass., U.S.A., September 16–18, 1936. Professor E. B. BAILEY attended and presented an Address.

The Society entertained the officials of the Assembly of the International Union of Geodesy and Geophysics to tea in the Rooms of the Society on Friday, September 18, 1936; and on Monday, September 21, 1936, the Rooms were opened to officials, delegates, and guests of the Assembly, and tea was served from 4.30 P.M. On Monday afternoon the ladies of the Assembly had a motor tour round Edinburgh provided for them jointly by this Society and the Royal Scottish Geographical Society. At the Reception on September 21, Sir WILLIAM HENRY BRAGG, Professor V. F. K. BJERKNES, Professor E. G. CONKLIN, and Dr HELLAND-HANSEN signed the Roll of the Society's Honorary Fellows.

The Society's congratulations were conveyed to Sir JAMES CRICHTON-BROWNE on the attainment of his 95th year. He was elected a Fellow in 1870, and is the senior Fellow of the Society.

Professor D. FRASER HARRIS was appointed to continue to serve for 1936, as the representative of the Society on the Committee of the Eugenics Society, London, dealing with the Leonard Darwin Studentships in Eugenics.

Emeritus Professor J. GRAHAM KERR and Professor JAMES RITCHIE were appointed to serve as representatives of the Society on the British Section of the International Committee for Bird Preservation.

During the session the sum of £250 has been expended on the binding of books in the Library.

Framed enlargements of photographs of our former President, Sir E. A. SHARPEY-SCHAFER, and our late General Secretary, Professor J. H. ASHWORTH, are now hung in the Rooms of the Society.

A considerable amount of painting has been done during the session in the West Staircase, Lecture Room, lavatories, and kitchen, and a new central heating boiler has been installed. A new carpet for the West Staircase is being provided.

The DAVID ANDERSON-BERRY PRIZE (1935) was awarded to CHARLES MELVILLE SCOTT, M.A., M.B., D.Sc., for his essay "On the Action of X- and Gamma-Rays on Living Cells."

The KEITH PRIZE for the period 1933 to 1935 was awarded to Professor LANCELOT T. HOGBEN, D.Sc., for his papers, published alone and in collaboration, which have appeared in the *Proceedings* of the Society during the period of the award; and the NEILL PRIZE for the period 1933 to 1935 was awarded to SAMUEL WILLIAMS, Ph.D., for his contribution to the Anatomy and Experimental Morphology of the Pteridophyta.

The BRUCE MEMORIAL PRIZE (1936) was awarded by the joint Committee of the Royal Society of Edinburgh, the Royal Physical Society, and the Royal Scottish Geographical Society to JAMES WILLIAM SLESSER MARR, M.A., B.Sc., for work in the Southern Ocean and more particularly for his monograph on the South Orkney Islands.

The acknowledgments of the Society are due to the Carnegie Trust for the Universities of Scotland for grants to authors towards the cost of tabular matter and illustrations of papers published in the *Transactions* and *Proceedings*, amounting to £189, 9s. 3d.; to the Department of Oceanography, University of Liverpool, for £4, 16s. 6d. towards the cost of the paper by Miss E. LOWE in the *Transactions*; to Dr THOMAS NICOL, University of Glasgow, for £23, 8s. towards the cost of his paper in the *Transactions*; to the Colston Research Fund, University of Bristol, for £10 toward the cost of the paper by Mr B. J. MARPLES in the *Transactions*; and for a sum of £250 received from the Royal Society of London, from the Government Publication Grant, in aid of the cost of the Society's publications for the session 1935–1936.

TREASURER'S REPORT:—

The TREASURER went over in a general way the items of income and expenditure and indicated that the financial position was satisfactory.

The Rev. Dr MITCHELL HUNTER moved the adoption of the Reports, and the reappointment of Messrs LINDSAY, JAMIESON and HALDANE, C.A., as auditors for the ensuing session. These motions were approved.

The Scrutineers reported that the Ballot Papers were in order, and the CHAIRMAN declared that the following Office-Bearers and Council had been duly elected:—

Professor D'ARCY W. THOMPSON, C.B., D.Litt., Hon. D.Sc., LL.D., F.R.S., President.

Principal O. CHARNOCK BRADLEY, M.D., D.Sc. Professor P. T. HERRING, M.D., F.R.C.P.E. The Most Hon. THE MARQUIS OF LINLITHGOW, P.C., K.T., G.C.I.E., D.L. Professor E. B. BAILEY, M.C., M.A., F.R.S. Professor F. A. E. CREW, M.D., D.Sc., Ph.D. Lt.-Col. A. G. M'KENDRICK, M.B., D.Sc., F.R.C.P.E.	Vice-Presidents.

Professor JAMES P. KENDALL, M.A., D.Sc., F.R.S., General Secretary.

ALEXANDER C. AITKEN, M.A., D.Sc., F.R.S. CHARLES H. O'DONOGHUE, D.Sc.	Secretaries to Ordinary Meetings.

JAMES WATT, W.S., LL.D., Treasurer.

LEONARD DOBBIN, Ph.D., Curator of Library and Museum.

MEMBERS OF COUNCIL.

Principal J. C. SMAIL, O.B.E., Companion Inst.E.E.
Sir HAROLD J. STILES, K.B.E., M.B., F.R.C.S.E., LL.D.
Professor JOHN WALTON, M.A., D.Sc.
EDWIN BRAMWELL, M.D., LL.D., F.R.C.P.E.
Emeritus Professor T. H. BRYCE, M.A., M.D., LL.D., F.R.S.
Professor I. DE BURGH DALY, M.A., M.D., B.Ch.
JOHN ALEXANDER INGLIS, K.C., M.A., LL.B.
Professor A. D. PEACOCK, D.Sc.
JOHN E. MACKENZIE, D.Sc.
Professor SYDNEY SMITH, M.D., F.R.C.P., D.P.H.
Professor RALPH STOCKMAN, M.D., LL.D., F.R.C.P.E.
E. MACLAGAN WEDDERBURN, O.B.E., M.A., D.Sc., W.S.

The CHAIRMAN thanked the Scrutineers for their services.

The CHAIRMAN intimated that the President was in the United States delivering the Harvard University Lowell Lectures. In the course of his remarks he gave a short review of the session's activities, and referred to the death of His Majesty King George V, the Patron of the Society. Reference was also made to the loss the Society had sustained by the deaths of distinguished Fellows, including the General Secretary, Professor J. H. ASHWORTH, F.R.S. The CHAIRMAN warmly thanked the retiring Office-Bearers and Members of Council for their services, and formally opened the session.

PROCEEDINGS OF THE ORDINARY MEETINGS, Session 1936–37.

FIRST ORDINARY MEETING.

Monday, November 2, 1936.

Professor F. A. E. Crew, M.D., D.Sc., Ph.D., Vice-President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read and signed.

Dr S. G. GIBBONS signed the Roll and was admitted to the Fellowship of the Society by the President.

The following Communications were submitted:—

1. Perthshire Tectonics: Schiehallion to Glen Lyon. By Professor E. B. BAILEY, M.C., F.R.S., and W. J. McCALLIEN, D.Sc. *Trans.*, vol. 59, pp. 79–117.

Read by title:—

2. On the Feeding Mechanism of *Apseudes talpa*, and the Evolution of the Peracaridan Feeding Mechanisms. By RALPH DENNELL, M.Sc. Communicated by Professor H. GRAHAM CANNON, F.R.S. *Trans.*, vol. 59, pp. 57–78.
3. A Study of the Vascular Supply to the Carpels in the Follicle-bearing Ranunculaceæ. By MABEL S. FRASER, Ph.D. Communicated by Sir WILLIAM WRIGHT SMITH, M.A. *Trans.*, vol. 59, pp. 1–56.
4. The Gravitational Field of a Distribution of Particles rotating about an Axis of Symmetry. By W. J. VAN STOCKUM. Communicated by Professor E. T. WHITTAKER, F.R.S. *Proc.*, vol. 57, pp. 135–154.
5. Some Formulæ for the Associated Legendre Functions of the Second Kind, with Corresponding Formulæ for the Bessel Functions. By Professor T. M. MACROBERT, D.Sc. *Proc.*, vol. 57, pp. 19–25.

SECOND ORDINARY MEETING.

Monday, December 2, 1936.

Professor E. B. Bailey, M.C., M.A., F.R.S., Vice-President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following Communications were submitted:—

1. Microphthalmia and other Eye-defects throughout Fourteen Generations of Albino Rats. By A. M. HAIN, M.A., Ph.D. Communicated by Professor F. A. E. CREW, M.D. *Proc.*, vol. 57, pp. 64–77.
2. Ovarian Rhythm in *Drosophila*. By H. P. DONALD, Ph.D., and Miss ROWENA LAMY. Communicated by Professor F. A. E. CREW, M.D. *Proc.*, vol. 57, pp. 78–96.
3. Studies in Clocks and Time-keeping. No. 6. The Arc Equation. By Professor R. A. SAMPSON, F.R.S. *Proc.*, vol. 57, pp. 55–63.

Read by Title:—

4. Quantitative Evolution in Compositæ. By Professor J. SMALL, D.Sc., and Miss I. K. JOHNSTON. *Proc.*, vol. 57, pp. 26–54.

THIRD ORDINARY MEETING.

Monday, January 11, 1937.

Professor D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Dr R. S. CLARK signed the Roll and was admitted to the Fellowship of the Society.

At the request of the Council an Address was delivered by Dr F. FRASER DARLING on "Observations on Animal Sociality."

Read by Title:—

On the Geometry of Dirac's Equations and their Expression in Tensor Form. By Professor H. S. RUSE, M.A., D.Sc. *Proc.*, vol. 57, pp. 97–127.

FOURTH ORDINARY MEETING.

Monday, February 1, 1937.

Professor D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The Secretary read the Loyal Address to be sent to His Majesty King George VI, on behalf of the President and Fellows of the Society (see p. 447).

Dr JOHN BERRY, Dr C. H. KEMBALL and Dr WILLIAM MCRAE signed the Roll and were admitted to the Fellowship of the Society.

At the request of the Council an Address was delivered by Professor JAMES KENDALL, M.A., D.Sc., F.R.S., on "Ions and Isotopes." *Proc.*, vol. 57, pp. 182–193.

Read by Title:—

On the Ciliary Currents on the Gills of some Tellinacea. By ALASTAIR GRAHAM, M.A., B.Sc. *Proc.*, vol. 57, pp. 128–134.

FIFTH ORDINARY MEETING.

Monday, March 1, 1937.

Professor D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Mr A. D. STEWART signed the Roll and was admitted to the Fellowship of the Society.

The Ballot for the election of Ordinary Fellows then took place. The Chairman nominated Mr HUGH WYLIE BROWN and Mr J. L. SOMERVILLE as Scrutineers. The following gentlemen were duly elected:—

ADAM JOHN GUILBERT BARNETT, KALIPADA BISWAS, JOSEPH JOHN BLACKIE, MAX BORN, GEORGE BERNARD BROOK, ARCHIBALD GRAY ROBERTSON BROWN, CHARLES JOHNSTONE COUSLAND, WILLIAM STUART MCRAE CRAIG, SASINDRA CHANDRA DHAR, HUGH PATERSON DONALD, DERRICK MELVILLE DUNLOP, ROBERT CAMPBELL GARRY, ANDREW RAE GILCHRIST, GEORGE GREEN, WILLIAM ALLAN FORSYTH HEPBURN, PERCIVAL ROBSON KIRBY, PEO CHARLES KOLLER, WILLIAM ORR LEITCH, HARRY WORK MELVILLE, JAMES MILLER, JOHN MORISON, ROY NASMITH, THOMAS T. PATERSON, JOHN DONALD POLLOCK, BADRI NARAYAN PRASAD, FRANCIS IAN GREGORY RAWLINS, MOWBRAY RITCHIE, JOHN WATSON ROBERTSON, The Hon. LORD ROBERTSON, WILLIAM RITCHIE RUSSELL, JAMES KIRKWOOD SLATER, ALEXANDER MARTIN SMITH, HORACE GEORGE SMITH, HENRY MARSHALL STEVEN, The Rt. Hon. THE EARL OF SUFFOLK AND BERKSHIRE, JOHN GUTHRIE TAIT, HENRY TOD, MATHEW MCKERROW TURNBULL, EDWARD MAITLAND WRIGHT, ANDREW WHITE YOUNG.

The following Communications were submitted:—

1. The Time Lag of the Vacuum Photo-Cell. By R. A. HOUSTOUN, M.A., D.Sc. *Proc.*, vol. 57, pp. 163–171.

2. Tests for Randomness in a Series of Numerical Observations. By W. O. KERMACK, M.A., D.Sc., and Lt.-Col. A. G. MCKENDRICK, M.B., D.Sc. *Proc.*, vol. 57, pp. 228–240.

3. Production of Large Broods in Certain Lamellibranchs in relation to Weather Conditions. By A. C. STEPHEN, D.Sc.

4. The Genetical and Mechanical Properties of Sex Chromosomes. Part III.—Man. By P. C. KOLLER, D.Sc. Communicated by Professor F. A. E. CREW, M.D. *Proc.*, vol. 57, pp. 194–214.

Read by Title:—

5. The Structure and Function of the Alimentary Canal of some Species of Polyplacophora (Mollusca). By VERA FRETTER, Ph.D. Communicated by Professor H. GRAHAM CANNON, F.R.S. *Trans.*, vol. 59, pp. 119–164.

6. The Revised Complete System of a Quadratic Complex. By Professor H. W. TURNBULL, F.R.S. *Proc.*, vol. 57, pp. 155–162.

7. Studies in Practical Mathematics. I.—The Evaluation, with Applications, of a Certain Triple Product Matrix. By A. C. AITKEN, D.Sc., F.R.S. *Proc.*, vol. 57, pp. 172–181.

SIXTH ORDINARY MEETING.

Monday, May 3, 1937.

Professor D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following signed the Roll and were admitted to the Fellowship of the Society: Professor MAX BORN, Mr GEORGE BERNARD BROOK, Mr ARCHIBALD GRAY ROBERTSON BROWN, Mr CHARLES JOHNSTONE COUSLAND, Professor ROBERT CAMPBELL GARRY, Mr WILLIAM ORR LEITCH, Mr ROY NASMITH, Dr JOHN DONALD POLLOCK, Dr WILLIAM RITCHIE RUSSELL, Dr ALEXANDER MARTIN SMITH, Dr HENRY MARSHALL STEVEN, The Rt. Hon. THE EARL OF SUFFOLK AND BERKSHIRE, Dr HENRY TOD, Mr MATHEW MCKERROW TURNBULL and Mr ANDREW WHITE YOUNG.

It was intimated that the following names had been proposed by the Council for Honorary Fellowship:—

FOREIGN.

C. U. ARIËNS KAPPERS, Director of the Central Institute of Brain Research, Amsterdam, and Professor of Comparative Neurology, University, Amsterdam.

MARSTON TAYLOR BOGERT, Professor of Organic Chemistry, Columbia University, City of New York.

MAX PLANCK, Nobel Laureate, Physics, 1918, Professor Ordinarius emeritus of Theoretical Physics, Director of the Institute for Theoretical Physics, University of Berlin.

BRITISH.

WILLIAM THOMAS CALMAN, C.B., F.R.S., lately Keeper of Zoology, British Museum. 24 Lexham Gardens, London, W. 8.

JOHN LOGIE BAIRD, Inventor of the Televisor, 3 Crescent Wood Road, Sydenham, London, S.E.

It was intimated that the Council had awarded the GUNNING VICTORIA JUBILEE PRIZE for the period 1932 to 1936 to Professor CHARLES GALTON DARWIN, F.R.S., Master of Christ's College, Cambridge, formerly Tait Professor of Natural Philosophy in the University of Edinburgh, for his distinguished contributions in Mathematical Physics, and the MAKDOUGALL-BRISBANE PRIZE for the period 1934 to 1936 to Dr ERNEST MASSON ANDERSON, formerly of H.M. Geological Survey (Scotland), for his paper "The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences," published in the Society's *Proceedings* within the period of the award.

It was intimated that, at the request of the Council, and in terms of the BRUCE-PRELLER LECTURE FUND, Professor HUGH STOTT TAYLOR, D.Sc., F.R.S., Department of Physical Chemistry, Princeton University, N.J., U.S.A., would deliver an Address on "Heavy Hydrogen in Scientific Research" at the Ordinary Meeting of the Society on June 7, 1937.

The following Communications were submitted:—

1. Metamorphic Correlation in the Polymetamorphic Rocks of the Valla Field Block, Unst, Shetland Islands. By Professor H. H. READ, D.Sc., A.R.C.Sc. *Trans.*, vol. 59, pp. 195–221.
2. Quantitative Evolution. II.—Compositæ Dp-ages in Relation to Time. III.—Dp-ages of Gramineæ. By Professor J. SMALL, D.Sc. *Proc.*, vol. 57, pp. 215–227.
3. On a New Long-headed Dipnoan Fish from the Upper Devonian of Scaumenac Bay, P.Q., Canada. By W. GRAHAM-SMITH, B.A., and T. S. WESTOLL, Ph.D. Communicated by A. C. STEPHEN, D.Sc. *Trans.*, vol. 59, pp. 241–266.

Read by Title:—

4. The Early Stages in the Development of the Ferret: The Formation of the Mesoblast and Notochord. By Professor W. J. HAMILTON, D.Sc., M.D. *Trans.*, vol. 59, pp. 165–193.
5. Some Distributions associated with a Randomly Arranged Set of Numbers. By W. O. KERMACK, M.A., D.Sc., and Lt.-Col. A. G. MCKENDRICK, M.B., D.Sc. *Proc.*, vol. 57, pp. 332–376.
6. On the Immature Stages of some Scottish and other Psyllidæ. By K. B. LAL, M.Sc., Ph.D. Communicated by A. E. CAMERON, M.A., D.Sc. *Proc.*, vol. 57, pp. 305–331.
7. The Benthic Amphipoda of the North-Western North Sea and Adjacent Waters. By D. S. RAITT, Ph.D. *Proc.*, vol. 57, pp. 241–254.

SEVENTH ORDINARY MEETING.

Monday, June 7, 1937.

Professor Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following signed the Roll and were admitted to the Fellowship of the Society: Dr H. P. DONALD, Dr P. C. KOLLER, Lt.-Col. J. MORISON, and the Hon. LORD ROBERTSON.

At the request of the Council, and in terms of the BRUCE-PRELLER LECTURE FUND, an Address was delivered by Professor HUGH STOTT TAYLOR, D.Sc., F.R.S., Chairman of Chemistry Department, Princeton University, U.S.A., on "Heavy Hydrogen in Scientific Research."

Read by Title:—

1. Studies in Practical Mathematics. II. The Evaluation of the Latent Roots and Latent Vectors of a Matrix. By A. C. AITKEN, D.Sc., F.R.S. *Proc.*, vol. 57, pp. 269–304.
2. "Spheroidal": a Mutant in *Drosophila funebris* affecting Egg Size and Shape, and Fecundity. By Professor F. A. E. CREW, M.D., D.Sc., and CHARLOTTE AUERBACH, Ph.D. *Proc.*, vol. 57, pp. 255–268.
3. Studies in the Cytology of Parthenogenetic Reproduction of *Hymenoptera Symphyta*. I. Chromosome Number and Individuality in Three Arrhenotokous Species. By FRANCIS GREENSHIELDS, Ph.D. Communicated by Professor A. D. PEACOCK, D.Sc.

EIGHTH ORDINARY MEETING.

Monday, July 5, 1937.

Principal O. Charnock Bradley, M.D., D.Sc., Vice-President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Mr F. I. G. RAWLINS signed the Roll and was admitted to the Fellowship of the Society.

Professor W. H. MCCREA and Dr THOMAS STEPHENSON were appointed Scrutineers for the election of Honorary Fellows.

The following, proposed by the Council for Honorary Fellowship, were declared unanimously elected:—

FOREIGN HONORARY FELLOWS.

C. U. ARIËNS KAPPERS, Amsterdam.
MARSTON TAYLOR BOGERT, New York.
MAX PLANCK, Berlin.

BRITISH HONORARY FELLOWS.

WILLIAM THOMAS CALMAN, London.
JOHN LOGIE BAIRD, London.

The Chairman thanked the Scrutineers for their services.

Visitors were admitted, and the presentation of the MAKDOUGALL-BRISBANE (1934–1936) and the GUNNING VICTORIA JUBILEE (1932–1936) PRIZES was made by the Vice-President:—

The MAKDOUGALL-BRISBANE PRIZE to Dr ERNEST MASSON ANDERSON, formerly of H.M. Geological Survey (Scotland), for his paper "The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences," published in the Society's *Proceedings* within the period of the award.

Principal BRADLEY made the presentation in the following terms:—

Dr E. M. ANDERSON was for many years a member of the Geological Survey and took advantage of its varied opportunities. His researches into the schistose rocks of Scotland have throughout been characterized by an interest in structure, whether large or small. Among other things they have advertised the prevalence of mineral lineation. His holiday work on Schiehallion reached a very high standard, and has greatly assisted in attacks upon the problems of neighbouring districts. The support he has given to the glacial interpretation of Schiehallion boulder bed has also attracted much attention. In the Carboniferous rocks of Ayrshire he has played a prominent part in deciphering faults, which, by affecting the sea bottom of the time, have introduced great contrasts in the thicknesses of sediments developed in adjacent districts. Lastly his mathematical analysis of the stresses involved in the production of the ring-dykes and cone-sheets, that are such characteristic features of Hebridean Tertiary igneous activity, is acknowledged as a masterpiece.

Apart from Geology he has contributed an interesting paper to our *Proceedings* (1913–14) dealing with the path of a ray of light in a rotating homogeneous and isotropic solid.

The Vice-President then presented the GUNNING VICTORIA JUBILEE PRIZE to Professor CHARLES GALTON DARWIN, F.R.S., Master of Christ's College, Cambridge, formerly Tait Professor of Natural Philosophy in the University of Edinburgh, for his distinguished contributions in Mathematical Physics.

His papers on the reflection of X-rays are now regarded as the fundamental researches on the subject. In a series of papers on statistical mechanics, written in collaboration with R. H. FOWLER, he published a new mathematical technique by which the relation of thermodynamics to statistics was placed on a satisfactory basis, and the way was opened for extensive subsequent developments.

Perhaps the most original and influential of Darwin's papers have been those concerned with quantum-mechanics. The quantum theory of the motion of an electron in an electromagnetic field, as it had been developed before 1925, gave results not in accordance with experiment. To obviate this, in 1925, the assumption was introduced that the electron has a mechanical angular momentum and a magnetic moment; but this conception did not admit of the application of wave-theory, since the waves corresponding directly to such an electron would be in space of six dimensions. It was Darwin who in February 1927 put forward the idea which led ultimately to the solution of these difficulties; he assumed that, just as there are two independent polarised components in a wave of light, so there are two independent components in the wave of an electron. He then constructed a pair of equations to represent the fine structure of the hydrogen spectrum, obtaining all the levels correctly except the *s*-levels, and showed how the equations could be expressed in vectorial form, so that, as he expressed it, the electron was a vector wave.

Among the quantum papers is also numbered the celebrated memoir on "Free Motion in Wave Mechanics," in which he took up the general question of aperiodic phenomena and dealt with it in masterly fashion, the principles of description and interpretation which are now universally accepted being formulated and applied to the free motion of electrons and atoms in various types of field.

Professor C. G. DARWIN, F.R.S., Master of Christ's College, Cambridge, was called upon to open a discussion on "The Origin of the Laws of Nature."

Professor DARWIN outlined the present position in theoretical physics with reference to recent correspondence in *Nature* regarding the theories of Milne and Eddington.

Professor W. H. MCCREA, Professor E. T. WHITTAKER, Professor MAX BORN, and Mr W. A. SINCLAIR, of the Department of Philosophy, University of Edinburgh, took part in the debate.

Read by Title:—

1. The Middle Devonian Fish Fauna of Achanarras. By C. FORSTER-COOPER, F.R.S. Communicated by Dr A. C. STEPHEN. *Trans.*, vol. 59, pp. 223–239.

2. An Histological Analysis of Eye Pigment Development in *Drosophila pseudo-obscura*. By Mrs FLORA COCHRANE. Communicated by Professor F. A. E. CREW, M.D., D.Sc. *Proc.*, vol. 57, pp. 385–399.

3. *Geonemertes dendyi* Dakin, a Land Nemertean in Wales. By A. R. WATERSTON, B.Sc., and H. E. QUICK, M.B., F.R.C.S., B.Sc. Communicated by Dr A. C. STEPHEN. *Proc.*, vol. 57, pp. 379–384.

4. On the Anatomy of *Ophelia cluthensis*. By R. S. BROWN. Communicated by Professor E. HINDLE, M.A., Sc.D., Ph.D. *Proc.*, vol. 58.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Ending the 154th Session, 1936–1937.

At the Statutory General Meeting, held in the Society's Rooms, 24 George Street, on Monday, October 25, 1937, at 4.30 P.M.

Professor Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the Statutory Meeting held on October 26, 1936, were read, approved, and signed.

The President nominated as Scrutineers, for the election of Office-Bearers and Council, Mr THOMAS ROWATT and Mr W. J. M. MENZIES.

The President, before proceeding to the Ballot, requested the Fellows, subject to their approval, to substitute the name of Dr G. W. TYRRELL on the Balloting List for that of Principal O. CHARNOCK BRADLEY, who had withdrawn his name.

The Ballot was then taken.

The General Secretary submitted the following report:—

GENERAL SECRETARY'S REPORT, OCTOBER 25, 1937.

The following Address was sent to His Majesty King George VI, on His Accession to the Throne:—

TO THE
KING'S MOST EXCELLENT MAJESTY

May it please Your Majesty,

We, the President and Fellows of the Royal Society of Edinburgh, tender to Your Majesty our loyal devotion to the Throne.

We recall with pleasure that Your Majesty as Duke of York accepted the Honorary Fellowship of our Society, and we express our gratitude that Your Majesty has been graciously pleased to continue the Sovereign's Patronage of the Society.

We earnestly hope that Your Majesty may long be spared to rule over a peaceful, prosperous, and happy nation, and that every blessing may attend Your Majesty, Her Majesty the Queen, and the Royal Princesses.

(*Signed*) D'ARCY W. THOMPSON, *President.*
JAMES KENDALL, *Secretary.*

January 1937.

An acknowledgment of the Address was received from the Secretary of State for Scotland.

The Society was represented at the Proclamation of His Majesty King George VI, at the Market Cross of Edinburgh, on December 14, 1936, by Lt.-Col. A. G. MCKENDRICK, Dr JAMES WATT, Dr A. C. AITKEN, and Professor JAMES KENDALL.

The President represented the Society at the Coronation of Their Majesties in Westminster Abbey on Wednesday, May 12, 1937.

The Society was also represented at the Coronation Service in St Giles' Cathedral on that day.

A communication was received from the Keeper of the Privy Purse stating that "His Majesty is pleased to intimate to those Societies and Institutions which were recently granted Patronage by King Edward VIII, that they may continue to show the Sovereign as their Patron during His Reign."

His Majesty was graciously pleased to sign the Roll of Fellows as an Honorary Fellow (Elected March 5, 1934), and as the Patron of the Society, on July 7, 1937, when His Majesty was in residence at the Palace of Holyroodhouse.

During the session three Addresses have been given by request of the Council: on January 11, 1937, by Dr F. FRASER DARLING, on "Observations on Animal Sociality"; on February 1, 1937, by Professor JAMES KENDALL, on "Ions and Isotopes"; and on June 7, 1937, in terms of the BRUCE-PRELLER LECTURE FUND, by Professor H. S. TAYLOR, on "Heavy Hydrogen in Scientific Research." On July 5, 1937, Professor C. G. DARWIN opened a discussion on "The

Origin of the Laws of Nature," which was taken part in by Professor W. H. McCREA, Professor E. T. WHITTAKER, Professor MAX BORN, and Mr W. A. SINCLAIR, Department of Philosophy, University of Edinburgh. Professor KENDALL'S Address was published in the *Proceedings* of the Society, vol. 57, 1937, pp. 182–193.

33 papers were read, as compared with 21 in the previous session. The papers were in the following subjects: Mathematics, 8; Physics, 2; Geology, 2; Palæontology, 2; Botany, 4; Zoology, 8; Animal Genetics, 6; and Anatomy, 1. 24 papers have been published in the *Proceedings* and 8 in the *Transactions*. 7 papers have been withdrawn, and several papers are still under the consideration of the Council.

The Society has lost by death 24 Fellows and 6 Honorary Fellows. 40 Fellows and 3 Foreign and 2 British Honorary Fellows were elected.

Invitations were received and the Society was or will be represented as follows on the occasions mentioned:—

1. Centenary of the Army Medical Library, Washington, November 16, 1936. Lt.-Col. EDGAR ERSKINE HUME.
2. 150th Anniversary of the College of Physicians of Philadelphia, May 14, 1937. Letter sent.
3. Lord High Commissioner's Levee at the Palace of Holyroodhouse, May 19, 1937. Three Vice-Presidents and three Members of Council.
4. 250th Anniversary of the Foundation of the K. Leopold Carolinisch d. Deutsche Akademie d. Naturforscher, Halle a Saale, May 28–29, 1937. Letter sent.
5. Inauguration of the Pontifical Academy of Sciences, Citta Vaticana, May 31, 1937. Engrossed letter sent.
6. 23rd Annual Conference of the National Association for the Prevention of Tuberculosis, Bristol, July 1–3, 1937. Professor Sir ROBERT PHILIP.
7. Jubilee of the Buchan Club, Peterhead, August 7, 1937. Professor JAMES RITCHIE and Dr A. BREMNER.
8. Congrès Mondial de la Documentation Universelle, Paris. Août 16–21, 1937. Professor M. CAULLERY was asked, but could not be present. A letter was sent.
9. International Congress of Geology (XVII) Session, Moscow, July to September 1937. Dr G. W. TYRRELL.
10. 150th Anniversary of the Birth of J. E. PURKYNE, Prague, September 24–28, 1937. Letter sent.
11. Réunion Internationale de Physique de Chemie et Biologie. Paris, September 30 to October 7, 1937. Professor MARCELLIN BOULE, Hon. F.R.S.E.
12. 200th Anniversary of the Birth of LUIGI GALVANI, University and City of Bologna, October 1937. President attended and presented an Address.
13. Semi-Jubilee of the Edinburgh Workers' Educational Association at Newbattle Abbey College, October 23, 1937. Dr J. DONALD POLLOCK.
14. International Congress of Geography, Amsterdam, July 18–28, 1938. Professor ALAN G. OGILVIE.

The undernoted were appointed by the Council to fill vacancies in the Society's representation on the following British National Committees:—

Astronomy . . .	Sir FRANK DYSON.
Biology . . .	Professor J. RITCHIE.
Geodesy and Geophysics	Sir JOHN FLETT.
Geography . . .	Mr J. BARTHOLOMEW.
Physics . . .	Professor J. A. CARROLL.
Scientific Radio . .	Emeritus Professor C. T. R. WILSON.

Dr A. CRICHTON MITCHELL was re-appointed to serve as the Society's representative on the Governing Body of the Heriot-Watt College for three years as from January 1, 1937.

Dr L. DOBBIN represented the Society at a Conference in the Rooms of the Royal Society of London, on December 15, 1936, to discuss the question of a Standard System of Bibliographical References and Abbreviations to be used in Scientific Publications.

Professor F. A. E. CREW was appointed by the Council to succeed the late Professor D. F. FRASER-HARRIS as the representative of the Society on the Leonard Darwin Studentship Committee of the Eugenics Society, London.

Professor T. J. MACKIE was reappointed as the joint representative of the University of Edinburgh and the Society on the India Office Consultative Committee to assist in questions connected with the recruitment of medical research workers for service in India, for three years, as from February 24, 1936.

Professor J. WALTON was appointed as the Society's representative on the Scientific and Technical Committee of the Empire Exhibition, Scotland, 1938.

Dr A. LANDSBOROUGH-THOMSON was appointed to act as a representative of the Society,

with Professor JAMES RITCHIE, on the British Section of the International Committee for the Preservation of Birds, in succession to Professor J. GRAHAM KERR, who resigned.

Dr D. A. BANNERMAN (British Museum, Natural History), through our representatives, was appointed to represent the Society at the meeting of the European Continental Section of the International Committee for the Preservation of Birds, in Vienna, from July 3 to 6, 1937.

A Committee was appointed to make arrangements jointly with the Edinburgh Mathematical Society for the Celebration of the Tercentenary of JAMES GREGORY in July 1938.

During the session the sum of £300 was expended on the binding of books in the Library. Arrears of binding are being gradually overtaken.

The oil portraits in the possession of the Society have been cleaned and renovated by an expert under the eye of Mr STANLEY CURSITER, O.B.E., R.S.A., Director of the National Galleries of Scotland.

The GUNNING VICTORIA JUBILEE PRIZE (1932–1936) was presented to Professor C. G. DARWIN, Master of Christ's College, Cambridge, and formerly Tait Professor of Natural Philosophy in the University of Edinburgh, for his distinguished contributions in mathematical physics; and the MAKDOUGALL-BRISBANE PRIZE (1934–1936) to Dr E. M. ANDERSON, formerly of H.M. Geological Survey (Scotland), for his paper on "The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences," published in the Society's *Proceedings* within the period of the award.

The acknowledgments of the Society are due to the Carnegie Trust for the Universities of Scotland for grants to authors towards the cost of illustrations, tabular matter, etc., of papers published in the *Transactions* and *Proceedings*, amounting to £133, 9s. 11d.; to Queen's University, Belfast, for a grant of £15 towards the publication of Professor J. SMALL's joint paper in the *Proceedings*; to Birkbeck College, London, for £5 towards the cost of the illustrations of Dr VERA FRETTER's paper in the *Transactions*; and for £300 received from the Royal Society of London, from the Government Publication Grant, in aid of the cost of the Society's publications for the session 1936–1937.

TREASURER'S REPORT:—

The TREASURER, in submitting the Accounts for the past year, mentioned briefly the leading items of Receipts and Expenditure and compared them with those of the previous year. He stated that a sum of £200, set aside in the Accounts for the session 1934–35, had now been absorbed. He indicated that the financial position of the Society was satisfactory.

Dr R. A. FLEMING moved the adoption of the reports, and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors for the ensuing session. The meeting approved.

The Scrutineers reported that the Ballot Papers were in order, and the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Professor Sir D'ARCY W. THOMPSON, Kt., C.B., D.Litt., Hon. D.Sc., LL.D., F.R.S., President.

Vice-Presidents:
Professor F. A. E. CREW, M.D., D.Sc., Ph.D.
Lt.-Col. A. G. McKENDRICK, M.B., D.Sc., F.R.C.P.E.
Principal J. C. SMAIL, O.B.E., Companion Inst.E.E.
Professor J. WALTON, M.A., D.Sc.
JAMES WATT, W.S., LL.D.
Professor E. T. WHITTAKER, M.A., Hon. Sc.D., LL.D., F.R.S.

Professor JAMES P. KENDALL, M.A., D.Sc., F.R.S., General Secretary.

Secretaries to Ordinary Meetings:
ALEXANDER C. AITKEN, M.A., D.Sc., F.R.S.
CHARLES H. O'DONOGHUE, D.Sc.

E. MACLAGAN WEDDERBURN, O.B.E., D.K.S., M.A., D.Sc., Treasurer.
LEONARD DOBBIN, Ph.D., Curator of Library and Museum.

MEMBERS OF COUNCIL.

EDWIN BRAMWELL, M.D., LL.D., F.R.C.P. Lond. and Edin.
Emeritus Professor T. H. BRYCE, M.A., M.D., LL.D., F.R.S.
Professor I. DE BURGH DALY, M.A., M.D., B.Ch.
JOHN ALEXANDER INGLIS, K.C., M.A., LL.B.
Professor A. D. PEACOCK, D.Sc.
JOHN E. MACKENZIE, D.Sc.
Professor SYDNEY SMITH, M.D., F.R.C.P., D.P.H.
Emeritus Professor RALPH STOCKMAN, M.D., LL.D., F.R.C.P.E.
Professor LANCELOT T. HOGBEN, M.A., D.Sc., F.R.S.
Professor JAMES RITCHIE, M.A., D.Sc.
G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S.
Emeritus Professor C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.

The Scrutineers were thanked for their services.

The PRESIDENT paid a tribute to the valuable services rendered to the Society by Dr JAMES WATT, who had retired by rotation after eleven years service as Treasurer, and who that day had been elected a Vice-President.

The retiring Office-Bearers and Members of Council were also warmly thanked for their services.

The PRESIDENT, Emeritus Professor C. T. R. WILSON, and Professor MAX BORN spoke in high appreciation of the scientific achievements of the late Lord Rutherford of Nelson, and expressed the Society's sorrow at his untimely death. It was agreed that a minute of appreciation and condolence should be sent by the Council to Lady Rutherford.

In presenting the BRUCE PRIZE (1936) to Mr JAMES WILLIAM SLESSER MARR, the PRESIDENT said:—

> A Committee consisting of representatives of the Royal Society of Edinburgh, the Royal Physical Society and the Royal Scottish Geographical Society, has awarded the DR W. S. BRUCE MEMORIAL PRIZE (1936) to Mr JAMES W. S. MARR, M.A., B.Sc., who first went to Polar Regions with Sir Ernest Shackleton in the *Quest* in 1921, sailing as a Boy Scout. On Shackleton's death the Expedition continued under Commander Worsley into the Weddell Sea. MARR next sailed with Commander Worsley in 1925 to Spitsbergen and White Island in the Algarsson Expedition. In 1927 he joined the staff of the R.R.S. *Discovery*; and since then, with brief intervals at home, he has spent his time in the Southern Ocean, partly in the old *Discovery* and partly in *Discovery II*. From 1929 to 1931 he was in *Discovery* when she was lent to Sir Douglas Mawson for the British-Australian-New Zealand Expedition which found many new stretches of the coast-line of Antarctica. In November 1935 MARR published in the "*Discovery*" *Reports* a monograph on the South Orkney Islands, which greatly extended the original researches of Dr W. S. BRUCE on that Antarctic Group. He has recently returned from another Antarctic voyage in *Discovery II*.

The PRESIDENT then called upon Mr MARR to deliver his Address on "Antarctic Surveys: The Work of the 'Discovery' Investigations."

The PRESIDENT, in name of the Society, cordially thanked Mr MARR for his Address.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING VICTORIA JUBILEE, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and about £30 in Money, will be awarded in the Session 1939–1940 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society of Edinburgh during the Sessions 1937–1938 and 1938–1939." Preference will be given to a Paper containing a discovery. (See also Council's resolutions at the end of these regulations.)

II. MAKDOUGALL-BRISBANE PRIZE.

(*Amended June* 7, 1926.)

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some Paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded before the close of the Session 1938–1939, for an Essay, Paper, or other work having reference to any branch of scientific inquiry, either material or mental.
2. It is open to all men of science.
3. The specific subjects taken into consideration in the current award are governed by the resolutions of the Council as stated at the end of these regulations.
4. For the current period the Committee is representative of Group *B*.
5. The Committee will consider Papers presented to the Society within the Sessions 1936–1937 and 1937–1938, and will make a recommendation.

It is empowered to recommend either:—

(*a*) An award to the Author of an Essay or Paper considered as above, or

(*b*) That no award be made on the ground that, within its group, no Paper of sufficient merit has been presented, or

* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a Paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

(*c*) That the Prize be awarded to some distinguished man of learning, who may not have presented a Paper to the Society within the period considered, but who is willing to deliver an address.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:—

1. The NEILL PRIZE, consisting of a Gold Medal, will be awarded during the Session 1939–1940.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1939,—or failing presentation of a Paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award. (See also resolutions of Council at the end of these regulations.)

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented * to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. The next award will be made in Session 1940–1941.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

V. JAMES SCOTT PRIZE.

This Prize, founded in the year 1918 by the Trustees of the JAMES SCOTT Bequest, is to be awarded triennially, or at such intervals as the Council of the Royal Society of Edinburgh may decide, "for a lecture or essay on the fundamental concepts of Natural Philosophy."

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a Paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

VI. BRUCE PRIZE.

The Society is trustee of a fund, instituted in 1923, to commemorate the work of Dr W. S. Bruce as an explorer and scientific investigator in polar regions.

The Committee of Award is appointed jointly by the Royal Society of Edinburgh, the Royal Physical Society, and the Royal Scottish Geographical Society.

The Prize consists of a Bronze Medal and sum of Money. It is open to workers of all nationalities, with a preference, *ceteris paribus*, for those of Scottish birth or origin, and is to be awarded biennially for some notable contribution to Natural Sciences, such as Zoology, Botany, Geology, Meteorology, Oceanography, and Geography; the contribution to be in the nature of new knowledge, the outcome of a personal visit to polar regions on the part of the recipient. The recipient shall preferably be at the outset of his career as an investigator.

The next award will be made in 1940. Papers for the consideration of the Committee should be in the hands of the General Secretary of the Royal Society, 22 George Street, Edinburgh 2, not later than March 31 of that year.

VII. BRUCE-PRELLER LECTURE FUND.

The Council of the Royal Society of Edinburgh having received in 1929 the bequest of the late Dr Charles Du Riche Preller of the sum of £500, decided that the income thereof be applied by the Council biennially as an honorarium for a special Bruce-Preller Lecture or Address by an outstanding man of science, its subject to be Geology or Electrical or Physical Science, or in the discretion of the Council some other branch of science. The next award will be made in Session 1938–1939.

VIII. DAVID ANDERSON-BERRY FUND.

The Council of the Royal Society of Edinburgh having received in the year 1930, free of duty, the capital sum of one thousand pounds (£1000), to be used in terms of the will of the late Dr David Anderson-Berry, dated 23rd April 1926, decided that the income thereof be applied triennially, "in the first place in the presentation of a gold medal, and in the second place in the payment of a sum of money to the winner for the year of such gold medal, the winner being the person who, in the opinion of the Society, shall be the producer for the year of the best essay on the nature of X-rays and their therapeutical effect on human diseases."

The third award will be made in July 1941.

RESOLUTIONS OF COUNCIL IN REGARD TO THE MODE OF AWARDING PRIZES.

(*See Minutes of Meeting of January* 18, 1915.)

I. With regard to the Keith and Makdougall-Brisbane Prizes, which are open to all Sciences, the mode of award will be as follows:—

1. Papers or essays to be considered shall be arranged in two groups, *A* and *B*—Group *A* to include Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, and Physics; Group *B* to include Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, and Zoology.
2. These two prizes shall be awarded to each group in alternate biennial periods, provided Papers worthy of recommendation have been communicated to the Society.
3. Prior to the adjudication the Council shall appoint, in the first instance, a Committee composed of representatives of the group of Sciences which did not receive the award in the immediately preceding period. The Committee shall consider the Papers which come within their group of Sciences, and report in due course to the Council.
4. In the event of the aforesaid Committee reporting that within their group of subjects there is, in their opinion, no Paper worthy of being recommended for the award, the Council, on accepting this report, shall appoint a Committee representative of the alternate group to consider Papers coming within their group and to report accordingly.
5. Papers to be considered by the Committees shall fall within the period dating from the last award in Groups *A* and *B* respectively.

II. With regard to the Neill Prize, the term "Naturalist" shall be understood to include any student in the Sciences composing Group *B*, namely, Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND.

I. KEITH PRIZE.

1ST BIENNIAL PERIOD, 1827–29.—Dr BREWSTER, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2ND BIENNIAL PERIOD, 1829–31.—Dr BREWSTER, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3RD BIENNIAL PERIOD, 1831–33.—THOMAS GRAHAM, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4TH BIENNIAL PERIOD, 1833–35.—Professor J. D. FORBES, for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5TH BIENNIAL PERIOD, 1835–37.—JOHN SCOTT RUSSELL, Esq., for his researches "on Hydrodynamics," published in the Transactions of the Society.

6TH BIENNIAL PERIOD, 1837–39.—Mr JOHN SHAW, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7TH BIENNIAL PERIOD, 1839–41.—Not awarded.

8TH BIENNIAL PERIOD, 1841–43.—Professor JAMES DAVID FORBES, for his papers "on Glaciers," published in the Proceedings of the Society.

9TH BIENNIAL PERIOD, 1843–45.—Not awarded.

10TH BIENNIAL PERIOD, 1845–47.—General Sir THOMAS BRISBANE, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11TH BIENNIAL PERIOD, 1847–49.—Not awarded.

12TH BIENNIAL PERIOD, 1849–51.—Professor KELLAND, for his papers "on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13TH BIENNIAL PERIOD, 1851–53.—W. J. MACQUORN RANKINE, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14TH BIENNIAL PERIOD, 1853–55.—Dr THOMAS ANDERSON, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15TH BIENNIAL PERIOD, 1855–57.—Professor BOOLE, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16TH BIENNIAL PERIOD, 1857–59.—Not awarded.

17TH BIENNIAL PERIOD, 1859–61.—JOHN ALLAN BROUN, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18TH BIENNIAL PERIOD, 1861–63.—Professor WILLIAM THOMSON, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19TH BIENNIAL PERIOD, 1863–65.—Principal FORBES, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20TH BIENNIAL PERIOD, 1865–67.—Professor C. PIAZZI SMYTH, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21ST BIENNIAL PERIOD, 1867–69.—Professor P. G. TAIT, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22ND BIENNIAL PERIOD, 1869–71.—Professor CLERK MAXWELL, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23RD BIENNIAL PERIOD, 1871–73.—Professor P. G. TAIT, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24TH BIENNIAL PERIOD, 1873–75.—Professor CRUM BROWN, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25TH BIENNIAL PERIOD, 1875–77.—Professor M. FORSTER HEDDLE, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26TH BIENNIAL PERIOD, 1877–79.—Professor H. C. FLEEMING JENKIN, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II having appeared in the volume for 1877–78.

27TH BIENNIAL PERIOD, 1879–81.—Professor GEORGE CHRYSTAL, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28TH BIENNIAL PERIOD, 1881–83.—THOMAS MUIR, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29TH BIENNIAL PERIOD, 1883–85.—JOHN AITKEN, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30TH BIENNIAL PERIOD, 1885–87.—JOHN YOUNG BUCHANAN, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

31ST BIENNIAL PERIOD, 1887–89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.

32ND BIENNIAL PERIOD, 1889–91.—R. T. OMOND, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv of the Society's Transactions.

33RD BIENNIAL PERIOD, 1891–93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv, xxxvi, and xxxvii of the Society's Transactions.

34TH BIENNIAL PERIOD, 1893–95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.

35TH BIENNIAL PERIOD, 1895–97.—Dr THOMAS MUIR, for his continued communications on Determinants and Allied Questions.

36TH BIENNIAL PERIOD, 1897–99.—Dr JAMES BURGESS, for his paper "on the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$, with extended Tables of Values," printed in vol. xxxix of the Transactions of the Society.

37TH BIENNIAL PERIOD, 1899–1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.

38TH BIENNIAL PERIOD, 1901–03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., etc., for his memoirs entitled "A Contribution to the Craniology of the People of Scotland," published in the Transactions of the Society, and for his "Contributions to the Craniology of the People of the Empire of India," Parts I, II, likewise published in the Transactions of the Society.

39TH BIENNIAL PERIOD, 1903–05.—THOMAS H. BRYCE, M.A., M.D., for his two papers on "The Histology of the Blood of the Larva of *Lepidosiren paradoxa*," published in the Transactions of the Society within the period.

40TH BIENNIAL PERIOD, 1905–07.—ALEXANDER BRUCE, M.A., M.D., F.R.C.P.E., for his paper entitled "Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord," published in the Transactions of the Society within the period.

41ST BIENNIAL PERIOD, 1907–09.—WHEELTON HIND, M.D., B.S., F.R.C.S., F.G.S., for a paper published in the Transactions of the Society, "On the Lamellibranch and Gasteropod Fauna found in the Millstone Grit of Scotland."

42ND BIENNIAL PERIOD, 1909–11.—Professor ALEXANDER SMITH, B.Sc., Ph.D., of New York, for his researches upon "Sulphur" and upon "Vapour Pressure," appearing in the Proceedings of the Society.

43RD BIENNIAL PERIOD, 1911–13.—JAMES RUSSELL, Esq., for his series of investigations relating to magnetic phenomena in metals and the molecular theory of magnetism, the results of which have been published in the Proceedings and Transactions of the Society, the last paper having been issued within the period.

44TH BIENNIAL PERIOD, 1913–15.—JAMES HARTLEY ASHWORTH, D.Sc., for his papers on "Larvæ of Lingula and Pelagodiscus," and on "Sclerocheilus," published in the Transactions of the Society, and for other papers on the Morphology and Histology of Polychæta.

45TH BIENNIAL PERIOD, 1915–17.—ROBERT C. MOSSMAN, Esq., for his work on the Meteorology of the Antarctic Regions, which originated with the important series of observations made by him during the voyage of the "Scotia" (1902–1904), and includes his paper "On a Sea-Saw of Barometric Pressure, Temperature, and Wind Velocity between the Weddell Sea and the Ross Sea," published in the Proceedings of the Society.

46TH BIENNIAL PERIOD, 1917–19.—JOHN STEPHENSON, Lt.-Col., I.M.S., for his series of papers on the Oligochæta and other Annelida, several of which have been published in the Transactions of the Society.

47TH BIENNIAL PERIOD, 1919–21.—RALPH ALLEN SAMPSON, F.R.S., for his Astronomical Researches, including the papers "Studies in Clocks and Time-keeping: No. 1, Theory of the Maintenance of Motion; No. 2, Tables of the Circular Equation," published in the Proceedings of the Society within the period of the award.

48TH BIENNIAL PERIOD, 1921–23.—JOHN WALTER GREGORY, F.R.S., for his papers published in the Transactions of the Society, and in recognition of his numerous contributions to Geology, extending over a period of thirty-six years.

49TH BIENNIAL PERIOD, 1923–25.—HERBERT WESTREN TURNBULL, M.A., for the papers on "Hyper-Algebra," "Invariant Theory," and "Algebraic Geometry," three of which have been published in the Proceedings within the period of award.

50TH BIENNIAL PERIOD, 1925–27.—THOMAS JOHN JEHU, M.A., M.D., F.G.S., and ROBERT MELDRUM CRAIG, M.A., B.Sc., F.G.S., for the joint series of papers which have recently appeared in the Transactions of the Society on the "Geology of the Outer Hebrides."

51ST BIENNIAL PERIOD, 1927–29.—CHRISTINA C. MILLER, B.Sc., Ph.D., for her papers on the "Slow Oxidation of Phosphorus Trioxide," published in the Proceedings within the period of the award, and in consideration of subsequent developments on "Slow Oxidation of Phosphorus," published elsewhere.

52ND BIENNIAL PERIOD, 1929–31.—ALAN WILLIAM GREENWOOD, M.Sc., Ph.D., for his papers on the "Biology of the Fowl," several of which have appeared in the Proceedings within the period of award.

53RD BIENNIAL PERIOD, 1931–33.—A. CRICHTON MITCHELL, D.Sc., for his paper "On the Diurnal Incidence of Disturbance in the Terrestrial Magnetic Field," published in the Transactions within the period of award.

54TH BIENNIAL PERIOD, 1933–35.—LANCELOT T. HOGBEN, D.Sc., F.R.S., for his papers on genetical subjects, published alone and in collaboration, which have appeared in the Proceedings within the period of award.

II. MAKDOUGALL-BRISBANE PRIZE.

1ST BIENNIAL PERIOD, 1859.—Sir RODERICK IMPEY MURCHISON, on account of his Contributions to the Geology of Scotland.

2ND BIENNIAL PERIOD, 1860–62.—WILLIAM SELLER, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3RD BIENNIAL PERIOD, 1862–64.—JOHN DENIS MACDONALD, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4TH BIENNIAL PERIOD, 1864–66.—Not awarded.

5TH BIENNIAL PERIOD, 1866–68.—Dr ALEXANDER CRUM BROWN and Dr THOMAS RICHARD FRASER, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6TH BIENNIAL PERIOD, 1868–70.—Not awarded.

7TH BIENNIAL PERIOD, 1870–72.—GEORGE JAMES ALLMAN, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8TH BIENNIAL PERIOD, 1872–74.—Professor LISTER, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9TH BIENNIAL PERIOD, 1874–76.—ALEXANDER BUCHAN, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10TH BIENNIAL PERIOD, 1876–78.—Professor ARCHIBALD GEIKIE, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11TH BIENNIAL PERIOD, 1878–80.—Professor PIAZZI SMYTH, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1880–82.—Professor JAMES GEIKIE, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13TH BIENNIAL PERIOD, 1882–84.—EDWARD SANG, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculations of Logarithms both of Numbers and Trigonometrical Ratios—the former communication being published in the Proceedings of the Society.

14TH BIENNIAL PERIOD, 1884–86.—JOHN MURRAY, Esq., LL.D., for his papers "On the Drainage Areas of Continents and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15TH BIENNIAL PERIOD, 1886–88.—ARCHIBALD GEIKIE, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

16TH BIENNIAL PERIOD, 1888–90.—Dr LUDWIG BECKER, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in vol. xxxvi, Part I, of the Society's Transactions.

17TH BIENNIAL PERIOD, 1890–92.—HUGH ROBERT MILL, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I being already published in vol. xxxvi of the Society's Transactions.

18TH BIENNIAL PERIOD, 1892–94.—Professor JAMES WALKER, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx, pp. 255–263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.

19TH BIENNIAL PERIOD, 1894–96.—Professor JOHN G. M'KENDRICK, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society's publications.

20TH BIENNIAL PERIOD, 1896–98.—Dr WILLIAM PEDDIE, for his papers on the Torsional Rigidity of Wires.

21ST BIENNIAL PERIOD, 1898–1900.—Dr RAMSAY H. TRAQUAIR, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in vol. xxxix of the Transactions of the Society.

22ND BIENNIAL PERIOD, 1900–02.—Dr ARTHUR T. MASTERMAN, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in vol. xl of the Transactions of the Society.

23RD BIENNIAL PERIOD, 1902–04.—JOHN DOUGALL, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in vol. xli of the Transactions of the Society.

24TH BIENNIAL PERIOD, 1904–06.—JACOB E. HALM, Ph.D., for his two papers entitled "Spectroscopic Observations of the Rotation of the Sun," and "Some Further Results obtained with the Spectroheliometer," and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.

25TH BIENNIAL PERIOD, 1906–08.—D. T. GWYNNE-VAUGHAN, M.A., F.L.S., for his papers, 1st, "On the Fossil Osmundaceæ," and 2nd, "On the Origin of the Adaxially-curved Leaf-trace in the Filicales," communicated by him conjointly with Dr R. Kidston.

26TH BIENNIAL PERIOD, 1908–10.—ERNEST MACLAGAN WEDDERBURN, M.A., LL.B., for his series of papers bearing upon "The Temperature Distribution in Fresh-water Lochs," and especially upon "The Temperature Seiche."

27TH BIENNIAL PERIOD, 1910–12.—JOHN BROWNLEE, M.A., M.D., D.Sc., for his contributions to the Theory of Mendelian Distributions and cognate subjects, published in the Proceedings of the Society within and prior to the prescribed period.

28TH BIENNIAL PERIOD, 1912–14.—Professor C. R. MARSHALL, M.A., M.D., for his studies "On the Pharmacological Action of Tetra-alkyl-ammonium Compounds."

29TH BIENNIAL PERIOD, 1914–16.—ROBERT ALEXANDER HOUSTOUN, Ph.D., D.Sc., for his series of papers on "The Absorption of Light by Inorganic Salts," published in the Proceedings of the Society.

30TH BIENNIAL PERIOD, 1916–18.—Professor A. ANSTRUTHER LAWSON, for his Memoirs on "The Prothalli of *Tmesipteris Tannensis* and of *Psilotum*," published in the Transactions of the Society, together with previous papers on Cytology and on The Gametophytes of various Gymnosperms.

31ST BIENNIAL PERIOD, 1918–20.—Professor J. H. MACLAGAN WEDDERBURN of Princeton University, for his Memoirs in Universal Algebra, etc., published in the Transactions and Proceedings of the Society, and elsewhere.

32ND BIENNIAL PERIOD, 1920–22.—Professor W. T. GORDON, M.A., D.Sc., for his paper on "Cambrian Organic Remains from a Dredging in the Weddell Sea," published in the Transactions of the Society within the period, and for his investigations on the Fossil Flora of the Pettycur Limestone, previously published in the Transactions.

33RD BIENNIAL PERIOD, 1922–24.—Professor H. STANLEY ALLEN, D.Sc., for his papers on the "Quantum and Atomic Theory," published in the Society's Proceedings within the period.

34TH BIENNIAL PERIOD, 1924–26.—CHARLES MORLEY WENYON, C.M.G., C.B.E., F.R.S., for his distinguished work in Protozoology extending over a period of twenty-one years.

35TH BIENNIAL PERIOD, 1926–28.—W. O. KERMACK, M.A., D.Sc., for his contributions to Chemistry, published in the Society's Proceedings and elsewhere.

36TH BIENNIAL PERIOD, 1928–30.—NELLIE B. EALES, D.Sc., for her papers in the Society's Transactions on "The Anatomy of a Fœtal African Elephant."

37TH BIENNIAL PERIOD, 1930–32.—A. C. AITKEN, M.A., D.Sc., for various contributions to Mathematics, published in the Society's Proceedings and elsewhere.

38TH BIENNIAL PERIOD, 1932–34.—A. E. CAMERON, M.A., D.Sc., for his publications in Entomology, including his paper in the Transactions, "The Life-History and Structure of *Hæmatopota pluvialis* Linné (Tabanidæ)."

39TH BIENNIAL PERIOD, 1934–36.—E. M. ANDERSON, M.A., D.Sc., for his paper "The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences," published in the Society's Proceedings within the period.

III. THE NEILL PRIZE.

1ST TRIENNIAL PERIOD, 1856–59.—Dr W. LAUDER LINDSAY, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2ND TRIENNIAL PERIOD, 1859–62.—ROBERT KAYE GREVILLE, LL.D., for his contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3RD TRIENNIAL PERIOD, 1862–65.—ANDREW CROMBIE RAMSAY, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4TH TRIENNIAL PERIOD, 1865–68.—Dr WILLIAM CARMICHAEL M'INTOSH, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5TH TRIENNIAL PERIOD, 1868–71.—Professor WILLIAM TURNER, for his papers "on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6TH TRIENNIAL PERIOD, 1871–74.—CHARLES WILLIAM PEACH, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7TH TRIENNIAL PERIOD, 1874–77.—Dr RAMSAY H. TRAQUAIR, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8TH TRIENNIAL PERIOD, 1877–80.—JOHN MURRAY, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9TH TRIENNIAL PERIOD, 1880–83.—Professor W. A. HERDMAN, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10TH TRIENNIAL PERIOD, 1883–86.—B. N. PEACH, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.

11TH TRIENNIAL PERIOD, 1886–89.—ROBERT KIDSTON, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.

12TH TRIENNIAL PERIOD, 1889–92.—JOHN HORNE, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.

13TH TRIENNIAL PERIOD, 1892–95.—ROBERT IRVINE, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.

14TH TRIENNIAL PERIOD, 1895–98.—Professor J. COSSAR EWART, for his recent Investigations connected with Telegony.

15TH TRIENNIAL PERIOD, 1898–1901.—Dr JOHN S. FLETT, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in vol. xxxix of the Transactions of the Society.

16TH TRIENNIAL PERIOD, 1901–04.—Professor J. GRAHAM KERR, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.

17TH TRIENNIAL PERIOD, 1904–07.—FRANK J. COLE, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a Study of Myxine," published in the Transactions of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes.

1ST BIENNIAL PERIOD, 1907–09.—FRANCIS J. LEWIS, M.Sc., F.L.S., for his papers in the Society's Transactions "On the Plant Remains of the Scottish Peat Mosses."

2ND BIENNIAL PERIOD, 1909–11.—JAMES MURRAY, Esq., for his paper on "Scottish Rotifers collected by the Lake Survey (Supplement)," and other papers on the "Rotifera" and "Tardigrada," which appeared in the Transactions of the Society—(this Prize was awarded after consideration of the papers received within the five years prior to the time of award: see Neill Prize Regulations).

3RD BIENNIAL PERIOD, 1911–13.—W. S. BRUCE, LL.D., in recognition of the scientific results of his Arctic and Antarctic explorations.

4TH BIENNIAL PERIOD, 1913–15.—ROBERT CAMPBELL, D.Sc., for his paper on "The Upper Cambrian Rocks at Craigeven Bay, Stonehaven," and "Downtonian and Old Red Sandstone Rocks of Kincardineshire," published in the Transactions of the Society.

5TH BIENNIAL PERIOD, 1915–17.—W. H. LANG, F.R.S., M.B., D.Sc., for his paper in conjunction with Dr R. KIDSTON, F.R.S., on *Rhynia Gwynne-Vaughani*, Kidston and Lang, published in the Transactions of the Society, and for his previous investigations on Pteridophytes and Cycads.

6TH BIENNIAL PERIOD, 1917–19.—JOHN TAIT, D.Sc., M.D., for his work on Crustacea, published in the Proceedings of the Society, and for his papers on the blood of Crustacea.

7TH BIENNIAL PERIOD, 1919–21.—Sir EDWARD A. SHARPEY-SCHAFER, F.R.S., for his recent contributions to our knowledge of Physiology, and in recognition of his published work extending over a period of fifty years.

8TH BIENNIAL PERIOD, 1921–23.—JOHN M'LEAN THOMPSON, M.A., D.Sc., University of Liverpool, for his series of Memoirs on Staminal Zygomorphy, and on the Anatomy of the Filicales.

9TH BIENNIAL PERIOD, 1923–25.—FREDERICK ORPEN BOWER, F.R.S., for his recent contributions to Botanical knowledge and in recognition of his published work extending over a period of forty-five years.

10TH BIENNIAL PERIOD, 1925–27.—ARTHUR ROBINSON, M.D., M.R.C.S., for his contributions to Comparative Anatomy and Embryology.

11TH BIENNIAL PERIOD, 1927–29.—EDWARD BATTERSBY BAILEY, M.C., F.R.S., in recognition of his valuable contributions to the Geology of Scotland, two of which have recently appeared in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1929–31.—CHARLES HENRY O'DONOGHUE, D.Sc., for his papers on the "Blood Vascular System," and for his earlier work on the "Morphology of the *corpus luteum*."

13TH BIENNIAL PERIOD. 1931–33.—GEORGE WALTER TYRRELL, A.R.C.S., D.Sc., for his contributions to the Geology and Petrology of Sub-Arctic and Sub-Antarctic Lands.

14TH BIENNIAL PERIOD, 1933–35.—SAMUEL WILLIAMS, Ph.D., for his contributions to the Anatomy and Experimental Morphology of the Pteridophyta.

IV. GUNNING VICTORIA JUBILEE PRIZE.

1ST TRIENNIAL PERIOD, 1884–87.—Sir WILLIAM THOMSON, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices which have been communicated to the Society.

2ND TRIENNIAL PERIOD, 1887–90.—Professor P. G. TAIT, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3RD TRIENNIAL PERIOD, 1890–93.—ALEXANDER BUCHAN, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's publications.

4TH TRIENNIAL PERIOD, 1893–96.—JOHN AITKEN, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1ST QUADRENNIAL PERIOD, 1896–1900.—Dr T. D. ANDERSON, for his discoveries of New and Variable Stars.

2ND QUADRENNIAL PERIOD, 1900–04.—Sir JAMES DEWAR, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

3RD QUADRENNIAL PERIOD, 1904–08.—Professor GEORGE CHRYSTAL, M.A., LL.D., for a series of papers on "Seiches," including "The Hydrodynamical Theory and Experimental Investigations of the Seiche Phenomena of Certain Scottish Lakes."

4TH QUADRENNIAL PERIOD, 1908–12.—Professor J. NORMAN COLLIE, Ph.D., F.R.S., for his distinguished contributions to Chemistry, Organic and Inorganic, during twenty-seven years, including his work upon Neon and other rare gases. Professor Collie's early papers were contributed to the Transactions of the Society.

5TH QUADRENNIAL PERIOD, 1912–16.—Sir THOMAS MUIR, C.M.G., LL.D., F.R.S., for his series of Memoirs upon "The Theory and History of Determinants and Allied Forms," published in the Transactions and Proceedings of the Society between the years 1872 and 1915.

6TH QUADRENNIAL PERIOD, 1916–20.—C. T. R. WILSON, Esq., F.R.S., in recognition of his important discoveries in relation to Condensation Nuclei, Ionisation of Gases and Atmospheric Electricity.

7TH QUADRENNIAL PERIOD, 1920–24.—Sir J. J. THOMSON, O.M., F.R.S., in recognition of his great discoveries in Physical Science.

8TH QUADRENNIAL PERIOD, 1924–28.—Professor E. T. WHITTAKER, F.R.S., in recognition of his distinguished contributions to Mathematical Science, and of his promotion of Mathematical Research in Scotland.

9TH QUADRENNIAL PERIOD, 1928–32.—Emeritus Professor Sir J. WALKER, F.R.S., for numerous contributions to Physical and General Chemistry.

10TH QUADRENNIAL PERIOD, 1932–36.—Professor C. G. DARWIN, F.R.S., for his distinguished contributions in Mathematical Physics.

V. JAMES SCOTT PRIZE.

1ST AWARD, 1918–22.—Professor A. N. WHITEHEAD, F.R.S., for his lecture delivered on June 5, 1922, on "The Relatedness of Nature."

2ND AWARD, 1922–27.—Sir JOSEPH LARMOR, M.A., D.Sc., LL.D., F.R.S., for his lecture delivered on July 4, 1927, on "The Grasp of Mind on Nature."

3RD AWARD, 1927–30.—Professor NIELS BOHR, for his lecture delivered on May 26, 1930, on "Philosophical Aspects of Atomic Theory."

4TH AWARD, 1930–33.—Professor Dr ARNOLD SOMMERFELD, for his lecture delivered on May 1, 1933, on "Ways to the Knowledge of Nature."

VI. BRUCE PRIZE.

1ST AWARD, 1926.—JAMES MANN WORDIE, M.A., for his Oceanographical and Geological work in both Polar Regions.

2ND AWARD, 1928.—H. U. SVERDRUP, for his contributions to the knowledge of the Meteorology, Magnetism, and Tides of the Arctic, as an outcome of his travels with the Norwegian Expedition in the "Maud" from 1918 to 1925.

3RD AWARD, 1930.—N. A. MACKINTOSH, M.Sc., A.R.C.S., for his researches into the Biology of Whales in the Waters of the Falkland Islands Dependencies.

4TH AWARD, 1932.—HENRY GINO WATKINS, for important contributions to the topography of Spitsbergen, Labrador, and East Greenland, and investigation of the Ice Cap of Greenland.

5TH AWARD, 1936.—JAMES WILLIAM SLESSER MARR, M.A., B.Sc., for his work in the Southern Ocean and more particularly for his monograph on the South Orkney Islands.

VII. BRUCE-PRELLER LECTURE FUND.

1ST AWARD, 1931.—Professor E. T. WHITTAKER, F.R.S., for his lecture, "James Clerk Maxwell and Mechanical Descriptions of the Universe."

2ND AWARD, 1933.—Professor C. H. LANDER, C.B.E., for his lecture on October 23, 1933, on "The Liquefaction of Coal."

3RD AWARD, 1935.—Professor W. L. BRAGG, O.B.E., F.R.S., for his lecture on February 4, 1935, on "The New Crystallography."

4TH AWARD, 1937.—Professor H. S. TAYLOR, F.R.S., for his lecture on June 7, 1937, on "Heavy Hydrogen in Scientific Research."

VIII. DAVID ANDERSON-BERRY FUND.

1ST AWARD, 1935.—CHARLES MELVILLE SCOTT, M.A., M.B., D.Sc., for his essay "On the Action of X- and Gamma-Rays on Living Cells."

ABSTRACT

OF

THE ACCOUNTS

OF

THE ROYAL SOCIETY OF EDINBURGH,

SESSION—1st October 1936 to 30th September 1937.

JAMES WATT, LL.D., W.S.,
Treasurer.

I. GENERAL FUND

CHARGE.

1. Arrears of Contributions at 30th September 1936		£144 18 0
Less—Written off as irrecoverable		22 1 0
		£122 17 0
2. Contributions for current Session:—		
a. 453 Fellows at £3, 3s. each	£1426 19 0	
b. Fees of Admission of forty New Fellows (including one Life Fellow)	126 0 0	
c. First Annual Contribution of thirty-nine New Fellows	122 17 0	
		1675 16 0
3. Commutation Fee Fund:—		
a. 1 Fellow elected Session 1933–34	£49 7 0	
b. 1 Fellow elected Session 1936–37	52 10 0	
	£101 17 0	
4. Extra Contributions for 1936–37 under Amended Law VI.:—		
a. Voluntary Contributions	£65 1 0	
b. Commutation	10 10 0	
		75 11 0
5. Interest Received, Untaxed:—		
a. General Fund—		
On £2100 2½% Consolidated Stock	£52 10 0	
On £9742, 12s. 5d. 2¾% Guaranteed Stock (1933)	267 18 4	
On Deposit Receipts	8 18 5	
	£329 6 9	
b. Special Subscription Fund:— On £936, 0s. 11d. 2¾% Guaranteed Stock (1933)	25 14 8	
c. R. M. Smith Legacy— On £554, 13s. 7d. 2¾% Guaranteed Stock (1933)	15 5 0	
d. Publication Fund— On £2568, 5s. 1d. 2¾% Guaranteed Stock (1933)	70 12 4	
e. Commutation Fee Fund— On Deposit Receipt	0 10 2	
		441 8 11
Forward		£2315 12 11

		Forward .	. £2315 12 11
6. *Transactions* and *Proceedings* sold			124 2 9
7. Grants:—			
a. Annual Grant from Government		£600 0 0	
b. Grant from Royal Society—from Government Publication Grant		300 0 0	
c. Grants from Trustees of the Carnegie Trust for the Universities of Scotland	£133 9 11		
Other Sources	20 0 0		
		153 9 11	
			1053 9 11
		Amount of the Charge .	**£3493 5 7**

DISCHARGE.

1. Expenses of Transactions and Proceedings:—		
a. Transactions	£578 10 7	
b. Proceedings	874 5 4	
	£1452 15 11	
Less—Reserve made last Session now absorbed .	200 0 0	
		£1252 15 11
2. Books, Periodicals, Newspapers, etc.		258 9 5
3. Library Binding		300 0 2
4. General Upkeep of Society's Rooms:—		
a. Insurance and Water Rates	£35 6 1	
b. Repairs and Furnishings	24 5 4	
c. Heating and Lighting	64 9 6	
d. Cleaning.	12 2 6	
e. Caretaker's Salary and Uniform	163 4 4	
f. Charwoman's Wages	66 9 5	
		365 17 2
5. Management:—		
a. Honorarium to General Secretary	£100 0 0	
b. Salaries	525 0 0	
c. Fee to Treasurer's Clerk	35 0 0	
d. Society's contribution to Staff Pension Fund under Universities' Scheme	57 10 0	
e. State Insurance	4 2 4	
f. General Printing and Stationery	120 19 4	
g. Telephone	18 12 1	
h. Audit Fee	10 10 0	
i. Tea Expenses at Meetings	32 6 6	
j. Travelling Expenses	12 10 6	
k. Postages and Petty Outlays	57 16 9	
l. Miscellaneous	13 19 3	
		988 6 9
6. Extraordinary Expenditure:—		
a. Renovating Society's Oil Paintings and Pictures . .	£96 2 1	
b. Staircase (West) Carpet—Payment to Account . .	55 0 0	
c. Renovating Paintwork in Society's Rooms . . .	48 13 0	
d. Roneo Duplicator	21 10 0	
e. Donation to Scottish Development Council (Empire Exhibition—Scotland—1938)	10 10 0	
		231 15 1
7. Commutation Fee Fund:—		
Transferred to Special Fund (p. 465)	£101 17 0	
	Forward .	£3397 4 6

Brought forward				£3397	4	6
8. Arrears of Contributions outstanding at 30th September 1937:—						
Present Session	£75	12	0			
Previous Sessions	40	19	0			
				£116	11	0
Amount of the Discharge				£3513	15	6

Amount of the Charge	£3493	5	7
Amount of the Discharge	3513	15	6
Excess of Discharge transferred to Special Subscription Fund	£20	9	11

SPECIAL SUBSCRIPTION FUND.

Year to 30th September 1937.

Charge.

Total Subscriptions towards Fund				£1128	17	9
Surplus on Realisation of War Loan	£45	9	2			
Less—Written off War Loan in Session 1926–27	7	12	0			
				37	17	2
				£1166	14	11
Less—Transfers to General Fund to meet Deficits up to 30th September 1936	£372	4	4			
Deficit for year to 30th September 1937	20	9	11			
				392	14	3
Amount of the Charge				£774	0	8

Discharge.

Balance of Fund—						
£936, 0s. 11d. 2½% Guaranteed Stock (1933) at cost				£797	5	2
Cash—Imprest Amount				10	0	0
				£807	5	2
Less—Due to the Union Bank of Scotland, Ltd., on Current Account	£23	15	6			
Contributions in Advance for 1937–38	9	9	0			
				33	4	6
Amount of the Discharge				£774	0	8

Commutation Fee Fund.

Year to 30th September 1937.

Charge.

1. Due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936				£52	10	0
2. Fees Received for Life Membership—						
1 Fellow elected 1933–34	£49	7	0			
1 Fellow elected 1936–37	52	10	0			
				101	17	0
3. Interest received on Deposit Receipt				0	10	2
Amount of the Charge				£154	17	2

Discharge.

1. Transferred to General Fund (Interest on Deposit Receipt)	£0	10	2
2. Investment made—			
£198, 8s. 11d. 2½% Guaranteed Stock (1933) at 77¼ plus Brokerage	154	7	0
Amount of the Discharge	£154	17	2

II. KEITH FUND

Year to 30th September 1937.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£42 19 7
2. INTEREST RECEIVED:—		
On £801, 13s. 4d. 2½% Guaranteed Stock (1933), Untaxed	£22 0 10	
On Deposit Receipts	0 9 11	
		22 10 9
		£65 10 4

DISCHARGE.

BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	£65 10 4

III. NEILL FUND

Year to 30th September 1937.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£18 6 10
2. INTEREST RECEIVED:—		
On £370 2½% Guaranteed Stock (1933), Untaxed	£10 3 6	
On Deposit Receipts	0 3 10	
		10 7 4
		£28 14 2

DISCHARGE.

BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	£28 14 2

IV. MAKDOUGALL-BRISBANE FUND

Year to 30th September 1937.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£60 10 9
2. INTEREST RECEIVED:—		
On £493, 6s. 7d. 2½% Guaranteed Stock (1933), Untaxed	£13 11 2	
On Deposit Receipts	0 10 8	
		14 1 10
		£74 12 7

DISCHARGE.

1. Dr E. M. Anderson 1934–36 Award	£15 0 0	
Cost of Gold Medal	21 0 0	
		£36 0 0
2. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937		38 12 7
		£74 12 7

V. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

Year to 30th September 1937.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£137 1 3
2. INTEREST RECEIVED:—		
On £308, 6s. 9d. 2¾% Guaranteed Stock (1933), Untaxed	£8 9 6	
On Deposit Receipts	1 7 6	
		9 17 0
		£146 18 3

DISCHARGE.

BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	£146 18 3

VI. GUNNING VICTORIA JUBILEE PRIZE FUND

Year to 30th September 1937.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£162 1 11
2. INTEREST RECEIVED:—		
On £739, 12s. 5d. 2¾% Guaranteed Stock (1933), Untaxed	£20 6 8	
On Deposit Receipts	1 8 7	
		21 15 3
		£183 17 2

DISCHARGE.

1. Professor C. G. Darwin 1932–36 Award	£105 0 0
2. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	78 17 2
	£183 17 2

VII. JAMES SCOTT PRIZE FUND

Year to 30th September 1937.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£28 4 6
2. INTEREST RECEIVED:—		
On £305, 5s. 2¾% Guaranteed Stock (1933), Untaxed	£8 7 10	
On Deposit Receipts	0 5 9	
		8 13 7
		£36 18 1

DISCHARGE.

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	£36 18 1

VIII. PUBLICATION FUND

(Comprising Peter Guthrie Tait Memorial Fund and Dr John Aitken Fund)

Year to 30th September 1937.

CHARGE.

1. Peter Guthrie Tait Memorial Fund:—	
Year's Interest on £1929, 17s. 1d. 2¾% Guaranteed Stock (1933), Untaxed	£53 1 4
2. Dr John Aitken Fund:—	
Year's Interest on £638, 8s. 2¾% Guaranteed Stock (1933), Untaxed .	17 11 0
	£70 12 4

DISCHARGE.

Transferred to General Fund to meet Cost of Publications (see General Fund Charge, 5 *d*)	£70 12 4

IX. DR W. S. BRUCE MEMORIAL FUND

Year to 30th September 1937.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£49 10 2
2. Interest Received:—		
On £289, 1s. 5d. 2¾% Guaranteed Stock (1933), Untaxed	£7 18 10	
On Deposit Receipts	0 9 9	
		8 8 7
		£57 18 9

DISCHARGE.

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	£57 18 9

X. BRUCE-PRELLER LECTURE FUND

Year to 30th September 1937.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£59 6 4
2. Dividend and Interest Received:—		
On £140, 9s. Royal Bank of Scotland Stock, less Tax £5, 16s. 5d.	£18 1 1	
On Deposit Receipts	0 9 6	
		18 10 7
3. Repayment of Income Tax for year to December 1936		5 13 6
		£83 10 5

DISCHARGE.

1. Professor H. S. Taylor 1934–36 Award		£50 0 0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937		33 10 5
		£83 10 5

XI. DR DAVID ANDERSON-BERRY FUND

Year to 30th September 1937.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1936		£35 3 8
2. Interest Received:—		
On £1528, 0s. 4d. Local Loans 3% Stock, Untaxed .	£45 16 8	
On Deposit Receipts	0 11 3	
		46 7 11
		£81 11 7

DISCHARGE.

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1937	£81 11 7

STATE OF THE FUNDS BELONGING TO THE ROYAL SOCIETY OF EDINBURGH

As at 30th September 1937.

1. GENERAL FUND—

1. £9742, 12s. 5d. 2¾% Guaranteed Stock (1933) at cost			£8298 1 9
2. £2100 2½% Consolidated Stock at cost			1113 0 0
3. £554, 13s. 7d. 2¾% Guaranteed Stock (1933) at cost. Robert Mackay Smith Legacy			472 8 9
4. £2568, 5s. 1d. 2¾% Guaranteed Stock (1933) at cost. Publication Fund—(Comprising Peter Guthrie Tait Memorial Fund and Dr John Aitken Fund)			2187 9 1
5. £198, 8s. 11d. 2¾% Guaranteed Stock (1933) at cost. Life Membership, Commutation Fee Fund			154 7 0
6. Balance of Special Subscription Fund—			
£936, 0s. 11d. 2¾% Guaranteed Stock (1933) at cost.		£797 5 2	
Cash—Imprest Amount		10 0 0	
		£807 5 2	
Less—			
Due to Union Bank of Scotland, Ltd., on Current Account	£23 15 6		
Contributions in Advance for 1937–38	9 9 0		
		33 4 6	
			774 0 8
7. Due by Union Bank of Scotland, Ltd., on Deposit Receipt . . .			5 6 10
8. Arrears of Contributions, as per preceding Abstract of Accounts . .			116 11 0
9. Contributions in Advance for 1937–38			9 9 0
Amount . . .			£13,130 14 1

In addition the Society owns the Library, Museum, Pictures, etc., and Furniture in the Rooms at George Street, Edinburgh.

2. KEITH FUND—

1. £801, 13s. 4d. 2½% Guaranteed Stock (1933) at cost.	£682 16 1
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	65 10 4
AMOUNT	£748 6 5

3. NEILL FUND—

1. £370 2½% Guaranteed Stock (1933) at cost	£315 2 9
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	28 14 2
AMOUNT	£343 16 11

4. MAKDOUGALL-BRISBANE FUND—

1. £493, 6s. 7d. 2½% Guaranteed Stock (1933) at cost	£420 3 8
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	38 12 7
AMOUNT	£458 16 3

5. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—

1. £308, 6s. 9d. 2½% Guaranteed Stock (1933) at cost	£262 12 5
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	146 18 3
AMOUNT	£409 10 8

6. GUNNING VICTORIA JUBILEE PRIZE FUND—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £739, 12s. 5d. 2½% Guaranteed Stock (1933) at cost	£629 19 2
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	78 17 2
AMOUNT	£708 16 4

7. JAMES SCOTT PRIZE FUND—

1. £305, 5s. 2½% Guaranteed Stock (1933) at cost	£259 19 10
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	36 18 1
AMOUNT	£296 17 11

8. DR W. S. BRUCE MEMORIAL FUND—

1. £289, 1s. 5d. 2½% Guaranteed Stock (1933) at cost	£246 4 2
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	57 18 9
AMOUNT	£304 2 11

9. BRUCE-PRELLER LECTURE FUND—

1. £140, 9s. Royal Bank of Scotland Stock, taken over at 350%	£491 11 6
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	33 10 5
AMOUNT	£525 1 11

10. DR DAVID ANDERSON-BERRY FUND—

1. £1528, 0s. 4d. Local Loans 3% Stock at cost	£1000 0 0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	81 11 7
AMOUNT	£1081 11 7

Note.—Under the Will of the late Prof. Charles Piazzi Smyth and his wife, the Society will, on the expiry of certain liferents, become entitled to payment of the residue to be applied as set out in the will.

EDINBURGH, 15*th October* 1937.—We have examined the preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1936–1937, and have found them to be correct. The securities for the various Investments, as noted in the foregoing Statement of Funds, have been verified by us at 30th September 1937.

LINDSAY, JAMIESON & HALDANE, C.A.,
Auditors.

VOLUNTARY CONTRIBUTORS who have made a Single Payment under Law VI (end of para. 3), up to 30th September 1937.

	£	s.	d.
Professor LAWRENCE CRAWFORD	£10	10	0
Dr C. H. MILNE	10	10	0
Dr R. A. HOUSTOUN	10	10	0

VOLUNTARY CONTRIBUTORS under Law VI (end of para. 3), to 30th September 1937.

	£	s.	d.
Sir JAMES BARR	£1	1	0
Sir T. HUDSON BEARE	1	1	0
Emeritus Professor F. O. BOWER, F.R.S.	1	1	0
Dr G. S. BROCK	1	1	0
J. W. BUTTERS, Esq.	1	1	0
Emeritus Professor E. W. W. CARLIER, 1935–36, 1936–37	2	2	0
Col. DAVID CARNEGIE	1	1	0
Professor E. G. COKER, F.R.S.	1	1	0
Emeritus Professor J. N. COLLIE, F.R.S.	1	1	0
Sir J. CRICHTON-BROWNE, F.R.S.	1	1	0
D. B. DOTT, Esq.	1	1	0
Dr OSWALD FERGUS	1	1	0
Dr R. A. FLEMING	1	1	0
J. S. FORD, Esq.	1	1	0
ALEXANDER FRASER, Esq.	1	1	0
Dr W. F. HUME	1	1	0
Dr G. R. JEFFREY	1	1	0
Emeritus Professor J. GRAHAM KERR, F.R.S.	1	1	0
Dr ALEXANDER LAUDER	1	1	0
F. H. LIGHTBODY, Esq.	1	1	0
	£22	1	0
Carried forward,	£22	1	0
Dr GEORGE M'GOWAN	1	1	0
Emeritus Professor JAMES MACKINNON	1	1	0
Dr J. S. MCKENDRICK	1	1	0
Dr D. J. MACKINTOSH	1	1	0
Dr H. R. MILL	1	1	0
Professor JOHN MILLER	1	1	0
Dr ALEXANDER MORGAN	1	1	0
Emeritus Professor J. T. MORRISON	1	0	0
R. C. MOSSMAN, Esq.	1	1	0
Sir GEORGE NEWMAN	1	1	0
Professor WM. PEDDIE	1	1	0
A. G. RAMAGE, Esq.	1	1	0
Professor R. A. SAMPSON, F.R.S.	1	1	0
EDWARD SMART, Esq.	1	1	0
STEPHEN SMITH, Esq.	1	1	0
CHARLES A. STEVENSON, Esq.	1	1	0
Dr H. F. STOCKDALE	1	1	0
GEORGE S. THOMSON, Esq.	1	1	0
R. TATLOCK THOMSON, Esq.	1	1	0
Rev. Dr L. MACLEAN WATT	1	1	0
W. WILLIAMSON, Esq.	1	1	0
Total,	£44	1	0

	£	s.	d.
Single payments	£31	10	0
Other payments	44	1	0
Total	£75	11	0

THE COUNCIL OF THE SOCIETY.

25th October 1937.

PRESIDENT.

PROFESSOR SIR D'ARCY WENTWORTH THOMPSON, Kt., C.B., D.Litt., Hon. D.Sc., LL.D., F.R.S.

VICE-PRESIDENTS.

PROFESSOR F. A. E. CREW, M.D., D.Sc., Ph.D.
LT.-COL. A. G. M'KENDRICK, M.B., D.Sc., F.R.C.P.E.
PRINCIPAL J. C. SMAIL, O.B.E., Companion Inst.E.E.
PROFESSOR JOHN WALTON, M.A., D.Sc.
JAMES WATT, W.S., LL.D.
PROFESSOR E. T. WHITTAKER, M.A., Hon. Sc.D., LL.D., F.R.S.

GENERAL SECRETARY.

PROFESSOR JAMES P. KENDALL, M.A., D.Sc., F.R.S.

SECRETARIES TO ORDINARY MEETINGS.

ALEXANDER C. AITKEN, M.A., D.Sc., F.R.S.
CHARLES H. O'DONOGHUE, D.Sc.

TREASURER.

E. MACLAGAN WEDDERBURN, O.B.E., D.K.S., M.A., D.Sc.

CURATOR OF LIBRARY AND MUSEUM.

LEONARD DOBBIN, Ph.D.

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PROFESSOR LANCELOT T. HOGBEN, M.A., D.Sc., F.R.S.

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G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S.

EMERITUS PROFESSOR C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.

OFFICE STAFF.

Assistant Secretary and Librarian, G. A. STEWART.
Assistant Librarian, R. J. B. MUNRO.
Housekeeper, S. HEDDLE.

PATRON.

HIS MOST EXCELLENT MAJESTY THE KING.

FELLOWS OF THE SOCIETY,

Corrected to 25th October 1937.

N.B.—*Those marked * are Annual Contributors.*
„ „ † *have commuted Voluntary Contribution (see 3rd Paragraph, Law VI).*

M-B.	prefixed to a name indicates that the Fellow has received a	Makdougall-Brisbane Medal.
K.	„ „ „ „	Keith Medal.
N.	„ „ „ „	Neill Medal.
V. J.	„ „ „ „	the Gunning Victoria Jubilee Prize.
B.	„ „ „ „	Bruce Medal.
B-P.	„ „ „ „	Bruce-Preller Lectureship.
C.	„ „ „	has contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election			Service on Council, etc.
1925	M-B. C.	* Aitken, Alexander Craig, M.A., D.Sc., F.R.S. (SECRETARY TO ORDINARY MEETINGS), Lecturer in Actuarial Science, University of Edinburgh (16 Chambers Street). 54 Braid Road, Edinburgh 10	1934–36. Sec. 1936–
1889		† Alison, John, M.A., LL.D., formerly Head Master, George Watson's College. 126 Craiglea Drive, Edinburgh 10	
1927	C.	* Allan, Douglas Alexander, D.Sc., Director, City of Liverpool Public Museums, William Brown Street, Liverpool	
1920	C. M-B.	* Allen, Herbert Stanley, M.A. (Cantab.), D.Sc. (London), F.R.S., Professor of Natural Philosophy, University of St Andrews	1921–24.
1936		* Alty, Thomas, D.Sc. (Liverpool), Ph.D. (Cantab.), F.R.S.C., Professor of Applied Physics, University of Glasgow. 287 Wilton Street, Glasgow	
1920	C. M-B.	* Anderson, Ernest Masson, M.A., D.Sc., F.G.S., 62 Greenbank Crescent, Edinburgh	
1905		Anderson, William, M.A., formerly Head Science Master, George Watson's College, Edinburgh. 6 Lockharton Crescent, Edinburgh 11	
1905		Andrew, George, C.B.E., M.A., B.A., H.M.I.S., Royal Technical College, George Street, Glasgow. Hamewith, Kilmacolm, Renfrewshire	
1930		* Annan, William, M.A., C.A., Professor of Accounting and Business Method, University of Edinburgh (South Bridge). Tofthill, Ferry Road West, Edinburgh 5	
1915		Anthony, Charles, M.Inst.C.E., M.Am.Soc.C.E., F.R.San.I., F.R.Met.S., F.R.A.S., F.C.S., Springcroft, Les Croutes, St Peter Port, Guernsey, Channel Islands	
1906		Appleton, Colonel Arthur Frederick, F.R.C.V.S. (no permanent address until further notice)	
1910	C.	Archibald, E. H., B.Sc., Professor of Chemistry, University of British Columbia, Vancouver, Canada	
1933		* Arnot, Frederick Latham, B.Sc. (Sydney), Ph.D. (Cantab.), Lecturer in Natural Philosophy, University of St Andrews. Yirdlag, Hepburn Gardens, St Andrews	
1921		* Arthur, William, M.A., Lecturer in Mathematics, University of Glasgow. 148 Carmunnock Road, Cathcart, Glasgow	
1920		* Bagnall, Richard Siddoway, Hon. D.Sc., F.R.E.S., 3 St Helen's Terrace, Low Fell, Co. Durham	
1920	C. N.	* Bailey, Edward Battersby, M.C., M.A., D.Sc. (Harvard), F.R.S., F.G.S. Director, Geological Survey of Great Britain and Museum of Practical Geology, Exhibition Road, London, S.W. 7	1932–35. V-P 1935–37.
1896	C.	† Baily, Francis Gibson, M.A., M.Inst.E.E., Emeritus Professor of Electrical Engineering, Heriot-Watt College, Edinburgh. Newbury, Juniper Green, Midlothian	1909–12, 1920–23. V-P 1929–32.
1934		* Bain, David, M.Sc. (Manch.), D.Sc. (Edin.), Lecturer in Technical Chemistry, University of Edinburgh (West Mains Road). 87 Cluny Gardens, Edinburgh 10	

Date of Election			Service on Council, etc.
1931	C.	* Bain, William Alexander, Ph.D., Reader in Pharmacology, School of Medicine, University of Leeds. 26 Weetwood Road, Headingley, Leeds 6	
1931		* Baird, Sir William Macdonald, Kt., J.P., Fellow and Past President of the Faculty of Surveyors of Scotland, F.S.A.Scot. Dalveen, Barnton Avenue, Davidson's Mains, Edinburgh 4	
1921		* Baker, Bevan Braithwaite, M.A., D.Sc., Professor of Mathematics, Royal Holloway College, Englefield Green, London	
1928	C.	* Baker, Edwin Arthur, D.Sc. (Edin.), Assistant at the Royal Observatory, Edinburgh. 17 Ladysmith Road, Edinburgh 9	
1905	C.	Balfour-Browne, William Alexander Francis, M.A., F.Z.S., F.L.S., F.R.E.S., F.R.M.S., Barrister-at-Law, formerly Professor of Entomology, Imperial College of Science, London. Hook Place, Burgess Hill, Sussex	
1933		* Banerjee, Prabodh Chandra, L.R.C.P.E., L.R.C.S.E., F.R.F.P.S.G., F.A.C.S., Major, I.M.S. C/o Lloyds Bank, Ltd., 101–1 Clive Street, Calcutta, India	
1928		Barbour, George Brown, M.A. (Edin.), M.A. (Cantab.), Ph.D., F.G.S., Department of Geology, University of Cincinnati, Ohio, U.S.A.	
1886		Barclay, A. J. Gunion, M.A., 112 Chandos Avenue, Oakleigh Park, London, N.	
1903		Bardswell, Noël Dean, M.V.O., M.D., M.R.C.P. (Edin. and Lond.) (no permanent address until further notice)	
1922		* Barger, George, M.A., D.Sc., Dr *h. c.* (Padua), Hon. D.Sc. (Liverp.), Hon. M.D. (Heidelberg), LL.D. (Michigan), F.R.S., Regius Professor of Chemistry, University of Glasgow, W. 2., 76 John Street, Helensburgh	1925–28.
1929		* Barker, Sydney George, O.B.E., D.I.C., F.Inst.P., Scientific Advisor, Indian Jute Mills Association. 191 Coombe Lane, Wimbledon, London, S.W. 20	
1914	C.	* Barkla, Charles Glover, M.A., D.Sc., F.R.S., Professor of Natural Philosophy, University of Edinburgh (Drummond Street), Nobel Laureate, Physics, 1917. The Hermitage of Braid, Edinburgh	1915–18. 1924–27.
1937		* Barnett, Adam John Guilbert, B.Sc., Ph.D. (Edin.), Lecturer in Chemistry, Education Department, Lagos, Nigeria, B.W.A.	
1927		* Barnett, John, F.F.A., C.A., Scottish Widows' Fund Life Assurance Society, 9 St Andrew Square, Edinburgh 2	
1904		Barr, Sir James, C.B.E., M.D., LL.D., F.R.C.P., Hindhead Brae, Hindhead, Surrey	
1921		* Bartholomew, John, M.C., M.A., F.R.G.S., Geographical Institute, Duncan Street, Edinburgh. The Manor House, Inveresk	1925–28.
1927		* Bastow, Stephen Everard, M.Inst.E.E., M.Inst.Min.E. Northwood, Russell Place, Trinity, Edinburgh 5	
1929		* Bath, Frederick, Ph.D., Lecturer in Mathematics, University of Edinburgh (16 Chambers Street)	
1936		* Bayliss, Leonard Ernest, B.A., Ph.D., Lecturer in Biophysics, University of Edinburgh (Teviot Place). 52 Palmerston Place, Edinburgh 12	
1913		† Beard, Joseph, F.R.C.S.E., M.R.C.S., L.R.C.P., D.P.H. (Cantab.), formerly Medical Officer of Health, City of Carlisle. 8 Carlton Gardens, Carlisle	
1888		Beare, Sir Thomas Hudson, Kt., J.P., D.L., B.A., B.Sc., LL.D. (Edin.), M.Inst.C.E., Hon. M.I.Mech.E., Professor of Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 10 Regent Terrace, Edinburgh 7	1907–09. V-P 1909–15, 1923–26.
1897	C.	Beattie, Sir John Carruthers, K.B., D.Sc., LL.D., Vice-Chancellor and Principal, The University, Cape Town	
1893	C. M-B.	Becker, Ludwig, Ph.D., Emeritus Regius Professor of Astronomy, University of Glasgow. The Observatory, Dowanhill, Glasgow	
1933		* Begg, James Livingstone, F.G.S. (Treasurer, Geological Society of Glasgow). Elms, Mount Vernon, Glasgow	
1916		* Bell, Robert John Tainsh, M.A., D.Sc., LL.D. (Glas.), Professor of Mathematics, University of Otago, Dunedin, New Zealand	
1929		* Bennet, George, A.H.-W.C., B.Sc., A.M.I.Mech.E., Lecturer in Mechanical Engineering, Heriot-Watt College. 68 Arden Street, Edinburgh 10	
1936		Berry, John, M.A. (Cantab.), Ph.D. (St Andrews), Research Officer, Avon Biological Research, University College, Southampton. Tayfield, Newport, Fife	

Date of Election			Service on Council, etc.
1893	C.	Berry, Sir George A., M.B., C.M., LL.D., F.R.C.S.E., King's Knoll, North Berwick	1916–19. V-P 1919–22.
1897	C.	Berry, Richard J. A., M.D., F.R.C.S.E., Director of Medical Services, Stoke Park Colony, Stapleton, Bristol. Rufford, Canford Lane, Westbury-on-Trym, Bristol	
1932		* Bhatia, Sohan Lal, M.C., M.A., M.D. (Cantab.), M.R.C.P., Major, I.M.S., Professor of Physiology and Dean, Grant Medical College, Bombay. Two Gables, Mount Pleasant Road, Malabar Hill, Bombay, India	
1937		* Biswas, Kalipada, M.A., Curator of the Herbarium, Royal Botanic Garden, Calcutta, India	
1937		* Blackie, Joseph John, Ph.D. (Edin.), F.C.S., F.I.C., a Partner in Messrs Duncan, Flockhart and Co., Chemical Manufacturers. 104 Holyrood Road, Edinburgh 8	
1936		* Blair, Duncan MacCallum, M.B., Ch.B. (Glas.), D.Sc. (Lond.), Regius Professor of Anatomy, University of Glasgow. 2 The University, Glasgow, W. 2	
1933		* Bolam, Thomas Robert, M.Sc. (Bristol), D.Sc. (Edin.), Lecturer in Chemistry, University of Edinburgh (West Mains Road). 8 Wilton Road, Edinburgh 9	
1915		* Boon, Alfred Archibald, D.Sc., B.A., F.I.C., Emeritus Professor of Chemistry, Heriot-Watt College, Edinburgh	
1937	C.	* Born, Max, M.A. (Cantab.), Hon. D.Sc. (Bristol), Dr.phil. (Göttingen), Tait Professor of Natural Philosophy, University of Edinburgh (Drummond Street). 84 Grange Loan, Edinburgh 9	
1925		* Bose, Sahay Ram, M.A., D.Sc., F.L.S., Professor of Botany, Carmichael Medical College, Belgachia, Calcutta, India	
1886	C. N.	Bower, Frederick Orpen, M.A., D.Sc., LL.D., F.R.S., F.L.S., Emeritus Regius Professor of Botany, University of Glasgow. 2 The Crescent, Ripon, Yorks	1887–90, 1893–96, 1907–09, 1917–19. V-P 1910–16. P 1919–24.
1903	C.	Bradley, O. Charnock, M.D., D.Sc., Principal, Royal (Dick) Veterinary College, Edinburgh 9 (*Died 21st November, 1937*)	1907–10, 1915–17. V-P 1934–37.
1926		* Braid, Kenneth William, M.A. (Cantab.), B.Sc., Professor of Botany, West of Scotland Agricultural College, 6 Blythswood Square, Glasgow	
1907		† Bramwell, Edwin, M.D., LL.D., F.R.C.P. Edin. and Lond., formerly Professor of Clinical Medicine, University of Edinburgh (Royal Infirmary). 23 Drumsheugh Gardens, Edinburgh 3	1935–
1932		* Brash, James Couper, M.C., M.A., M.B., Ch.B. (Edin.), M.D. (Birm.), Professor of Anatomy, University of Edinburgh (Teviot Place)	1932–35.
1918		* Bremner, Alexander, M.A., D.Sc., formerly Headmaster, Demonstration School, Training Centre, Aberdeen. 13 Belgrave Terrace, Aberdeen	
1895		Bright, Sir Charles, M.Inst.C.E., M.Inst.E.E., F.R.Ae.S., F.Inst.Radio E., Little Brewers', Hatfield Heath, Harlow, Essex (*Died 20th November, 1937*)	
1893		Brock, G. Sandison, M.D., F.R.C.P.E., Greenbanks, St Saviour's, Jersey, C.I.	
1937		* Brook, George Bernard, F.I.C., F.C.S., Chief Chemist to the British Aluminium Co., Ltd., London. The Auld Manse, Onich, Inverness-shire	
1934		* Brough, Patrick, M.A., B.Sc. (Glas.), D.Sc. (Sydney), Lecturer in Botany, University of Sydney, N.S.W.	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, University, Cape Town	
1936		* Brown, Andrew Johnstone, F.R.C.S., L.R.C.P., L.D.S. (Edin.), Lecturer on Metallurgy, Tutor in charge of Anæsthetic Department and Demonstrator in Radiology, Edinburgh Dental Hospital and School. 51 Minto Street, Edinburgh 9	
1937		* Brown, Archibald Gray Robertson, F.F.A., Manager and Actuary of the Life Association of Scotland. Barnshalloch, Colinton, Midlothian	
1928		* Brown, Hugh Wylie, F.I.A., F.F.A., 1 Cobden Crescent, Edinburgh 9	
1924	C.	* Brown, Thomas Arnold, M.A., B.Sc., Professor of Mathematics, University College, Exeter	
1923		* Brown, Walter, M.A., B.Sc., Professor of Mathematics, University, Hong Kong, China	

Date of Election			Service on Council, etc.
1935		* Brownlie, James Law, M.D. (Glas.), D.P.H., M.R.C.P.E., formerly Chief Medical Officer, Department of Health for Scotland. C/o Mr Ross, Rannoch, 17 Ophir Road, Bournemouth, Hants	
1921		* Bruce, Alexander, B.Sc. (Edin.), Government Agricultural Chemist and City Analyst, The Laboratory, Turret Road S., Colombo, Ceylon	
1912		Bruce, Alexander Ninian, D.Sc., M.D., 8 Ainslie Place, Edinburgh 3	
1936		* Bruce, Sir Robert, Kt., D.L., J.P., LL.D., formerly Editor of the *Glasgow Herald*. Brisbane House, 9 Rowan Road, Glasgow, S. 1	
1898	C. K.	† Bryce, Thomas Hastie, M.A., M.D. (Edin.), LL.D., F.R.S., Emeritus Professor of Anatomy, University of Glasgow. The Loaning, Peebles	1911–14, 1922–25, 1935– V-P 1925–28.
1936		* Bryden, William, M.Sc., B.A., Ph.D. (Edin.), Union House, The University, Melbourne, S. Australia	
1887		† Burnet, Sir John James, R.A., R.S.A., LL.D., Corresponding Member, Institute of France, Woodhall Cottage, Woodhall Road, Colinton, Edinburgh	
1888		Burns, Rev. Thomas, C.B.E., J.P., D.D., F.S.A.Scot., Minister of Lady Glenorchy's Parish Church. Croston Lodge, Chalmers Crescent, Edinburgh 9	
1917		* Burnside, George Barnhill, M.Inst.Mech.E., Fairhill, Dullatur	
1930	C.	* Burt, David Raitt Robertson, B.Sc. (St Andrews), F.L.S., Lecturer in Zoology, Ceylon University College, Colombo	
1896		Butters, John W., M.A., B.Sc., formerly Rector of Ardrossan Academy. 116 Comiston Drive, Edinburgh 10	
1929	C.	* Calder, Alexander, Ph.D., Chief Marketing Officer, Pig Marketing Board, Thames House, Millbank, London, S.W. 1	
1910		Calderwood, Rev. Robert Sibbald, D.D., formerly Minister of Cambuslang. 84 Findhorn Place, Edinburgh 9	
1893	C.	Calderwood, W. L., I.S.O., formerly Inspector of Salmon Fisheries of Scotland. New Club, Princes Street, Edinburgh	1923–26.
1926	C. M-B.	* Cameron, Alfred Ernest, M.A., D.Sc. (Aberd.), Steven Lecturer in Agricultural and Forest Zoology, University of Edinburgh (10 George Square). 8 West Savile Road, Edinburgh 9	
1933		* Cameron, Finlay James, F.F.A., F.I.A., General Manager, Caledonian Insurance Company. Beech Knowe, Barnton, near Edinburgh	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S., formerly Professor of Anatomy, Dalhousie University, Halifax, Nova Scotia. Balmashanner, Grove Road, East Cliff, Bournemouth	
1921		* Campbell, Andrew, Advisory Chemist, c/o Burmah Oil Co., Ltd., Research Laboratory, Fairlawn, Honor Oak Road, Forrest Hill, London, S.E. 30 Foxgrove Road, Beckenham, Kent	
1918		* Campbell, John Menzies, D.D.S. (Toronto), L.D.S. (Glas.), L.D.S. (Ontario), F.I.C.D., 14 Buckingham Terrace, Glasgow, W.	
1915	C. N.	* Campbell, Robert, M.A., D.Sc., F.G.S., Reader in Petrology, University of Edinburgh (Grant Institute of Geology; West Mains Road). Maryton, Colinton	1920–23.
1927	C.	* Cannon, Herbert Graham, M.A., Sc.D. (Cantab.), D.Sc. (Lond.), M.Sc. (Manc.), F.R.S., F.L.S., Beyer Professor of Zoology, University of Manchester. Hollin Knowle, Chapel-en-le-Frith, Derbyshire	
1899	C.	Carlier, Edmund W. W., B. ès Sc., M.Sc., M.D., F.R.E.S., Emeritus Professor of Physiology, University of Birmingham. Morningside, Dorridge, near Birmingham	
1910		Carnegie, Col. David, C.B.E., J.P., M.Inst.C.E., The Haven, Seasalter, Whitstable	
1931		* Carroll, John Anthony, M.A., Ph.D. (Cantab.), Professor of Natural Philosophy, University of Aberdeen. Marischal College, Aberdeen	
1920	C.	* Carruthers, R. G., F.G.S., District Geologist, H.M. Geological Survey, High Barn, Stocksfield-on-Tyne	
1905	C.	Carse, George Alexander, M.A., D.Sc., Reader in Natural Philosophy, University of Edinburgh (Drummond Street). 3 Middleby Street, Edinburgh 9	
1901		Carslaw, Horatio Scott, M.A., Sc.D. (Cantab.), D.Sc., LL.D. (Glas.), Emeritus Professor of Mathematics, University of Sydney, New South Wales. Burradoo, New South Wales	

Date of Election			Service on Council, etc.
1933		* Carswell, John Irvine, B.Sc., Ph.D., A.M.Inst.C.E., A.M.Inst.Mech.E., Lecturer in Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 43 Mansionhouse Road, Edinburgh 9	
1925		* Carter, George Stuart, M.A., Ph.D., Corpus Christi College, Cambridge	
1932		* Cathcart, Edward Provan, C.B.E., M.D., D.Sc., LL.D., F.R.S., Professor of Physiology, University of Glasgow. 28 Hillhead Street, Glasgow	
1932	C.	* Childe, Vere Gordon, B.A., B.Litt., D.Litt. (Harvard), Hon. D.Sc. (Pennsylvania), F.R.A.I., F.S.A., Professor of Prehistoric Archæology, University of Edinburgh (14 Chambers Street)	
1925	C.	* Chumley, James, M.A., Ph.D., formerly Lecturer in Oceanography, Department of Zoology, University of Glasgow. Thalassa, Milton Road East, Portobello, Midlothian	
1928	C.	* Clark, Alfred Joseph, M.C., B.A., M.D., F.R.S., Professor of Materia Medica, University of Edinburgh (Teviot Place). 67 Braid Avenue, Edinburgh 10	1932–35.
1933		* Clark, Arthur Melville, M.A. (Edin.), D. Phil. (Oxon.), Lecturer in English Literature, University of Edinburgh. 34 Bruntsfield Gardens, Edinburgh 10	
1891		Clark, John Brown, C.B.E., J.P., M.A., LL.D., formerly Head Master of George Heriot's School. Garleffin, 146 Craiglea Drive, Edinburgh 10	1928–1931. V-P 1931–34.
1935		* Clark, Robert Selbie, M.A., D.Sc. (Aberd.), Scientific Superintendent, Fishery Board for Scotland. The Cottage, Murtle, Aberdeenshire	
1932		* Clark, Sir Thomas, Bart., Publisher, Head of T. & T. Clark, Ltd. 6 Wester Coates Road, Edinburgh 12	
1903		Clarke, William Eagle, I.S.O., LL.D., F.L.S., Honorary Supervisor of the Bird Collection and formerly Keeper of the Natural History Collections, Royal Scottish Museum, Edinburgh	
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Greenesfield House, Gateshead-on-Tyne	
1932		* Clouston, David, C.I.E., M.A., B.Sc. (Agric.), D.Sc., formerly Director, Imperial Agricultural Research Institute, Pusa. Forthview, Boswall Road, Edinburgh 5	
1936	C.	* Cockburn, Alexander Murray, Ph.D. (Edin.), Assistant, Geological Department, University of Edinburgh (Grant Institute of Geology, West Mains Road). 53 Ladysmith Road, Edinburgh 9	
1904	C.	Coker, Ernest George, M.A. (Cantab.), D.Sc. (Edin.), Hon. D.Sc. (Sydney and Louvain), M.Sc. (McGill), F.R.S., M.Inst.C.E., M.I.Mech.E., Emeritus Professor of Civil and Mechanical Engineering, University of London. Engineering Laboratories, 3 Farnley Road, Chingford, London, E. 4	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	V. J. C.	Collie, John Norman, Ph.D., D.Sc., LL.D., F.R.S., F.C.S., F.I.C., Emeritus Professor of Organic Chemistry, University College, Gower Street, London. 20 Gower Street, London, W.C. 1	
1909	C.	Comrie, Peter, M.A., B.Sc., LL.D., formerly Rector, Leith Academy. 19 Craighouse Terrace, Edinburgh 10	
1936		* Cooper, The Rt. Hon. Thomas Mackay, O.B.E., K.C., M.P., M.A., LL.B., His Majesty's Advocate for Scotland. 7 Abercromby Place, Edinburgh 3	
1924	C.	* Copson, Edward Thomas, M.A. (Oxon.), D.Sc. (Edin.), Professor of Mathematics, University College, Dundee (University of St Andrews). 14 Balmyle Road, Broughty Ferry, Dundee	
1937		* Cousland, Charles Johnstone, formerly President, Royal Scottish Society of Arts. Achray, Kinnear Road, Edinburgh 4	
1928		* Coutie, Rev. Alexander, Ph.D., 13 Mayfield Gardens, Edinburgh 9	
1914		* Coutts, William Barron, M.A., B.Sc., Senior Lecturer in Range Finding and Optics, Military College of Science, Red Barracks, Woolwich, S.E. 18. 11 Coleraine Road, Blackheath, S.E. 3	
1911		Cowan, Alexander, M.A. (Cantab.), Papermaker, Valleyfield, Penicuik, Midlothian	
1931		* Cowan, John Macqueen, M.A., D.Sc. (Edin.), B.A. (Oxon.), F.L.S., Assistant Keeper, Royal Botanic Garden, Edinburgh. 17 Inverleith Place, Edinburgh 4	
1935		* Cowan, Samuel Hunter, Lt.-Col. R. E., Lecturer in Forestry Engineering, University of Edinburgh (George Square). New Club, Edinburgh 2	

Date of Election			Service on Council, etc.
1916	C.	† Craig, E. H. Cunningham, B.A. (Cantab.), Geologist and Mining Engineer, The Dutch House, Beaconsfield	
1908		Craig, James Ireland, M.A., B.A., 88 Sharia Kasr el Eini, Cairo, Egypt	
1925	C. K.	* Craig, Robert Meldrum, M.A., D.Sc., F.G.S., Lecturer in Economic Geology, University of Edinburgh (Grant Institute of Geology, West Mains Road)	
1937		* Craig, William Stuart McRae, B.Sc. (Glas.), M.D. (Edin.), F.R.C.P.E., Medical Officer, Ministry of Health, Whitehall, London, S.W. 1. 7 Western Gardens, Ealing Common, London, W. 5	
1933		* Craig-Bennett, Arthur Lancelot, M.A., Ph.D. (Cantab.), Chief Fisheries Officer, Haifa, Palestine	
1903		† Crawford, Lawrence, M.A., D.Sc., Professor of Pure Mathematics, University, Cape Town	
1922	C.	* Crew, Francis Albert Eley, M.D., D.Sc., Ph.D. (Vice-President), Professor of Animal Genetics and Director of the Institute of Animal Genetics, University of Edinburgh (West Mains Road). 41 Mansionhouse Road, Edinburgh 9	1928–31. Sec. 1931–36. V-P 1936–
1931		* Crichton, John, M.A., B.Sc. (Edin.), 17 Willows Avenue, Morden, Surrey	
1870		Crichton-Browne, Sir James, Kt., M.D., LL.D., D.Sc., F.R.S., Vice-President of the Royal Institution of Great Britain. 45 Hans Place, London, S.W. 1	
1936		* Cronshaw, Cecil John Turrell, B.Sc. (Vict.), Managing Director, Dyestuffs Group of Imperial Chemical Industries, Ltd. (including Scottish Dyes, Ltd.). Hexagon House, Blaikley, Manchester	
1929		* Cruickshank, Ernest William Henderson, M.D., D.Sc., Ph.D., Regius Professor of Physiology, University of Aberdeen	
1914		* Cumming, Alexander Charles, O.B.E., D.Sc., Red Latches, King's Drive, Caldy, Cheshire	
1928		* Cumming, William Murdoch, D.Sc. (Glas.), F.I.C., M.Inst.Chem.E., "Young" Professor of Technical Chemistry, Royal Technical College, Glasgow. Bonnieblink, 4 Newlands Road, Newlands, Glasgow, C. 1	
1917		* Cunningham, Brysson, D.Sc., B.E., M.Inst.C.E., Editor *The Dock and Harbour Authority*, formerly Lecturer in Waterways, Harbours, and Docks, University College, London. 141 Copers Cope Road, Beckenham, Kent	
1930		* Cunningham, John, C.I.E., B.A., M.D., Lt.-Colonel, I.M.S. (retired). South Bank, Grange Loan, Edinburgh 10	
1934		* Daly, Ivan de Burgh, M.A., M.D., B.Ch., Professor of Physiology, University of Edinburgh (Teviot Place). Cooliney, Spylaw Avenue, Edinburgh	1935–
1934		* Darling, Frank Fraser, Ph.D. (Edin.), N.D.A., Dundonnell, by Garve, Wester Ross	
1924	V. J.	* Darwin, Charles Galton, M.A., Sc.D., F.R.S., formerly Tait Professor of Natural Philosophy, University of Edinburgh, Master of Christ's College, Cambridge	1925–28. Sec. 1928–33. V-P 1933–36.
1935		* Davidson, Charles Findlay, B.Sc. (St Andrews), F.G.S., Geologist, H.M. Geological Survey of Great Britain, Geological Survey and Museum, Exhibition Road, South Kensington, London, S.W. 7	
1932	C.	* Davidson, Leybourne Stanley Patrick, B.A. (Cantab.), M.D. (Edin.), F.R.C.P.E., Regius Professor of Medicine, University of Aberdeen. 55 Queen's Road, Aberdeen	
1935		* Davidson, Maxwell, B.Sc. (Eng.), Ph.D., M.I.Mech.E., Lecturer in Heat Engines and Thermodynamics, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 47 Liberton Drive, Edinburgh 9	
1930	C.	* Davies, Lewis Merson, M.A., F.G.S., F.R.A.I., Lt.-Colonel, Royal Artillery (retired). 8 Garscube Terrace, Murrayfield, Edinburgh 12	
1928		Dawson, Warren Royal, F.R.S.L., F.S.A.Scot., Honorary Librarian of Lloyd's, London, E.C. 3. Simpson House, Simpson, Bletchley, Bucks	
1923		* Deane, Arthur, M.R.I.A., Curator, Public Art Gallery and Museum, Belfast. Threave, 57 Cranmore Park, Belfast	
1894		† Denny, Sir Archibald, Bart., LL.D., 5 St Helen's Place, London, E.C. 4	
1935		* Desai, Bhimbhai Nichhabhai, B.A., M.Sc. (Bombay), Ph.D. (Edin.), Assistant Meteorologist, Government of India, Meteorological Office, Karachi-Saddar (Sind), India	

Date of Election			Service on Council, etc.
1925		Dey, Alexander John, Managing Director of T. & H. Smith, Ltd., Manufacturing Chemists, Edinburgh. Rothiemay, Corstorphine, Edinburgh 12	
1937		* Dhar, Sasindra Chandra, D.Sc. (Edin.), Head of the Department of Mathematics, College of Science, Nagpur, C.P., India	
1924		* Dinham, C. H., B.A., H.M. Geological Survey. Edgemoor, 19 Highfield Road, Northwood, Middlesex	
1923		* Dixon, Ronald Audley Martineau, of Thearne, F.G.S., F.S.A.Scot., F.R.G.S., Thearne Hall, near Beverley	
1881	C.	Dobbin, Leonard, Ph.D. (CURATOR OF LIBRARY AND MUSEUM), formerly Reader in Chemistry, University of Edinburgh. Faladam, Blackshiels, Midlothian	1904–07, 1913–16. Curator 1934–
1918		* Dodd, Alexander Scott, Ph.D., F.I.C., F.C.S., City Analyst for Edinburgh. 20 Stafford Street, Edinburgh 3	
1925		* Donald, Alexander Graham, M.A., F.F.A., F.S.A.Scot., Secretary of the Scottish Provident Institution, Edinburgh. 18 Carlton Terrace, Edinburgh 7	
1937	C.	* Donald, Hugh Paterson, B.Sc., M.Agr.Sc. (N.Z.), Ph.D. (Edin.), Research Assistant, Institute of Animal Genetics, University of Edinburgh (West Mains Road)	
1882	C.	Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1921	M-B. C.	* Dougall, John, M.A., D.Sc., 47 Airthrey Avenue, Glasgow, W. 4	
1901 & 1918		Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow. Clonbeith, Milngavie, by Glasgow	
1910		Douglas, Loudon MacQueen, Newpark, Mid-Calder, Midlothian	
1934		* Dow, David Rutherford, M.D., D.P.H., F.R.C.P.E., Professor of Anatomy, University of St Andrews (University College, Dundee). 16 Windsor Street, Dundee	
1932		* Drennan, Alexander Murray, M.D. (Edin.), F.R.C.P.E., Professor of Pathology, University of Edinburgh (Teviot Place)	
1923	C.	* Drever, James, M.A., B.Sc., D.Phil., Professor of Psychology, University of Edinburgh (South Bridge). Ivybank, Wardie Road, Edinburgh 5	1929–32.
1901		Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall, Edinburgh	
1923		* Drummond, J. Montagu F., M.A. (Cantab.), Harrison Professor of Botany, University of Manchester	1928–31.
1925		* Dryerre, Henry, M.R.C.S., Ph.D., Professor of Physiology, Royal (Dick) Veterinary College; Physiological Biochemist, Animal Diseases Research Association. Kenmore, Lasswade	
1921		* Drysdale, Charles Vickery, C.B., O.B.E., D.Sc. (Lond.), M.I.E.E., F.Inst.P., formerly Director of Scientific Research to the Admiralty. No. 151 Wick Hall, Furge Hill, Hove, Sussex	
1937	C.	* Dunlop, Derrick Melville, B.A. (Oxon.), M.D. (Edin.), F.R.C.P.E., Christison Professor of Therapeutics, University of Edinburgh (Teviot Place). 1 Moray Place, Edinburgh 3	
1904		Dunlop, William Brown, M.A., 4A St Andrew Square, Edinburgh 2. Seton Castle, Longniddry, E. Lothian	
1892	C.	Dunstan, Malcolm James Rowley, M.A., F.I.C., F.C.S., formerly Principal, Royal Agricultural College, Cirencester. Windyacres, Wrotham, Kent	
1933		* Dymond, Edmund Gilbert, M.A., Lecturer in Natural Philosophy, University of Edinburgh (Drummond Street). 7 Greenhill Gardens, Edinburgh 10	
1906	C.	Dyson, Sir Frank Watson, K.B.E., M.A., D.Sc., LL.D., F.R.S., formerly Astronomer Royal, Royal Observatory, Greenwich. 27 Westcombe Park Road, Blackheath, S.E. 3	
1925		* Eastwood, George Samuel, B.Sc., Principal Teacher of Mathematics, Beath Secondary School, Cowdenbeath, Fife. Craigie Lea, Cowdenbeath, Fife	
1934		* Edge, William Leonard, M.A., Sc.D. (Cantab.), Lecturer in Mathematics, University of Edinburgh (16 Chambers Street)	
1931		* Eggleton, Philip, D.Sc., Lecturer in Biochemistry, Department of Physiology, University of Edinburgh (Teviot Place). 5 Corbiehill Avenue, Davidson's Mains	

Date of Election		
1924		* Elliot, Right Hon. Walter Elliot, P.C., M.C., M.B., Ch.B., D.Sc., LL.D., M.P., F.R.S., Secretary of State for Scotland. 17 North Street, London, S.W. 1
1933		* Erskine, John Maxwell, General Manager of the Commercial Bank of Scotland, Ltd. Hazlieburn, West Linton
1934		* Etherington, Ivor Malcolm Haddon, B.A. (Oxon.), Ph.D. (Edin.), Lecturer in Mathematics, University of Edinburgh (16 Chambers Street). 41 Scotland Street, Edinburgh 3
1924		* Evans, Arthur Humble, M.A., Sc.D., Lecturer in English History. Cheviot House, Crowthorne, Berks
1924		Evans, William Edgar, B.Sc., Assistant in charge of Herbarium, Royal Botanic Garden, Edinburgh 4
1902		Ewen, John Taylor, O.B.E., J.P., B.Sc., M.I.Mech.E., H.M. Inspector of Schools (Emeritus), Pitscandly, Forfar
1900	C.	Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Cantab.), Emeritus Professor of Bacteriology, Guy's Hospital. 51 Portland Place, London, W. 1
1931		* Fairbairn, William Ronald Dodds, M.A., M.D., D.Psych. (Edin.), F.R.A.I. 18 Lansdowne Crescent, Edinburgh 12
1936		* Fairweather, James Falconer, W.S., N.P., Fiscal of the Society of Writers to the Signet. 14 Henderland Road, Edinburgh 12
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., formerly Director of the Geological Survey of Nigeria. The Cedars, Hatton Road, Harlington, Hayes, Middlesex
1923		* Feldman, William Moses, M.D., B.S., F.R.C.P., F.R.A.S., Senior Physician, St Mary's Hospital for Women and Children, Plaistow. 851 Finchley Road, London, N.W. 11
1928		* Fenton, Edward Wyllie, M.A., D.Sc. (Aberd.), F.L.S., Head of Botany Department, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 8
1907		Fergus, Edward Oswald, c/o Messrs M'Kay & Boyd, Solicitors, 50 Wellington Street, Glasgow
1933		* Ferguson, Thomas, M.D., D.P.H. (Edin.), F.R.C.P.E., D.Sc., Medical Officer, Department of Health for Scotland. 4 Park Street, Falkirk
1925	C.	* Ferrar, William Leonard, M.A., Fellow and Tutor of Hertford College, Oxford
1932		* Findlay, Sir John Edmund Ritchie, Bart., B.A. (Oxon.), M.P., Proprietor of the *Scotsman*. Aberlour House, Aberlour, Banffshire
1927	C.	* Finlay, Thomas Matthew, M.A., D.Sc. (Edin.), Lecturer in Palæontology, University of Edinburgh (Grant Institute of Geology, West Mains Road). 11 Dudley Terrace, Leith 6
1911		† Fleming, John Arnold, F.C.S., Pottery Manufacturer, Locksley, Helensburgh, Dumbartonshire
1906		Fleming, Robert Alexander, M.A., M.D., LL.D., F.R.C.P.E., Consulting Physician, Royal Infirmary. 10 Chester Street, Edinburgh 3
1900	C. N.	Flett, Sir John S., K.B.E., M.A., D.Sc., LL.D., F.R.S., formerly Director of the Geological Survey of Great Britain and Museum of Practical Geology, Exhibition Road, South Kensington, London, S.W. 7
1892		Ford, John Simpson, F.I.C., 7 Corrennie Drive, Edinburgh 10
1921		* Forrest, George Topham, F.R.I.B.A., F.G.S., formerly Architect to the London County Council. Devonshire Club, 50 St James's Street, London, S.W. 1
1928	C.	* Forrest, James, M.A., B.Sc. (Glas.), D.Sc. (St Andrews), Lecturer in Physics, University College, Dundee. Cumbrae, Oxford Street, Blackness, Dundee
1933		* Forrester, Charles, A.H.-W.C., Ph.D., F.I.C., F.Inst.F., Principal and Professor of Chemistry, Indian School of Mines, Dhanbad, India
1920	C.	* Franklin, Thomas Bedford, B.A. (Cantab.), 28 Kingshill Drive, Kenton, Middlesex
1910		Fraser, Alexander, Actuary, 5 St Margaret's Road, Edinburgh 9
1929		* Fraser, David Kennedy, M.A., B.Sc., Psychologist to Glasgow Education Authority. Edge o' the Moor, Milngavie, Dumbartonshire
1934		* Fraser, George, Chartered Civil Engineer, M.Inst.C.E., M.I.Struct.E. 25 Murrayfield Gardens, Edinburgh 12
1928		* Fraser, Sir John, K.C.V.O., M.C., M.D., Ch.M., F.R.C.S.E., Regius Professor of Clinical Surgery, University of Edinburgh (Royal Infirmary). 20 Moray Place, Edinburgh 3
1928		* Fraser, Kenneth, M.D. (Edin.), D.P.H. (Cantab.), D.T.M. (Edin.), County Medical Officer of Health, Cumberland. The Croft, Scotby, near Carlisle

Date of Election			Service on Council, etc.
1914		* Fraser, William, Managing Director, Neill & Co., Ltd., Printers, 212 Causewayside, Edinburgh 9	
1907		Galbraith, Alexander, Ravenswood, Dalmuir, Glasgow	
1933		* Galbraith, Augustus William de Rohan, M.Inst.C.E., M.Inst.C.E.I., F.S.E., City Engineer, Christchurch, New Zealand. The Spur, Sumner, New Zealand	
1888	C.	Galt, Alexander, D.Sc., formerly Keeper of the Department of Technology, Royal Scottish Museum, Edinburgh. C/o Clydesdale Bank, 1 Melville Place, Edinburgh 3	
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Jaipur, India	
1933		* Gardner, Alfred Charles, M.Inst.C.E., M.Inst.Mech.E., M.Inst.E.E., F.G.S., Chief Engineer, Clyde Navigation Trust, 16 Robertson Street, Glasgow. 117 Fotheringay Road, Glasgow, S. 1	
1926		* Gardner, John Davidson, B.Sc., A.M.Inst.C.E., Chief Assistant to D. A. Stevenson, Civil Engineer, Engineers to the Commissioners of Northern Lighthouses, Edinburgh. 84 George Street, Edinburgh 2	
1937		* Garry, Robert Campbell, M.B., Ch.B., D.Sc. (Glas.), Professor of Physiology, University College (University of St Andrews), Dundee. 58 Seafield Road, Broughty Ferry, Dundee	
1930	C.	* Geddes, Alexander Ebenezer M'Lean, O.B.E., M.A., D.Sc., Lecturer in Natural Philosophy, University of Aberdeen. 12 Louisville Avenue, Aberdeen	
1909	C.	† Geddes, Rt. Hon. Sir Auckland C., P.C., G.C.M.G., K.C.B., M.D., LL.D. Frensham, The Layne, Rolvenden, Kent	
1909		† Gentle, William, B.Sc., Head Master, George Heriot's School. 10 West Savile Road, Edinburgh 9	
1914		* Gibb, Sir Alexander, G.B.E., C.B., F.R.S., M.Inst.C.E., Queen Anne's Lodge, Westminster, London, S.W. 1	
1910	C.	Gibb, David, M.A., B.Sc., Reader in Mathematics, University of Edinburgh (16 Chambers Street). 45 Fountainhall Road, Edinburgh 9	
1936		* Gibbons, Sydney Guy, Ph.D. (Lond.), Naturalist, Fishery Board for Scotland, Aberdeen. Dunmore, Milltimber, Aberdeenshire	
1917	C.	* Gibson, Alexander, M.B., Ch.B., F.R.C.S., 620 Medical Arts Building, Winnipeg, Canada	
1921		* Gibson, Walcot, D.Sc., F.R.S., F.G.S., formerly Assistant Director, H.M. Geological Survey (Scotland). Pathways, Fairlight Road, Hythe, Kent	
1911		Gidney, Sir Henry A. J., Kt., J.P., M.L.A., F.R.C.S.E., Lt.-Col., I.M.S. (retired), c/o The Allahabad Bank, Ltd., Calcutta, India	
1937		* Gilchrist, Andrew Rae, M.D. (Edin.), F.R.C.P.E., Assistant Physician, Royal Infirmary, Edinburgh, and Lecturer on Therapeutics, University of Edinburgh (Teviot Place). 6 Lansdowne Crescent, Edinburgh 12	
1933		* Gillespie, Robert Pollock, M.A., B.Sc., Ph.D. (Cantab.), Lecturer in Mathematics, University of Glasgow. Ashcot, Kilbarchan Road, Johnstone, Renfrewshire	
1933		* Gillespie, Thomas Haining, Director-Secretary, Zoological Society of Scotland. Corstorphine Hill House, Murrayfield, Edinburgh 12	
1925		* Gillies, William King, M.A., B.A., F.E.I.S., LL.D. (Glas.), Rector of the Royal High School, Edinburgh. Davaar, 12 Suffolk Road, Edinburgh 9	
1909		Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Penpont, Dumfriesshire	
1911		Gladstone, Reginald John, M.D., F.R.C.S., Lecturer and Senior Demonstrator of Anatomy, King's College, University of London. 22 Court Lane Gardens, London, S.E. 21	
1934		* Glaister, John, M.D., D.Sc. (Glas.), Professor of Forensic Medicine, University of Glasgow. 5 Kew Terrace, Glasgow, W. 2	
1925	C.	* Goldie, Archibald Hayman Robertson, M.A., B.A., D.Sc., Superintendent, Meteorological Office, Air Ministry, Edinburgh. 6 Drumsheugh Gardens, Edinburgh 3	1929–32.
1901		Goodwillie, James, M.A., B.Sc., 239 Clifton Road, Aberdeen	
1913	C. M-B.	Gordon, William Thomas, M.A., D.Sc. (Edin.), M.A. (Cantab.), Professor of Geology, University of London, King's College, Strand, W.C.	
1897		Gordon-Munn, John Gordon, M.D., Croys, Castle Douglas	
1934		* Gorrie, Robert Maclagan, D.Sc. (Edin.), Silviculturalist to Punjab Government. C/o Chief Conservator of Forests, Lahore, Punjab, India	

Date of Election			Service on Council, etc.
1923		* Grabham, George Walter, O.B.E., M.A. (Cantab.), F.G.S., Government Geologist, Anglo-Egyptian Sudan. Box 178, Khartoum	
1924		Graham, Robert James Douglas, M.A., D.Sc., Professor of Botany, University of St Andrews	
1931		* Grant, Robert, Publisher (Oliver & Boyd), Edinburgh. 6 Kilgraston Road, Edinburgh 10	
1935	C.	* Grant, Ronald, Ph.D. (Edin.), Department of Zoology, University of Chicago, Ill., U.S.A.	
1918		* Gray, William Forbes, F.S.A.Scot., 8 Mansionhouse Road, Edinburgh 9	
1937	C.	* Green, George, M.A., D.Sc. (Glas.), Lecturer in Applied Physics, University of Glasgow. 64 Partickhill Road, Glasgow, W. 1	
1927	C. K.	* Greenwood, Alan William, D.Sc. (Melb.), Ph.D. (Edin.), Lecturer in the Institute of Animal Genetics, University of Edinburgh (West Mains Road)	
1922		* Greenwood, Rev. William Osborne, M.D. (Leeds), B.S. (Lond.), L.S.A., Clerk in Holy Orders, Woodroyd, 19 Ripon Road, Harrogate, Yorks	
1906		Greig, Edward David Wilson, C.I.E., M.D., D.Sc., Lt.-Col., I.M.S. (retired), 38 Coates Gardens, Edinburgh 12	
1931		* Greig, John Russell, Ph.D. (Edin.), Director, Moredun Institute, Animal Diseases Research Association. Wedderlie, Kirkbrae, Liberton 9	
1905		† Greig, Sir Robert Blyth, M.C., LL.D., formerly Secretary to the Department of Agriculture for Scotland. The Shaws, Barnton, Midlothian	1921–24. V-P 1924–27.
1935		* Grierson, Alexander Millar Meek, M.D. (Edin.), D.P.H. (Edin. and Glas.), Senior Assistant to the Medical Officer of Health, Manchester	
1910		Grimshaw, Percy Hall, I.S.O., F.R.E.S., formerly Keeper, Natural History Department, Royal Scottish Museum. 133 Liberton Brae, Edinburgh 9	
1899		† Guest, Edward Graham, J.P., M.A., B.Sc., 5 Newbattle Terrace, Edinburgh 10	
1927		* Gulland, John Masson, M.A. (Oxon.), D.Sc. (Edin.), Ph.D. (St Andrews), Sir Jesse Boot Professor of Chemistry, University College, Nottingham	
1907		Gulliver, Gilbert Henry, D.Sc., A.M.I.Mech.E., 99 Southwark Street, London, S.E.	
1930		* Guthrie, Douglas, M.D., F.R.C.S., Lecturer in Diseases of the Ear, Nose, and Throat, School of Medicine of the Royal Colleges, Edinburgh. 21 Clarendon Crescent, Edinburgh 4	
1933	C.	* Guthrie, William Gilmour, M.A. (Edin.), B.A. (Cantab.), Ph.D., Professor of Mathematics and Natural Philosophy, Magee College, Londonderry, Northern Ireland	
1911		Guy, William, F.R.C.S., L.R.C.P., L.D.S.Ed., LL.D. (Penn.), Consulting Dental Surgeon, Edinburgh Royal Infirmary; Lecturer on Human and Comparative Dental Anatomy and Physiology. 11 Wemyss Place, Edinburgh 3	
1934		* Haldane, David, B.Sc. (Edin.), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9. 6 Kilmaurs Road, Edinburgh 9	
1936	C.	* Hamilton, William James, D.Sc. (Glas.), M.D., B.Ch. (Belfast), Professor of Anatomy, St Bartholomew's Hospital Medical College, London, E.C. 1	
1922		* Hannay, Robert Kerr, M.A., LL.D., H.R.S.A., Fraser Professor of Scottish History and Palæography, University of Edinburgh (South Bridge). Historiographer-Royal for Scotland. 5 Royal Terrace, Edinburgh 7	
1923		* Hanneford-Smith, William, A.M.Inst.C.E., Hon. A.R.I.B.A. 1 The Avenue, Gravesend, Kent	
1918		* Hardie, Patrick Sinclair, M.A., B.Sc., formerly Head of the Physics Department, Medical School, Cairo. 19 Ardmillan Terrace, Edinburgh 11	
1928		* Harding, William Gerald, F.R.Hist.S., F.S.A.Scot., F.R.E.S., Peckwater House, Charing, Kent	
1923	C.	* Harris, Robert Graham, M.A., D.Sc. (Edin.), 44 Manor Road, Farnborough, Hants	
1914		Harrison, Edward Philip, Ph.D., F.Inst.P., Chief Scientist, H.M.S. "Vernon," Portsmouth	
1934		Harrison, John Vernon, D.Sc.(Glas.), F.G.S., 34 Rowallan Gardens, Glasgow, W. 1	
1921		* Harrison, John William Heslop, D.Sc. (Durham), F.R.S., Professor of Botany and Reader in Genetics, King's College, Newcastle-upon-Tyne. The Avenue, Birtley, Co. Durham	

Date of Election			Service on Council, etc.
1926		* Harvey, William Frederick, C.I.E., M.A., M.B., C.M., D.P.H., Lieut.-Col., I.M.S. (retired), Histologist, Research Laboratory, Royal College of Physicians, Edinburgh. 56 Garscube Terrace, Edinburgh 12	
1936		* Henderson, David Kennedy, M.D., F.R.C.P.E., F.R.F.P.S.G., Professor of Psychiatry, University of Edinburgh. Tipperlinn House, Edinburgh 10	
1931		* Henderson, John, F.C.I.I., Manager and Secretary, Edinburgh Assurance Co., Ltd. Seaforth Cottage, York Road, Trinity, Edinburgh 5	
1936		* Henderson, Thomas, B.Sc. (Lond.), Secretary to the Educational Institute of Scotland. 2 Hillview Terrace, Edinburgh 12	
1929		* Henderson, Thomas, C.B.E., J.P., F.S.A.Scot., Actuary of the Savings Bank of Glasgow. 5 Belmont Crescent, Glasgow, W. 2	
1908		Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Bristol. 77 Coldharbour Road, Bristol 6	
1937		* Hepburn, William Allan Forsyth, M.C., M.A., B.Ed. (Edin.), Director of Education, Ayrshire Education Authority. 2 St Leonard's Road, Ayr	
1925		* Heron, Alexander Macmillan, D.Sc. (Edin.), F.G.S., Director, Geological Survey of India, Calcutta, India	
1916		* Herring, Percy Theodore, M.D., F.R.C.P.E., Professor of Physiology, University of St Andrews. Linton, St Andrews	1917–20, 1931–34. V-P 1934–37.
1936		* Hewat, Andrew Fergus, M.D., F.R.C.P.E., Secretary, Royal College of Physicians, Edinburgh. 14 Chester Street, Edinburgh 3	
1922		Hindle, Edward, M.A., Sc.D. (Cantab.), Ph.D., A.R.C.S., Regius Professor of Zoology, University of Glasgow	
1904		Hobday, Sir Frederick T. G., Kt., C.M.G., Dr. Med. Vet. (Zurich), F.R.C.V.S., Principal, Royal Veterinary College, Camden Town, London, N.W. 1. 31 Argyll Road, Kensington, London, W. 8	
1928	C.	* Hobson, Alfred Dennis, M.A. (Cantab.), Professor of Zoology, King's College, Newcastle-upon-Tyne	
1928		* Hodge, William Vallance Douglas, M.A. (Edin.), M.A. (Cantab.), Lowndean Professor of Astronomy and Geometry, University of Cambridge. 28 Barrow Road, Cambridge	
1923	C. K.	* Hogben, Lancelot Thomas, M.A., D.Sc., F.R.S., Professor of Natural History, University of Aberdeen	1937–
1927		Holden, Henry Smith, D.Sc., F.L.S., Director of the East Midlands Forensic Science Laboratory, Burton Street, Nottingham	
1930		* Holland, Sir Thomas Henry, K.C.S.I., K.C.I.E., D.L., Hon. D.Sc., LL.D., F.R.S., Vice-Chancellor and Principal of the University of Edinburgh. Blackford Brae, Edinburgh 9	1931–32. V-P 1932–35.
1929	C.	* Hora, Sunder Lal, D.Sc. (Punjab et Edin.), F.L.S., F.Z.S., F.A.S.B., Senior Assistant Superintendent, Zoological Survey of India. Indian Museum, Calcutta	
1920	C.	* Horne, Alexander Robert, O.B.E., B.Sc., M.I.Mech.E., A.M.Inst.C.E., Professor of Mechanical Engineering, Heriot-Watt College, Edinburgh. 31 Queen's Crescent, Edinburgh 9	
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., Shelley Hall, Huddersfield.	
1912	C. M-B.	† Houstoun, Robert Alexander, M.A., Ph.D., D.Sc., F.Inst.P., Lecturer in Physical Optics, University of Glasgow. 45 Kirklee Road, Glasgow	1929–32.
1893		Howden, Robert, M.A., M.B., C.M., D.Sc., LL.D., Emeritus Professor of Anatomy, University of Durham. Rockearn, Perth Road, Crieff	
1933		* Hume, Edgar Erskine, D.S.M., M.A., M.D., LL.D., Lieut.-Col., U.S. Army, Librarian of the Army Medical Library, Washington. The Magnolias, Frankfort, Kentucky	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwân, Egypt. The Laurels, Rustington, Sussex	
1927		* Hunt, Owen Duke, B.Sc. (Manch.), Corrofell, Newton Ferrers, South Devon	
1923		* Hunter, Rev. Adam Mitchell, M.A., D.Litt., Librarian of New College, Edinburgh. 3 Suffolk Road, Edinburgh 9	
1932		* Hunter, Andrew, M.A., B.Sc., M.B., Ch.B., F.R.S.C., F.R.F.P.S.G., Professor of Pathological Chemistry, University of Toronto, Canada	
1928		* Hunter, Arthur, F.F.A., LL.D. (Edin.), Vice-President and Chief Actuary of the New York Life Insurance Co. 124 Lloyd Road, Montclair, N.J., U.S.A.	

Date of Election			Service on Council, etc.
1916		* Hunter, Charles Stewart, M.A., L.R.C.P.E., L.R.C.S.E., D.P.H., F.R.E.S., Cotswold, 36 Streatham Hill, London, S.W. 2	
1911		Hunter, Gilbert Macintyre, M.Inst.C.E., M.Inst.E.S., M.Inst.M.E., 27 Kilmaurs Road, Edinburgh 9	
1935		* Hutchinson, Arthur Cyril William, M.D.S. (Manch.), D.D.S. (Witwatersrand), Dean of the Edinburgh Dental Hospital and School. 12 Glencairn Crescent, Edinburgh 12	
1923	C.	* Ince, Edward Lindsay, M.A. (Cantab.), D.Sc. (Edin.), Lecturer in Technical Mathematics, University of Edinburgh (16 Chambers Street)	
1920		* Inglis, James Gall, Publisher. 36 Blacket Place, Edinburgh 9	
1927		* Inglis, John Alexander, of Auchindinny and Redhall, K.C., M.A. (Oxon.), LL.B. (Edin.), King's and Lord Treasurer's Remembrancer. 13 Randolph Crescent, Edinburgh 3	1935–
1912		Inglis, Robert John Mathieson, M.Inst.C.E., Chief Engineer, Southern Area, L.N.E.R. Dixton, Monken Hadley, Herts	
1936		* Innes, Donald Esme, M.C., M.A. (Oxon.), Professor of Geology, University of St Andrews. Cortina, St Andrews	
1917		* Irvine, Sir James Colquhoun, Kt., C.B.E., D.L., Ph.D. (Leipzig), D.Sc. (St Andrews), Hon. D.Sc. (Liverpool, Princeton), Hon. Sc.D. (Cantab., Yale, Pennsylvania), Hon. LL.D. (Glas., Aberd., Edin., and Toronto), Hon. D.C.L. (Durham), F.R.S., Hon. F.E.I.S., Vice-Chancellor and Principal of the University of St Andrews	1920–22. V-P 1922–25.
1930	C.	* Jack, David, M.A., B.Sc. (Edin.), Ph.D. (St Andrews), Lecturer in Natural Philosophy, United College, University of St Andrews. 22 Grange Road, St Andrews	
1923		* Jack, John Louttit, C.B.E., Solicitor, Assistant Secretary, Department of Health for Scotland, 121A Princes Street, Edinburgh 2	
1912	C.	Jeffrey, George Rutherford, M.D. (Glas.), F.R.C.P.E., 11 Langlands Gardens, Hampstead, London, N.W. 3	
1934		* Jeffrey, Sir John, K.C.B., C.B.E., formerly Under-Secretary of State for Scotland. 9 Cluny Gardens, Edinburgh 10	
1906	C. K.	Jehu, Thomas John, M.A., M.D., F.G.S., Professor of Geology, University of Edinburgh (Grant Institute of Geology, West Mains Road). 35 Great King Street, Edinburgh 3	1917–20, 1923–26. V-P 1929–32.
1900		† Jerdan, David Smiles, M.A., D.Sc., Ph.D., Avenel, Melrose	
1895		Johnston, Col. Henry Halcro, C.B., C.B.E., D.Sc., M.D., C.M. (Edin.), F.L.S., late Administrative Staff, Army. C/o Messrs Glyn, Mills & Co., Holts Branch, Kirkland House, London, S.W. 1	
1936		* Johnston, John McQueen, M.D. (Glas.), F.R.C.S.E., Pharmacologist, Department of Health for Scotland. 18 Berkeley Terrace, Glasgow, C. 3	
1934		* Johnston, Sir William Campbell, W.S., J.P., LL.D., formerly Deputy Keeper of the Signet. 43 Castle Street, Edinburgh 2	
1928		* Johnston-Saint, Percy Johnston, M.A. (Cantab.), Conservator, Wellcome Historical Medical Museum, 183–193 Euston Road, London, N.W. 1. 4 Wyndham Place, Bryanston Square, London, W. 1	
1928		* Johnstone, Robert William, C.B.E., M.A., M.D. (Edin.), F.R.C.S.E., M.R.C.P.E., Professor of Midwifery and Diseases of Women, University of Edinburgh. 26 Palmerston Place, Edinburgh 12	
1927		* Jones, Edward Taylor, D.Sc. (Lond.), Professor of Natural Philosophy, University of Glasgow	1927–30.
1928	C.	* Jones, Tudor Jenkyn, D.Sc., M.B., Ch.B. (Glas.); Lecturer in Anatomy (Embryology), University of Liverpool	
1922		* Juritz, Charles Frederick, M.A., D.Sc., F.I.C., Chief of the Union Department of Chemistry. Grenoble, Avenue Fresnaye, Sea Point, Cape Town, South Africa.	
1936		* Kemball, Charles Henry, H.D.D., L.D.S., D.D.S. (Univ. Pennsylvania), Dental Surgeon, Lecturer on Orthodontics in Edinburgh Dental Hospital and School. 20 Ainslie Place, Edinburgh 3	
1925	C.	* Kemp, Charles Norman, B.Sc., Technical Radiologist, Secretary of the Royal Scottish Society of Arts. Ivy Lodge, Laverockbank Road, Edinburgh 5	
1929	C.	* Kendall, James Pickering, M.A., D.Sc., F.R.S. (GENERAL SECRETARY), Professor of Chemistry, University of Edinburgh (West Mains Road). 14 Mayfield Gardens, Edinburgh 9	1931–33. Sec. 1933–36. Gen. Sec. 1936–

Date of Election			Service on Council, etc.
1912		† Kennedy, Robert Foster, M.D. (Belfast), M.B., B.Ch. (R.U.I.), Associate Professor of Neurology, Cornell University, New York. 410 East 57 Street, New York City, U.S.A.	
1927		* Kennedy, Walter Phillips, Ph.D. (Edin.), L.R.C.P. and S.E., A.I.C., Professor of Physiology, Royal College of Medicine, Baghdad, Iraq	
1935		* Kenneth, John Henry, M.A., Ph.D. (Edin.), Assistant, Imperial Bureau of Animal Genetics, University of Edinburgh (West Mains Road). University Union, Edinburgh 8	
1909		Kenwood, Henry Richard, C.M.G., M.B., C.M., Emeritus Chadwick Professor of Hygiene, University of London. Wadhurst, Queen's Road, Finsbury Park, London, N.	
1925	C. M-B.	* Kermack, William Ogilvy, M.A., D.Sc., LL.D., Chemist, Research Laboratory of the Royal College of Physicians, 2 Forrest Road, Edinburgh 1	
1903 & 1923	C. N.	Kerr, John Graham, M.A. (Cantab.), LL.D., F.R.S., Honorary Fellow of Christ's College, M.P., Scottish Universities, Emeritus Regius Professor of Zoology, University of Glasgow. Dalny Veed, Barley, near Royston, Herts	1904–07, 1913–16, 1924–27. V-P 1928–31.
1891		Kerr, Joshua Law, M.D., J.P. Waratah, Tasmania	
1926		* Khastgir, Satis Ranjan, M.Sc. (Calcutta), D.Sc. (Edin.), Physics Department, University, Dacca, India	
1907		King, Archibald, M.A., B.Sc., H.M. Inspector of Schools, The Cottage, Barassie, Ayrshire	
1925		* King, Leonard Augustus Lucas, M.A., Professor of Zoology, West of Scotland Agricultural College, Glasgow. 14 Bank Street, Glasgow, W. 2	
1918		* Kingon, Rev. John Robert Lewis, M.A., D.Sc., The Manse, Simonston, Cape Colony, South Africa	
1937		* Kirby, Percival Robson, M.A., D.Litt., F.R.C.M., Professor of Music and Musical History, University of the Witwatersrand, Johannesburg, South Africa	
1937	C.	* Koller, Peo Charles, Ph.D. (Budapest), D.Sc. (Edin.), Cytologist, Institute of Animal Genetics, University of Edinburgh (West Mains Road). 49 George Square, Edinburgh 8	
1927		* Lambie, Charles George, M.C., M.D., F.R.C.P.E., Bosch Professor of Medicine, University of Sydney. Capri, 4 Wyuna Road, Point Piper, Sydney, N.S.W., Australia	
1920	C.	* Lamont, John Charles, Lieut.-Col., I.M.S. (retired), C.I.E., M.B., C.M. (Edin.), M.R.C.S., formerly Professor of Anatomy, Medical College, Lahore, India. 7 Merchiston Park, Edinburgh 10	
1925	C. N.	* Lang, William Henry, M.B., C.M., D.Sc., LL.D. (Glas.), F.R.S., Barker Professor of Cryptogamic Botany, University of Manchester	
1931		* Langrishe, John du Plessis, D.S.O., M.B., B.Ch. (Dub.), D.P.H., Lt.-Col. R.A.M.C. (retired), Lecturer in Public Health, University of Edinburgh (Usher Institute of Public Health, Warrender Park Road). 2 South Gillsland Road, Edinburgh 10	
1910	C.	Lauder, Alexander, D.Sc., formerly Head of Chemistry Department, Edinburgh and East of Scotland College of Agriculture, and Lecturer in Agricultural Chemistry, University of Edinburgh. 78 Dalkeith Road, Edinburgh 9	1917–20. Sec. 1923–28.
1885	C.	Laurie, Arthur Pillans, M.A., D.Sc., LL.D., formerly Principal, Heriot-Watt College, Edinburgh. 38 Springfield Road, St John's Wood, London, N.W. 8	1908–11, 1913–16.
1905		Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1903		Leighton, Gerald Rowley, O.B.E., M.D., D.Sc., formerly Medical Officer (Foods), Department of Health for Scotland. Sharston, near Ramsey, Isle of Man	
1937		* Leitch, William Orr, M.Inst.C.E., formerly Chief Engineer and General Manager Peking Mukden Railway. 1 Gordon Terrace, Edinburgh 9	
1930		* Lelean, Percy Samuel, C.B., C.M.G., F.R.C.S., L.R.C.P., D.P.H., Professor of Public Health, University of Edinburgh (Usher Institute of Public Health, Warrender Park Road). 4 South Lauder Road, Edinburgh 9	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., Balmae, Manor Road, Littleover, Derby	
1916	C.	* Levy, Hyman, M.A., D.Sc., Professor of Mathematics, Imperial College of Science and Technology, London, S.W. 7	

Date of Election			Service on Council, etc.
1914	C. N.	Lewis, Francis John, D.Sc., F.L.S., Professor of Botany, Egyptian University, Abbissea, Cairo	
1918		* Lidstone, George James, F.F.A., F.I.A., LL.D., formerly Manager and Actuary, Scottish Widows' Fund Life Assurance Society. Hermiston House, Hermiston, Currie, Midlothian	1919–22.
1905		Lightbody, Forrest Hay, 53 Queen Street, Edinburgh 2	
1931		* Lightfoot, Nicholas Morpeth Hutchinson, M.A. (Cantab.), Lecturer in Mathematics, Heriot-Watt College, Edinburgh. 3 Park Gardens, Liberton, Edinburgh 9	
1923		* Lim, Robert Kho Seng, M.B., Ch.B., D.Sc., Peking Union Medical College, Department of Physiology, Peking, China	
1912		Lindsay, John George, M.A., B.Sc. (Edin.), Rector of Dunfermline High School	
1920	C.	* Lindsay, Thomas A., M.A., B.Sc., Head Master, Higher Grade School, Bucksburn, Aberdeenshire	
1912		Linlithgow, The Most Honourable the Marquis of, P.C., K.T., G.C.I.E., D.L., Viceroy and Governor-General of India. Hopetoun House, South Queensferry	V-P 1934–37.
1903		† Liston, William Glen, C.I.E., M.D., Lt.-Col., I.M.S. (retired), Milburn Tower, Gogar, Corstorphine, Edinburgh 12	
1929		* Little, John Robert, F.C.I.I., F.C.I.S., formerly Manager and Secretary, Century Insurance Co., Ltd. 5 Dalrymple Crescent, Edinburgh 9	
1932		* Lockhart, James Balfour, M.A., B.Sc. (Edin.), Mathematical Master, Edinburgh Academy. Westering, Inverleith Grove, Edinburgh 4	
1926		* Lorraine, Norman Stanley Rees, M.D., D.P.H. (Edin. and Glas.), Medical Officer of Health, Benfleet Urban District. 1 Burlescoombe Leas, Burlescoombe Road, Thorpe Bay, Southend-on-Sea	
1930		* Low, James Wotherspoon, B.Sc., Ph.D., 1 Hamilton Park Avenue, Glasgow, W. 2	
1934		* Low, R. Cranston, M.D., F.R.C.P.E., formerly Lecturer in Dermatology, University of Edinburgh. 1 Randolph Crescent, Edinburgh 3	
1923	C.	* Ludlam, Ernest Bowman, M.A., D.Sc., Lecturer in Chemistry, University of Edinburgh (West Mains Road)	
1923		* Lyford-Pike, James, M.A., B.Sc., Lecturer in Forestry, University of Edinburgh. Rosetta, 56 Kirkbrae, Liberton, Edinburgh 9	
1924		* Lyon, David Murray, M.D., F.R.C.P.E., D.P.H., D.Sc., Professor of Clinical Medicine, University of Edinburgh (Royal Infirmary). Druim, Colinton, Edinburgh 11	
1894		Mabbott, Walter John, M.A., formerly Rector of County High School, Duns, Berwickshire. The Hawthorn, Farnham Lane, Haslemere, Surrey	
1929		* M'Arthur, Donald Neil, D.Sc., F.I.C., Professor of Agricultural Chemistry, West of Scotland Agricultural College, Glasgow, C. 2. 35 Kersland Street, Glasgow, W. 2	
1921		* M'Arthur, Neil, M.A., B.Sc., Lecturer in Mathematics, University of Glasgow. The Whins, Heathfield Drive, Milngavie	
1926		* M'Bride, James Alexander, B.A. (Roy. Univ., Ireland), B.Sc. (Lond.), formerly Rector of Queen's Park Secondary School, Glasgow. Scottish Liberal Club, Princes Street, Edinburgh 2	
1883		M'Bride, Peter, M.D., F.R.C.P.E., 3 St Peter's Grove, York	
1931	C.	* McCallien, William John, D.Sc. (Glas.), Lecturer in Geology, University of Glasgow. Glenorchy, Tarbert, Argyll	
1935		* MacCallum, Peter, M.Sc., M.A. (New Zealand), M.B., Ch.B., D.P.H. (Edin.), Professor of Pathology, University of Melbourne. Blackrock House, Blackrock, Victoria, Australia	
1923		* M'Cracken, William, J.P., F.S.I., Englesea House, Crewe	
1931	C.	* M'Crea, William Hunter, M.A., Ph.D. (Cantab.), B.Sc. (Lond.), F.R.A.S., Professor of Mathematics, The Queen's University, Belfast. 61 Ulsterville Avenue, Belfast	
1918		* M'Culloch, Rev. James David, 3 Ardgowan Street, Greenock	
1920		* M'Donald, Stuart, M.A., M.D., F.R.C.P.E., Professor of Pathology, University of Durham. College of Medicine, Newcastle-upon-Tyne	
1928		* MacDonald, Thomas Logie, M.A., B.Sc. (Glas.), F.R.A.S., 9 Colebrooke Terrace, Glasgow, W. 2	
1886		Macdonald, William J., M.A., 15 Comiston Drive, Edinburgh 10	

Date of Election			Service on Council, etc.
1931		* M'Dougall, John Bowes, M.D. (Glas.), F.R.F.P.S.G., F.R.C.P.E., Medical Director, British Legion Village, Preston Hall, Kent. Preston Hall, Aylesford, Kent	
1901	C.	MacDougall, R. Stewart, M.A., D.Sc., LL.D. (Edin.), Emeritus Professor of Biology, Royal (Dick) Veterinary College, Edinburgh. Ivy Lodge, Gullane, East Lothian	1914–17.
1910		Macewen, Hugh Allen, O.B.E., M.B., Ch.B., D.P.H. (Lond. and Cantab.), Local Government Board, Ministry of Health, Whitehall, London, S.W.	
1888	C.	M'Fadyean, Sir John, Kt., M.B., B.Sc., LL.D., formerly Principal and Professor of Comparative Pathology, Royal Veterinary College, Camden Town, London. Highlands House, Leatherhead	
1885	C.	† Macfarlane, John M., D.Sc., LL.D., Emeritus Professor of Botany. 427 West Hansberry Street, Germantown, Pa., U.S.A.	
1897		MacGillivray of MacGillivray, Angus, M.D., C.M., D.Sc., LL.D., 23 South Tay Street, Dundee	
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, London, W. 5	
1932		* MacGregor, Archibald Gordon, M.C., B.Sc., Geologist, H.M. Geological Survey (Scotland). 1 Greenbank Terrace, Edinburgh 10	
1922		* Macgregor, Murray, M.A., D.Sc., F.G.S., Assistant Director (Scotland), H.M. Geological Survey, 19 Grange Terrace, Edinburgh 9	1930–33.
1903		M'Intosh, Donald C., M.A., D.Sc., formerly Education Officer, Elgin. Glenavon, Boat of Garten	
1911		M'Intosh, John William, M.R.C.V.S., Dollis Hill Farm, Cricklewood, London, N.W. 2	
1927	C.	* M'Intyre, Donald, M.B.E., M.D. (Glas.), F.R.C.S.E., Royal Samaritan Lecturer in Gynæcology, University of Glasgow. 9 Park Circus, Glasgow, C. 3	
1912	C.	M'Kendrick, Anderson Gray, M.B., D.Sc., F.R.C.P.E., Lt.-Col., I.M.S. (retired) (VICE-PRESIDENT), Superintendent, Research Laboratory, Royal College of Physicians, 2 Forrest Road, Edinburgh 1	1924–27. 1933–36. V-P 1936–
1914		* M'Kendrick, Archibald, J.P., F.R.C.S.E., D.P.H., L.D.S., 12 Rothesay Place, Edinburgh 3	
1900	C.	M'Kendrick, John Soutar, M.D., F.R.F.P.S. (Glas.), 2 Buckingham Terrace, Hillhead, Glasgow	
1910	C.	Mackenzie, Alister, M.A., M.D., D.P.H., Principal Medical Officer and Lecturer in Hygiene, Training Centre, Jordanhill, Glasgow. 22 Queen's Gate, Dowanhill, Glasgow	
1916	C.	* Mackenzie, John E., D.Sc., Reader in Chemistry, University of Edinburgh (West Mains Road). 2 Ramsay Garden, Edinburgh 1	1936–
1905		Mackenzie, Sir William Colin, K.B., M.D., F.R.C.S., Director, Australian Institute of Anatomy, Canberra, F.C.T., Australia	
1929	C.	* Mackie, John, M.A., D.Sc., Rector, Leith Academy. 7 York Road, Trinity, Leith 5	
1928		* Mackie, Thomas Jones, M.D., M.R.C.P.E., Professor of Bacteriology, University of Edinburgh (Teviot Place). 22 Mortonhall Road, Edinburgh 9	
1936		* M'Kinlay, Peter Laird, M.D., D.P.H., Medical Officer (Statistics), Department of Health for Scotland. 69 Montrose Street, Clydebank	
1910		MacKinnon, James, M.A., Ph.D., LL.D., Emeritus Professor of Ecclesiastical History, University of Edinburgh. 12 Lygon Road, Edinburgh 9	1933–36.
1904		Mackintosh, Donald James, C.B., M.V.O., D.L., M.B., C.M., LL.D., Superintendent, Western Infirmary, Glasgow	
1899		Maclean, Sir Ewen John, J.P., D.L., M.D., D.Sc. (Hon.), LL.D., F.R.C.P. (Lond.), Emeritus Professor of Obstetrics and Gynæcology, Welsh National Medical School. 12 Park Place, Cardiff	
1933		* Macleod, James, F.I.C., Manager, Glasgow Corporation Chemical Works Department. 16 Colebrooke Street, Glasgow, W. 2	
1916	C.	* M'Lintock, William Francis Porter, D.Sc. (Edin.), Geological Survey and Museum, Exhibition Road, South Kensington, London, S.W. 7	
1936		* M'Michael, John, M.D., Ch.B., M.R.C.P., Johnston and Lawrence (Royal Society) Research Fellow in Medicine. 32 Mount Vernon Road, Edinburgh 9	
1923		* Macmillan, Rt. Hon. Lord, P.C., G.C.V.O., LL.D., 44 Millbank, Westminster, S.W. 1	

Date of Election			Service on Council, etc.
1932		* M'Neil, Charles, M.A., M.D. (Edin.), F.R.C.P.E., Professor of Child Life and Health, University of Edinburgh (Royal Edinburgh Hospital for Sick Children). 44 Heriot Row, Edinburgh 3	
1917		* Macpherson, Rev. Hector Copland, M.A., Ph.D., F.R.A.S., Guthrie Memorial U.F. Church. 7 Wardie Crescent, Edinburgh 5	
1921		* M'Quistan, Dougald Black, M.A., B.Sc., Associate-Professor of Natural Philosophy, Royal Technical College, Glasgow. 29 Viewpark Drive, Rutherglen	
1936		* McRae, William, C.I.E., M.A., D.Sc. (Edin.), Late Agricultural Adviser to the Government of India, and Director, Agricultural Research Institute, Pusa. Daramona, Gamekeepers' Road, Barnton, Midlothian	
1921	C.	* MacRobert, Thomas Murray, M.A., D.Sc., Professor of Mathematics, University of Glasgow. 10 The University, Glasgow	1931–34.
1921	C.	* M'Whan, John, M.A. (Glasgow), Ph.D. (Gött.), Lecturer in Mathematics, University of Glasgow. 37 Airthrey Avenue, Jordanhill, Glasgow, W. 4	
1927		* Madwar, Mohamed Reda, Ph.D. (Edin.), A.M.Inst.C.E., Director, Helwân Observatory, Egypt	
1898	C.	† Mahâlânobis, S. C., B.Sc. (Edin.), Professor of Physiology, University of Calcutta. P. 45 New Park Street, Calcutta	
1913		Majumdar, Tarak Nath, D.P.H. (Cal.), L.M.S., F.C.S., Health Officer, IV, Calcutta. P. 235 Russa Road, P.O. Tollygunge	
1933		* Malcolm, John, M.D. (Edin.), Professor of Physiology, University of Dunedin, New Zealand, Medical School, King Street, Dunedin	
1917		* Malcolm, L. W. Gordon, M.Sc. (Cantab.), Ph.D., F.L.S., formerly Conservator, Wellcome Historical Medical Museum. Robins Herne, Northwood, Middlesex	
1908		Mallik, Devendranath, Sc.D., B.A., Principal, Carmichael College, Rungpur, Bengal, India	
1912		† Maloney, William Joseph, M.B.E., M.C., M.D. (Edin.), LL.D., formerly Professor of Neurology, Fordham University. Casa del Sale, Newport, Rhode Island, N.Y., U.S.A.	
1913		Marchant, Rev. Sir James, K.B.E., LL.D., F.R.A.S., F.L.S., Director, National Council for Promotion of Race-Regeneration. Pinegarth, Buccleuch Road, Bournemouth	
1901	C.	Marshall, Francis Hugh Adam, C.B.E., Sc.D., F.R.S., Reader in Agricultural Physiology, University of Cambridge. Christ's College, Cambridge	
1920	C.	* Marshall, John, M.A., D.Sc. (St Andrews), B.A. (Cantab.), University Reader in Mathematics, Bedford College, London. Logan House, 123 Torrington Park, London, N. 12	
1931		* Mason, John Huxley, F.R.C.V.S., Government Veterinary Laboratory, Onderstepoort, Pretoria, South Africa	
1913		Masson, George Henry, O.B.E., M.D., D.Sc., F.R.C.P.E., Carradale, Port of Spain, Trinidad, British West Indies	
1898	C. M-B.	Masterman, Arthur Thomas, M.A., D.Sc., F.R.S., formerly Superintending Inspector, H.M. Board of Agriculture and Fisheries. 3 Kedale Road, Seaford	1902–04.
1911		† Mathews, Gregory Macalister, M.B.O.U., Meadway, St Cross, Winchester, Hants	
1921		* Mathieson, John, F.R.S.G.S., Division Superintendent, Ordnance Survey (retired), 42 East Claremont Street, Edinburgh 7	
1906		Mathieson, Robert, F.C.S., St Serf's, Innerleithen	
1928		* Matthai, George, M.A. (Cantab.), Sc.D., F.Z.S., F.L.S., Professor of Zoology, Government College, Lahore, India	
1924		* Matthews, James Robert, M.A., F.L.S., Regius Professor of Botany, University of Aberdeen, and Keeper of the Cruickshank Botanic Garden	
1932		* Maxwell, William, Managing Director of R. & R. Clark, Ltd. 14 South Inverleith Avenue, Edinburgh 4	
1917		* Maylard, A. Ernest, M.B., B.Sc. (Lond.), F.R.F.P.S. (Glasgow), Kingsmuir, Peebles	
1922		* Meakins, Jonathan Campbell, M.D., LL.D., F.R.C.P.E., Professor of Medicine and Director of the Department of Medicine, McGill University, Montreal, Canada	
1931		* Mears, Frank Charles, A.R.S.A., F.R.I.B.A., 44 Queen Street, Edinburgh 2	
1937	C.	* Melville, Harry Work, Ph.D. (Edin., Cantab.), D.Sc. (Edin.), Fellow of Trinity College, Cambridge	

Date of Election			Service on Council, etc.
1901	C.	Menzies, Alan W. C., M.A., Ph.D., F.C.S., Professor of Chemistry, Princeton University, Princeton, New Jersey, U.S.A.	
1927		* Menzies, Sir Frederick Norton Kay, K.B.E., M.D., LL.D. (Edin.), F.R.C.P.E., D.P.H. (Lond.), Medical Officer of Health and School Medical Officer, Administrative County of London, County Hall, London, S.E.	
1933	C.	* Menzies, William John Milne, Inspector of Salmon Fisheries of Scotland, Fishery Board for Scotland. Tighbeag, Whitehouse Road, Cramond Bridge, near Edinburgh	
1929		* Mercer, Walter, M.B., Ch.B., F.R.C.S.E., Lecturer in Clinical Surgery, University of Edinburgh (Royal Infirmary). 12 Rothesay Terrace, Edinburgh 3	
1917		* Merson, George Fowlie, Manufacturing Technical Chemist, St John's Hill Works, Edinburgh 8. 7 Cumin Place, Edinburgh 9	
1902	C.	† Metzler, William H., A.B., D.Sc., Ph.D., formerly Dean of the New York State College for Teachers, Albany, N.Y., U.S.A. 5003 South Salina Street, Syracuse, N.Y., U.S.A.	
1885	C. M-B.	Mill, Hugh Robert, D.Sc., LL.D., Hill Crest, Dormans Park, E. Grinstead	
1937		* Miller, James, R.S.A., F.R.I.B.A., Randolphfield, Stirling	
1910		Miller, John, M.A., D.Sc., formerly Professor of Mathematics, Royal Technical College. 212 Wilton Street, Glasgow, N.W.	
1936		* Miller, The Very Rev. John Harry, C.B.E., M.A., D.D., Principal of St Mary's College, University of St Andrews. St Mary's, St Andrews	
1930		* Miller, William Christopher, M.R.C.V.S., Courtauld Professor of Animal Husbandry, Royal Veterinary College, Camden Town, London, N.W. 1. Blair Hyrne, Hadley Common, Barnet, Herts	
1905		Milne, Archibald, M.A., D.Sc., Deputy Director of Studies, Edinburgh Provincial Training College. 38 Morningside Grove, Edinburgh 10	
1905		† Milne, C. H., M.A., D.Litt., formerly Head Master, Daniel Stewart's College. 19 Merchiston Gardens, Edinburgh 10	
1904	C.	Milne, James Robert, D.Sc., Lecturer in Natural Philosophy, University of Edinburgh (Drummond Street). 7 Grosvenor Crescent, Edinburgh 12	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen	
1933	C.	Milne-Thomson, Louis Melville, M.A. (Cantab.), F.R.A.S., Professor of Mathematics, Royal Naval College, Greenwich. Gothic House, Maze Hill, London, S.E. 10	
1899		Milroy, Thomas Hugh, M.D., B.Sc., LL.D., formerly Professor of Physiology, Queen's University, Belfast. Woodville, North Berwick	
1889	C. K.	Mitchell, A. Crichton, D.Sc., Hon. Doc. Sc. (Genève), formerly Director of Public Instruction in Travancore, India. 246 Ferry Road, Edinburgh 5. (Society's Representative on Governing Body of Heriot-Watt College)	1915–16, 1930–33. Cur. 1916–26. V-P 1926–29.
1897		† Mitchell, Sir George Arthur, Kt., D.L., J.P., M.A., LL.D., M.I.M.E., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		Mitchell, James, M.A., B.Sc., Islay Lodge, Lochgilphead, Argyll	
1911		Modi, Edalji Manekji, D.Sc., LL.D., Litt.D., F.C.S. (No address)	
1890	C.	† Mond, Sir Robert Ludwig, Kt., M.A. (Cantab.), LL.D., F.C.S., 9 Cavendish Square, London, W. 1	
1936		* More, Francis, Chartered Accountant, of the firm of Lindsay, Jamieson and Haldane. 12 Albert Terrace, Edinburgh 10	
1896		Morgan, Alexander, O.B.E., M.A., D.Sc., LL.D., formerly Principal, Edinburgh Provincial Training College. 1 Midmar Gardens, Edinburgh 10	
1936		* Morgan, Daniel Owen, M.Sc. (Wales), Ph.D. (Lond.), Lecturer in Helminthology, Department of Zoology, University of Edinburgh (West Mains Road). 15 Mentone Terrace, Edinburgh 9	
1937		* Morison, John, C.I.E., M.B., Ch.B. (Glas.), D.P.H. (Cantab.), Lt.-Col. I.M.S. (retired). Engaged in Research in the Usher Institute of Public Health. 13 Cluny Drive, Edinburgh 10	
1930		* Morison, John Miller Woodburn, M.D., F.R.C.P.E., D.M.R. and E., Professor of Radiology, University of London, and Director of the Radiological Department, Cancer Hospital, Fulham Road, London, S.W. 3	
1926		* Morris, James Archibald, R.S.A., F.S.A.Scot., Savoy Croft, Ayr	

Date of Election			Service on Council, etc.
1919		* Morris, Robert Owen, O.B.E., M.A., M.D., C.M. (Edin.), D.P.H. (Liverpool). King Edward VII Welsh National Memorial Association (Tuberculosis). Hafod-ar-For, Aberdovey, N. Wales	
1892	C.	Morrison, J. T., M.A., D.Sc., Emeritus Professor of Mathematical Physics, University, Stellenbosch, South Africa	
1930		* Morton, Sir James, Kt., LL.D., Governing Director, Scottish Dyes, Ltd. Tuethur, Carlisle	
1901		Moses, O. St John, M.D., D.Sc., F.R.C.S., Lt.-Col., I.M.S. (retired), formerly Professor of Medical Jurisprudence, Medical College, Calcutta. 18 Manstone Road, Cricklewood, London, N.W. 2	
1892	C. K.	Mossman, Robert Cockburn, Lacar 4332, Villa Devoto F.C.P., Buenos Aires, Argentina	
1934		* Mowat, Magnus, C.B.E., T.D., M.Inst.C.E., M.I.Mech.E., Brigadier-General, Secretary of the Institution of Mechanical Engineers, Storey's Gate, London, S.W. 1	
1935	C.	* Mozley, Walter Alan, B.Sc. (Manitoba), Ph.D. (Edin.), Lecturer in Zoology, Johns Hopkins University, Baltimore, Md., U.S.A.	
1916		* Muir, Sir Robert, Kt., M.A., M.D., Sc.D., LL.D., F.R.S., F.R.C.P.E., Emeritus Professor of Pathology, University of Glasgow. 30 Victoria Crescent Road, Glasgow, W. 2	
1935		* Munnoch, James, F.S.A.Scot., formerly Controller, General Post Office, Edinburgh. 15 Liberton Drive, Edinburgh 9	
1933		* Murray, John, M.A. (Aberd.), Ph.D. (Edin.), Rector, The Academy, Annan. 9 Seaforth Avenue, Annan	
1932		* Murray, John Murdoch, B.Sc. (Edin.), Assistant Commissioner for Scotland, Forestry Commission. 76 Hillview Terrace, Corstorphine, Edinburgh 12	
1934		* Murray, Walter George Robertson, A.I.C., Technical Assistant, Department of Chemistry, University of Edinburgh (West Mains Road). 20 Montpelier Park, Edinburgh 10	
1935		* Narayana, Basudeva, M.Sc., M.B. (Calcutta), Ph.D. (Edin.), Professor of Physiology, University of Patna. Department of Physiology, Medical College, Patna, India	
1937		* Nasmith, Charles Roy, B.A., M.A. (Honorary), Colgate University, U.S.A., United States Consul in Edinburgh. Abden House, 1 Marchhall Crescent, Edinburgh 9	
1931		* Nelson, Alexander, B.Sc. (Glas.), Ph.D. (Edin.), N.D.A., Lecturer in Plant Physiology and Agricultural Botany, University of Edinburgh. 14 Netherby Road, Edinburgh 5	
1924		* Nelson, Philip, M.A., M.D., Ph.D., F.S.A., Beechwood, Calderstones, Liverpool	
1898		Newman, Sir George, G.B.E., K.C.B., M.D., D.C.L., Hon. D.Sc., LL.D., F.R.C.P., formerly Chief Medical Officer, Ministry of Health and Board of Education. Grims Wood, Harrow Weald, Middlesex	
1928		* Nichols, James Edward, M.Sc. (Dunelm), Ph.D. (Edin.), Professor of Agriculture, University of Western Australia, Perth, W.A.	
1933	C.	* Nicol, Thomas, M.D., Ch.B., F.R.C.S.E., D.Sc., Professor of Anatomy, University of London, King's College, Strand, London, W.C. 2	
1927		* Noble, Thomas Paterson, M.D. (Edin.), F.R.C.S., 27 Empire House, Thurloe Place, London, S.W. 3	
1934	C.	* Normand, Alexander Robert, M.A., B.Sc., Ph.D. (Edin.), formerly Professor of Chemistry, Wilson College, Bombay. 7 India Street, Edinburgh 3	
1928	C. N.	* O'Donoghue, Charles Henry, D.Sc. (Lond.) (SECRETARY TO ORDINARY MEETINGS), Reader in Zoology, University of Edinburgh (West Mains Road). Appin Lodge, Eskbank, Midlothian	Sec. 1936–
1925		* Ogg, William Gammie, M.A., Ph.D., Director, Macaulay Institute for Soil Research, Craigiebuckler, Aberdeen; Research Lecturer in Soil Science in the University of Aberdeen	
1923	C.	* Ogilvie, Alan G., O.B.E., M.A., B.Sc. (Oxon.), Professor of Geography, University of Edinburgh (High School Yards). 40 Fountainhall Road, Edinburgh 9	1932–35.
1929		* Ogilvie, Frederick Wolff, M.A., Vice-Chancellor and President of Queen's University, Belfast	
1886		Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women. 123 Harley Street, London, W.	

Date of Election			Service on Council, etc.
1895	C.	† Oliver, Sir Thomas, Kt., D.L., M.D., LL.D., F.R.C.P., formerly Vice-Chancellor of the University of Durham. 7 Ellison Place, Newcastle-upon-Tyne	
1930		* Oliver, William, B.Sc., A.M.Inst.C.E., Professor of Organisation of Industry and Commerce, University of Edinburgh (South Bridge). 70 Netherby Road, Trinity, Edinburgh 5	
1930		* O'Riordan, George Francis, B.Sc. (Eng.), M.I.Mech.E., Principal of Battersea Polytechnic, London, S.W. 7. Hesslewood, 3 Hazlewell Road, Putney, London, S.W. 15	
1924		* Orr, Sir John Boyd, Kt., D.S.O., M.C., M.A., D.Sc., M.D., LL.D., F.R.S., Director of the Rowett Research Institute for Research in Animal Nutrition, Aberdeen	
1915		* Orr, Lewis P., F.F.A., formerly General Manager of the Scottish Life Assurance Co., 3 Belgrave Place, Edinburgh 4	
1932		* Orr, Matthew Young, 38 Lennox Row, Edinburgh 5	
1908		Page, William Davidge, 13 Yarrell Mansions, Queen's Club Gardens, London, W. 14	
1934		* Pal, Rudrendra Kumar, D.Sc. (Edin.), M.R.C.P.E., L.R.C.S.E., formerly Professor of Physiology, Prince of Wales Medical College, Patna. Nayasarak, Sylhet, Assam, India	
1905		Pallin, Lt.-Col. William Alfred, C.B.E., D.S.O., F.R.C.V.S., 5 Tower Gardens, Hythe, Kent	
1933		Parsons, Charles Wynford, M.A., Lecturer in Zoology, University of Glasgow. Westbank, 1 Chapelton Terrace, Bearsden, Dumbartonshire	
1901		Paterson, David, Leewood, Rosslyn Castle, Midlothian	
1937		* Paterson, Thomas T., B.Sc. (Edin.), Fellow of Trinity College, Cambridge. Swanlea House, Buckhaven, Fife	
1936		* Paterson, William George Rogerson, B.Sc., Principal of the West of Scotland Agricultural College. Buckrigg, Beattock, Dumfriesshire	
1918		* Paterson, Rev. William Paterson, D.D., LL.D., Emeritus Professor of Divinity, University of Edinburgh (South Bridge). 17 South Oswald Road, Edinburgh 9	
1927		* Patterson, Charles, A.M.I.Mar.E., Lecturer in Mechanical Engineering Design, University of Edinburgh (Sanderson Engineering Laboratories, Mayfield Road). 22 Dudley Terrace, Trinity, Edinburgh 6	
1926		* Patton, Donald, M.A., B.Sc., Ph.D., Lecturer in Science, Glasgow Provincial College for the Training of Teachers. 15 Jordanhill Drive, Glasgow, W. 3	
1923	C.	* Peacock, Alexander David, D.Sc., Professor of Zoology, University College, Dundee	1935-
1914		Pearson, Joseph, D.Sc., F.L.S., formerly Director of the Colombo Museum, and Marine Biologist to the Ceylon Government. Director of the Tasmanian Museum, Hobart, Tasmania	
1904		Peck, James Wallace, C.B., M.A., Secretary to the Scottish Education Department, 14 Queen Street, Edinburgh	1926-28.
1887	C. M-B.	Peddie, William, D.Sc., Professor of Natural Philosophy, University College, Dundee. The Weisha, Ninewells, Dundee	1904-07, 1908-11, 1933-36. V-P 1919-22.
1925		* Penman, David, C.I.E., D.Sc., M.Inst.M.E., Chief Inspector of Mines in India, Geological Survey of India, 27 Chowringhee Road, Calcutta, India	
1931	C.	* Phemister, James, M.A., D.Sc. (Glas.), Petrographer, H.M. Geological Survey and Museum, Exhibition Road, London, S.W. 7	
1889		† Philip, Sir Robert William, Kt., M.A., M.D., LL.D., F.R.C.P.E., Professor of Tuberculosis, University of Edinburgh. 45 Charlotte Square, Edinburgh 2	V-P 1927-30.
1907	C.	† Phillips, Charles E. S., O.B.E., Castle House, Shooters Hill, Woolwich, S.E. 18	
1929	C.	* Phillips, John Frederick Vicars, D.Sc., F.L.S., Professor of Botany, University of the Witwatersrand, Johannesburg, Union of South Africa	
1935		* Pichamuthu, Charles Solomon, B.Sc. (Mysore), Ph.D. (Glas.), F.G.S., Assistant Professor of Geology, University of Mysore. Central College, Bangalore, India	
1932		* Pickard, James Nichol, B.A. (Cantab.), Ph.D. (Edin.), Lynburn, Carlops, Penicuik, Midlothian	

Date of Election		
1928		* Pilcher, Robert Stuart, General Manager, Manchester Corporation Tramways. 55 Piccadilly, Manchester
1908	C.	† Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., Research Pathologist and Bacteriologist, South African Institute for Medical Research. P.O. Box 1038, Johannesburg, South Africa
1911		Pirie, James Simpson, M.Inst.C.E., 25 Grange Road, Edinburgh 9
1906		Pitchford, Herbert Watkins, C.M.G., F.R.C.V.S., Victoria Club, Pietermaritzburg, South Africa
1934		* Plenderleith, Harold James, M.C., B.Sc., Ph.D. (St Andrews), F.C.S., Assistant Keeper, 1st Class, in Research Laboratory of British Museum, London. 134 The Vale, Golder's Green, N.W. 11
1937		* Pollock, John Donald, O.B.E. (Mil.), D.L., M.D., Hon. D.Sc. (Oxon.), LL.D. (Edin.). Manor House, Boswall Road, Edinburgh 5
1919		* Porritt, B. D., M.Sc. (Lond.), F.I.C., Director of Research, Research Association of British Rubber Manufacturers, 105 Lansdowne Road, Croydon, Surrey
1888		† Prain, Sir David, Kt., C.M.G., C.I.E., M.A., M.B., LL.D., F.R.S., F.L.S., Lt.-Col., I.M.S. (retired), formerly Director, Royal Botanic Gardens, Kew, Surrey. The Well Farm, Warlingham, Surrey
1937		* Prasad, Badri Narayan, Ph.D. (Edin.), M.Sc., M.B., D.T.M. (Cal.), Lecturer in Pharmacology, P. W. Medical College, Bankipore, P.O., Bihar, India
1932		* Prasad, Gorakh, D.Sc. (Edin.), Reader in Mathematics, University of Allahabad. Beli Road, Allahabad, India
1926	C.	* Prashad, Baini, D.Sc., Superintendent, Zoological Survey of India, Indian Museum, Calcutta
1933		* Preston, Frank Anderson Baillie, L.R.I.B.A., F.S.A.Scot., Lecturer in Municipal Engineering, Royal Technical College, Glasgow. Craigrownie, Briarwell Road, Milngavie
1915		† Price, Frederick William, M.D. (Edin.), Consulting Physician to the Royal Northern Hospital, London; Senior Physician to the National Hospital for Diseases of the Heart. 133 Harley Street, London, W.
1932		* Price, Thomas Slater, O.B.E., D.Sc., (Lond., Birm.), Ph.D. (Leip.), F.R.S., Professor of Chemistry, Heriot-Watt College, Edinburgh. 2 Cluny Drive, Edinburgh 10
1932		* Pringle, John, Hon. D.Sc., F.G.S., Palæontologist, Geological Survey of Great Britain, Exhibition Road, South Kensington, London, S.W. 7
1920	C.	* Purser, George Leslie, M.A. (Cantab.), F.Z.S., Lecturer in Embryology, University of Aberdeen
1898		Purves, John Archibald, D.Sc., Chilliswood, Trull, Taunton
1936	C.	* Raitt, Douglas Stewart, D.Sc., Ph.D. (Aberd.), F.L.S., Naturalist, Fishery Board for Scotland, Aberdeen. 41 Rosehill Drive, Aberdeen
1899	C.	Ramage, Alexander G., Lochcote, Linlithgowshire
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, University, Birmingham
1900		Raw, Nathan, C.M.G., M.D., 22 Ashworth Road, Maida Vale, London, W. 9
1937		Rawlins, Francis Ian Gregory, M.Sc. (Cantab.), Scientific Adviser to the Trustees of the National Gallery, London. 41 Rossmore Court, London, N.W. 1
1927	C.	* Read, Herbert Harold, D.Sc. (Lond.), A.R.C.S., F.G.S., George Herdman Professor of Geology, University of Liverpool
1929		* Read, Selwyn, B.A., Schoolmaster, Edinburgh Academy. Mackenzie House, Kinnear Road, Edinburgh 4
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., 90 Fitzjohns Avenue, Hampstead, London, N.W. 3
1913		Reid, Harry Avery, O.B.E., F.R.C.V.S., D.V.H., Veterinary Officer to the New Zealand Government. C/o High Commissioner for New Zealand, 415 Strand, London, W.C. 2
1924		* Reid, William Carstairs, Civil Engineer, 72A George Street, Edinburgh 2 (*Died 29th October, 1937*)
1936		* Renouf, Louis Percy Watt, B.A. (Cantab.), M.Sc. (Nat. Univ. Ireland), Professor of Zoology, University College, Cork, Director of the University of Cork Biological Station. St Philomena's, Tivoli, Cork, I.F.S.
1914		Renshaw, Graham, M.D., M.R.C.S., L.R.C.P., L.S.A., F.Z.S., Lecturer in Zoology, Extramural Department, University of Manchester. Editor of *Natureland*, Sale Bridge House, Sale, Manchester

Date of Election			Service on Council, etc.
1913		Richardson, Harry, O.B.E., M.C., M.Inst.E.E., M.Inst.M.E., 72 Oakwood Court, London, W. 14	
1908		Richardson, Linsdall, F.G.S., 104 Greenfield Road, Harbourne, Birmingham	
1927		* Richey, James Ernest, M.C., B.A., B.A.I. (T.C.D.), Sc.D., F.G.S., District Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9	
1930		* Ritchie, Allan Watt, M.B.E., F.R.San.I., Chief Sanitary Inspector, City of Edinburgh. 2 Queensferry Terrace, Edinburgh 4	
1916	C.	* Ritchie, James, M.A., D.Sc., Professor of Natural History, University of Edinburgh. 31 Mortonhall Road, Edinburgh 9	1921–24, 1926–28, 1937– Sec. 1928–31. V-P 1931–34.
1914	C.	* Ritchie, James Bonnyman, D.Sc., Rector, The Academy, Ayr. 28 Carrick Road, Ayr	
1937		* Ritchie, Mowbray, Ph.D., D.Sc. (Edin.), Assistant Lecturer in Chemistry, University of Sheffield	
1906	C.	Ritchie, William Thomas, M.D., F.R.C.P. Edin. and Lond., Professor of Medicine, University of Edinburgh (Teviot Place). 10 Douglas Crescent, Edinburgh 12	
1936		* Robb, James, M.A., B.D., LL.B., LL.D. (St Andrews), Secretary to the Carnegie Trust for the Universities of Scotland. 26 Ormidale Terrace, Edinburgh 12	
1929	C.	* Robb, Richard Alexander, M.A., B.Sc., M.Sc., Lecturer in Mathematics, University of Glasgow. 27 Moor Road, Eaglesham, Renfrewshire	
1931		* Robb, William, N.D.A., Director of Research, Scottish Society for Research in Plant Breeding. Craigs House, Corstorphine, Edinburgh 12.	
1898	C.	Roberts, Hon. Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa	
1919		* Roberts, Alfred Henry, O.B.E., M.Inst.C.E., formerly Superintendent and Engineer, Leith Docks. Dohnavur, Ravelston Dykes, Edinburgh 4	
1926		* Roberts, John Alexander Fraser, M.A. (Cantab.), M.B., D.Sc., Stoke Park Colony, Stapleton, Bristol	
1928	C.	* Roberts, Owen Fiennes Temple, M.C., M.A. (Cantab.), Lecturer in Astronomy and Meteorology, University of Aberdeen. 20 Belgrave Terrace, Aberdeen	
1937		* Robertson, John Watson, M.A., B.Sc. (Aberd.), Headmaster, Central School, Aberdeen. 21 Belvedere Crescent, Aberdeen	
1937		* Robertson, Thomas Graham, The Hon. Lord Robertson, Senator of the College of Justice. 14 India Street, Edinburgh 3	
1919		* Robertson, William Alexander, F.F.A., Century Insurance Co., Ltd., 18 Charlotte Square. Mardale, 3 Buckstone Park, Edinburgh 10	
1896	C.	Robertson, W. G. Aitchison, D.Sc., D.Litt., M.D., F.R.C.P.E., Barrister-at-Law, Lincoln's Inn. St Margarets, St Valerie Road, Bournemouth	
1932	C.	* Robson, John Michael, M.D., B.Sc., Lecturer in Materia Medica, University of Edinburgh (Teviot Place). Ellonville, Park Road, Eskbank	
1926		* Romanis, William Hugh Cowie, M.A., M.B., M.C. (Cantab.), F.R.C.S., Surgeon to St Thomas's Hospital, London. 120 Harley Street, London, W. 1	
1916		* Ronald, David, M.Inst.C.E., Chief Engineer, Scottish Board of Health, 125 George Street, Edinburgh 2	
1909	C.	Ross, Alexander David, M.A., D.Sc., F.Inst.P., F.R.A.S., Professor of Physics, University of Western Australia, Perth, Western Australia	
1921		* Ross, Edward Burns, M.A., formerly Professor of Mathematics, Madras Christian College, Madras. 41 Liberton Brae, Edinburgh 9	
1935		* Rowatt, Thomas, M.I.Mech.E., F.S.A.Scot., Director, Royal Scottish Museum. Spottiswoode, Spylaw Bank Road, Colinton, Edinburgh	
1931	C.	* Ruse, Harold Stanley, M.A. (Oxon.), D.Sc., Professor of Mathematics, University College, Southampton	
1906		Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School. 14 Heugh Street, Falkirk	
1930		Russell, David, LL.D., Paper Manufacturer, Silverburn, Leven, Fife	
1902	C. K.	Russell, James, 22 Glenorchy Terrace, Edinburgh 9	

Date of Election			Service on Council, etc.
1937		* Russell, William Ritchie, M.D. (Edin.), F.R.C.P.E., M.R.C.P., Assistant Physician to the Royal Infirmary and to the Deaconess Hospital, Edinburgh. 8 Randolph Cliff, Edinburgh 3	
1934		* Rutherford, Daniel Edwin, M.A., B.Sc. (St Andrews), D.Math. (Amsterdam), Lecturer in Applied Mathematics, United College, University of St Andrews. 5 John Street, St Andrews	
1925	C.	* Saddler, William, M.A., B.A., Professor of Mathematics, Canterbury College, Christchurch, N.Z.	
1906		Saleeby, Caleb Williams, M.D., 13 Greville Place, Hampstead, London, N.W. 6	
1916	C.	* Salvesen, The Rt. Hon. Lord, P.C., K.C., LL.D., Judge of the Court of Session (retired), Dean Park House, Edinburgh 4	1920–22. V-P 1922–25.
1934		* Salvesen, Harold Keith, M.A. (Oxon. and Harvard), Captain (retired, I.A.), Shipowner, Inveralmond, Cramond, Midlothian	
1914		* Salvesen, Theodore Émile, of Culrain, F.R.S.A., F.S.A.Scot., Chevalier de la Légion d'Honneur. 37 Inverleith Place, Edinburgh 4	
1912	C. K.	Sampson, Ralph Allen, M.A., D.Sc., LL.D., F.R.S., formerly Astronomer Royal for Scotland and Professor of Astronomy, University of Edinburgh. Greenhill, 20 Observatory Road, Edinburgh 9	1912–15, 1919–21. V-P 1915–18, 1933–36. Sec. 1922–23. Gen. Sec. 1923–33.
1927	C.	* Sandeman, Ian, M.A., B.Sc., Ph.D. (St Andrews), Acting Chief Inspector of Schools, Education Department, Colombo, Ceylon	
1930		* Sansome, Frederick Whalley, Ph.D., F.L.S., Senior Lecturer in Horticulture, Botany Department, University of Manchester	
1922		* Sarkar, Bijali Behari, M.Sc., D.Sc. (Edin.), Lecturer in Physiology, University, Calcutta. 33/3 Lansdowne Road, Calcutta	
1903		Sarolea, Charles, Ph.D., D.Litt., formerly Professor of French, University of Edinburgh. 21 Royal Terrace, Edinburgh 7	
1935		* Say, Maurice George, Ph.D., M.Sc. (Lond.), M.I.E.E., Professor of Electrical Engineering, Heriot-Watt College, Edinburgh. Dreghorn Loan, Colinton, Edinburgh	
1927	C.	* Schlapp, Robert, M.A. (Edin.), Ph.D. (Cantab.), Lecturer in Applied Mathematics, University of Edinburgh (Drummond Street). 40A Morningside Park, Edinburgh 10	
1885	C.	† Scott, Alexander, M.A., D.Sc., F.R.S., Director of Scientific Research at the British Museum. 117 Hamilton Terrace, London, N.W. 8	
1919		* Scott, Alexander, M.A., D.Sc., 3 Winton Terrace, Stoke-on-Trent	
1919		* Scott, Alexander Ritchie, B.Sc. (Edin.), D.Sc. (Lond.), Principal, London County Council, Beaufoy Institute, Prince's Road, London, S.E. 11	
1917		* Scott, Henry Harold, C.M.G., M.D., F.R.C.P., M.R.C.S., D.P.H., Director, Bureau of Hygiene and Tropical Diseases, Keppel Street, Gower Street, London, W.C. 1. 18 Ridgmount Gardens, Gower Street, London, W.C. 1	
1928		* Senior-White, Ronald, F.R.E.S., Malariologist, Bengal-Nagpur Railway, Kidderpore, P.O., Calcutta, India	
1930		* Shankland, Ernest Claud, F.R.Met.S., River Superintendent, Port of London Authority. Mariners, Balfour Gardens, Folkestone	
1927		* Sharpley, Forbes Wilmot, B.Sc. (Eng.) (Lond.), M.Inst.E.E., Professor of Electrical and Mechanical Engineering, Indian School of Mines, Dhanbad, India	
1931		* Shaw, John James M'Intosh, M.A., M.D., F.R.C.S., Lecturer in Surgery and Clinical Surgery, University of Edinburgh. Greenaway, Kinnear Road, Edinburgh 4	
1927		* Shearer, Ernest, M.A., B.Sc. (Edin.), Professor of Agriculture and Rural Economy, University of Edinburgh, and Principal, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 8	
1931		* Shearer, James Fleming, M.A., B.Sc., Ph.D., Lecturer in Natural Philosophy, University of Glasgow. 149 Queen's Drive, Wavertree, Liverpool 15	
1932		* Simpson, Alexander Rudolf Barbour, B.Sc. (Edin.), M.A. (Cantab.), F.R.G.S., Hillstone School, Malvern, Worcestershire	

Date of Election			Service on Council, etc.
1908		Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., J.P., 43 Manor Place, Edinburgh 3	
1932	C.	* Simpson, John Baird, D.Sc. (Aberd.), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9	
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. (Edin.), H.H. the Thakur Sahib of Gondal, Kathiawar, Bombay, India	
1903		† Skinner, Robert Taylor, J.P., M.A., F.S.A.Scot., formerly House-Governor, Donaldson's Hospital. 35 Campbell Road, Edinburgh 12	
1937		* Slater, James Kirkwood, M.B. (Edin.), F.R.C.P.E., Assistant Physician, Royal Infirmary, Edinburgh, and Physician to the Deaconess Hospital, Edinburgh. 7 Walker Street, Edinburgh 3	
1930	C.	* Slater, Robert Henry, D.Sc., Ph.D. (Edin.), F.I.C., Department of Chemical Pathology, St Mary's Hospital, London, W. 2	
1929		* Smail, James Cameron, O.B.E., Companion Inst.E.E. (VICE-PRESIDENT), Principal, Heriot-Watt College, Edinburgh. 1 Grange Terrace, Edinburgh 9	1934–37. V-P 1937–
1926	C.	* Small, James, D.Sc., Professor of Botany, Queen's University, Belfast. Orkla, 50 Myrtlefield Park, Belfast	
1901		Smart, Edward, B.A., B.Sc., Tillyloss, Tullylumb Terrace, Perth	
1920		* Smellie, William Robert, M.A., D.Sc., Geologist on the Staff of the Anglo-Persian Oil Company. Baron Cliff, Cove, Dumbartonshire	
1937		* Smith, Alexander Martin, Ph.D., D.Sc. (Edin.), A.I.C., Lecturer in Agricultural Chemistry, University of Edinburgh, 13 George Square. 1 Mortonhall Road, Edinburgh 9	
1928		Smith, Alick Drummond Buchanan, M.A., B.Sc. (Agric.) (Aberd.), M.S.A. (Iowa), Lecturer, Institute of Animal Genetics, University of Edinburgh (West Mains Road)	
1937		* Smith, Horace George, B.Sc. (Glas.), Assistant, Department of Natural History, University of Aberdeen. Strathcona Club, Bucksburn, Aberdeen	
1921		* Smith, Norman Kemp, M.A., D.Phil., D.Litt., LL.D., Professor of Logic and Metaphysics, University of Edinburgh (South Bridge). Ellerton, Grange Loan, Edinburgh 9	
1923		* Smith, Percy James Lancelot, M.A. (Oxon.), F.I.C., F.C.S., Science Master, Loretto School. 47 Dalrymple Loan, Musselburgh	
1911		Smith, Stephen, B.Sc., 34 Craigmillar Park, Edinburgh 9	
1929		* Smith, Sydney, M.D., F.R.C.P., D.P.H., Professor of Forensic Medicine, University of Edinburgh (Teviot Place). 10 Oswald Road, Edinburgh 9	1936–
1919		* Smith, Sir William Wright, Kt., M.A., D. ès Sc., Regius Professor of Botany, University of Edinburgh, Regius Keeper of the Royal Botanic Garden, and King's Botanist in Scotland. Inverleith House, Edinburgh 4	Sec. 1923–28. V-P 1928–31.
1932		* Sneeden, Jean-Baptiste Octave, B.Sc., Ph.D. (Glas.), Lecturer on Heat Engines, Royal Technical College, Glasgow. 39 Kingshouse Avenue, Glasgow, S. 4	
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. (Cantab.), Barrister-at-Law, formerly Medical Officer of Health, Coventry. 3 Eaton Road, Coventry	
1933		* Somerville, John Livingston, C.A., Auditor, University of Edinburgh. 8 Ravelston Park, Edinburgh 4	
1929		* Southwell, Thomas, D.Sc., A.R.C.S., Walter Myers Lecturer in Parasitology, School of Tropical Medicine, University of Liverpool. 53 Greenhill Road, Mossley Hill, Liverpool 18	
1925		* Staig, Robert Arnot, M.A., Ph.D., Lecturer in Zoology, University of Glasgow. Glenlea, Lasswade, Midlothian	
1891		Stanfield, Richard, A.R.S.M., M.Inst.C.E., Emeritus Professor of Mechanics and Engineering, Heriot-Watt College, Edinburgh. 24 Mayfield Gardens, Edinburgh 9	1926–29.
1923		* Stebbing, Edward Percy, M.A., Professor of Forestry, University of Edinburgh (George Square)	
1923		* Stenhouse, Andrew G., F.G.S., 191 Newhaven Road, Edinburgh 6	
1929	C.	* Stephen, Alexander Charles, D.Sc., Keeper, Natural History Department, Royal Scottish Museum, Edinburgh. Eastcroft, Cramond Bridge, Edinburgh 4	
1910		Stephenson, Thomas, D.Sc., F.C.S., 65 Castle Street, Edinburgh 2	
1931		* Steven, George Alexander, B.Sc. (Edin.), Assistant Naturalist, Marine Laboratory, Plymouth. 1 Seaview Villas, Pentyre Terrace, Plymouth, Devon	

Date of Election			Service on Council, etc.
1937	C.	* Steven, Henry Marshall, B.Sc., Ph.D. (Edin.), M.A., Hon. Causa (Oxon.), Divisional Officer, Forestry Commission. 75 Fountainhall Road, Aberdeen	
1925		* Stevens, Alexander, M.A., B.Sc., Lecturer in Geography, University of Glasgow	
1886	C.	Stevenson, Charles A., B.Sc., M.Inst.C.E., Radella, North Berwick	
1884		† Stevenson, David Alan, B.Sc., M.Inst.C.E., Troqueer, Kingsknowe, Slateford, Edinburgh 11	1928–31.
1919		* Stevenson, David Alan, B.Sc., M.Inst.C.E., 22 Glencairn Crescent, Edinburgh 12	
1935		* Stevenson, Eric, B.Sc. (Edin.), A.M.I.Mech.E., Lecturer in Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 7 Beauchamp Road. Edinburgh 9	
1936		* Stewart, Alexander Dron, C.I.E., M.B., Ch.B. (Edin.), F.R.C.S.E., Lt.-Col. I.M.S. (retired), Superintendent of the Royal Infirmary, Edinburgh. Meadow Walk House, Edinburgh 3	
1931		* Stewart, Corbet Page, Ph.D. (Edin.), Lecturer in General Biochemistry, University of Edinburgh (Teviot Place). 17 Orchard Road South, Edinburgh 4	
1925		* Stewart, David Smith, Ph.D., M.Inst.C.E., Lecturer on Structural Engineering Drawing, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 82 Lasswade Road, Edinburgh 9	
1924		* Stiles, Sir Harold Jalland, K.B.E., Kt., M.B., F.R.C.S.E., D.Sc. (Hon.), LL.D., Emeritus Professor of Clinical Surgery, University of Edinburgh. Whatton Lodge, Gullane, E. Lothian	1934–37.
1902		Stockdale, Herbert Fitton, LL.D., formerly Director of the Royal Technical College, Glasgow. Clairinch, Upper Helensburgh, Dumbartonshire	
1889	C.	Stockman, Ralph, M.D., LL.D., F.R.C.P.E., F.F.P.S.G., Emeritus Professor of Materia Medica and Therapeutics, University of Glasgow. White Lodge, Barnton Avenue, Edinburgh 4	1903–05, 1936–
1926		* Stokoe, William Norman, B.Sc., Ph.D. (Lond.), Chief Chemist, Craigmillar Creamery Co., Ltd. 8 Cobden Road, Edinburgh 9	
1906		Story, Fraser, O.B.E., formerly Professor of Forestry, University College, Bangor, North Wales. The Wall House, Yorke Road, Reigate, Surrey	
1907		Strong, John, C.B.E., M.A., LL.D., Emeritus Professor of Education, University of Leeds. C/o The Librarian, The University, Leeds	
1930	C.	* Stump, Claude Witherington, M.D., D.Sc., Professor of Embryology and Histology, University of Sydney	
1937		* Suffolk and Berkshire, The Rt. Hon. The Earl of, Glencairn, Eskbank, Midlothian	
1903		Sutherland, David W., C.I.E., M.D., M.R.C.P., Lt.-Col., I.M.S. (retired), Braeside, Belhaven, Dunbar	
1935		* Sutherland, John Derg, B.Sc., B.Ed., Ph.D. (Edin.), M.B., Ch.B., Lecturer in Psychology, University of Edinburgh. 2 Windsor Street, Edinburgh 7	
1930		* Sutherland, Sir John Donald, C.B.E., LL.D., Legion of Honour (France), Order of Leopold (Belgium), formerly Forestry Commissioner, Scotland. 11 Inverleith Row, Edinburgh 4	
1925		Sutton, Richard L., M.D., D.Sc., LL.D., 1308 Bryant Building, 1102 Grand Avenue, Kansas City, Mo., U.S.A.	
1932		* Swinton, William Elgin, B.Sc., Ph.D. (Glas.), F.L.S., F.Z.S., F.G.S., Curator of Fossil Reptiles and Amphibia, British Museum (Natural History), South Kensington, London, S.W. 7	
1933		* Tait, John Barclay, B.Sc. (Edin.), Ph.D., A.H.-W.C., Senior Naturalist (Hydrographer), Marine Laboratory, Fishery Board for Scotland, Aberdeen. 23 Cromwell Road, Aberdeen	
1937		* Tait, John Guthrie, Scholar of Peterhouse College, Cambridge, B.A. (Cantab.), Barrister-at-Law, Lincoln's Inn, Hon. Fellow, Madras University, formerly Principal, Central College, Bangalore, India. 38 George Square, Edinburgh 8	
1890	C.	† Tanakadate, Aikitu, Hon. Professor of Natural Philosophy, Imperial University of Japan. Koisikawa, Zôsigayamati, 144, Tokyo, Japan	
1899		Taylor, James, M.A., formerly Mathematical Master, Edinburgh Academy. 18 Hillview, Blackhall, Edinburgh 4	
1933		* Taylor, George, D.Sc. (Edin.), F.L.S., Assistant Keeper, Department of Botany, British Museum. Ballochmyle, Loudwater, Rickmansworth, Herts	

Date of Election			Service on Council, etc.
1885	C.	Thompson, Sir D'Arcy Wentworth, Kt., C.B., M.A., LL.D. (Aberd., Edin.), Hon. D.Sc. (Dublin, Witwatersrand), D.Litt., F.R.S. (PRESIDENT), Professor of Natural History, University, St Andrews. 44 South Street, St Andrews	1892–95, 1896–99, 1907–10, 1912–15, 1922–25. V-P 1916–19. Curator 1926–34. P 1934–
1932		Thompson, Harold William, D.Litt. (Edin.), A.M., Ph.D. (Harvard), F.S.A. Scot., Professor of English, N.Y. State College, Albany, N.Y., U.S.A.	
1917	C. N.	* Thompson, John M'Lean, M.A., D.Sc., F.L.S., Professor of Botany, University of Liverpool	
1896		Thomson, George Ritchie, C.M.G., M.B., C.M., formerly Professor of Surgery, University of the Witwatersrand, Johannesburg, Transvaal. Hordle Grange, Hordle, Hants	
1903		Thomson, George S., Kenmore Farm, Whelpley Hill, Chesham, Bucks	
1906		Thomson, Gilbert, M.A., M.Inst.C.E., 164 Bath Street, Glasgow, C. 2	
1926		* Thomson, Godfrey Hilton, D.Sc., Ph.D., Professor of Education, University of Edinburgh (Moray House)	1931–34.
1926	C.	* Thomson, John, M.A., Ph.D. (Glas.), Lecturer in Plant Physiology, University of Glasgow. 2 Chartwell Terrace, Bearsden, Glasgow	
1934		* Thomson, Matthew Sydney, M.A., M.D., B.Ch. (Cantab.), F.R.C.P., M.R.C.S., Physician for Diseases of the Skin, King's College Hospital, Belgrave Hospital for Children. 106 Harley Street, London, W. 1	
1899		Thomson, R. Tatlock, F.I.C., 156 Bath Street, Glasgow	
1912	C.	Thomson, Robert Black, M.B. (Edin.), Aliwal North, Cape Province, S.A.	
1882		Thomson, Sir William, Kt., M.A., B.Sc., LL.D., formerly Principal, University of the Witwatersrand. Dunedin, Glencairn, Simonstown, South Africa	
1917		* Thorneycroft, Wallace, J.P., Chalmington, Dorchester	
1933		* Timms, Geoffrey, Ph.D. (Cantab.), Lecturer in Mathematics, University of St Andrews. Deanscourt, St Andrews	
1937		* Tod, Henry, B.Sc., Ph.D. (Edin.), Biochemist, The Royal Edinburgh Hospital. 35 Oxgangs Road, Edinburgh 10	
1920		* Todd, John Barber, B.Sc., Ph.D., M.I.Mech.E., Reader in Engineering, University of Edinburgh. 4 Bright's Crescent, Edinburgh 9	
1935		Touche, John Edward, M.I.Mech.E., 11 Blackford Road, Edinburgh 9	
1917		* Tovey, Sir Donald Francis, Kt., B.A. (Oxon.), M.Mus. (Hon.), Birmingham, Professor of Music, University of Edinburgh (Reid School of Music). 39 Royal Terrace, Edinburgh 7	
1914		† Tredgold, Alfred Frank, M.D. (Durham), F.R.C.P. (Lond.), Lecturer on Mental Deficiency, University of London. St Martins, Guildford	
1915		* Trotter, George Clark, M.D. (Edin.), D.P.H. (Aberdeen), F.S.A.Scot., Medical Officer of Health, Metropolitan Borough, Islington. Braemar, 17 Haslemere Road, Crouch End, London, N. 8	
1922	C. K.	* Turnbull, Herbert Westren, M.A., F.R.S., Professor of Mathematics, University of St Andrews. Randa, Hepburn Gardens, St Andrews	1928–31.
1937		* Turnbull, Mathew McKerrow, Lecturer in Banking, University of Edinburgh (South Bridge). 48 Inverleith Place, Edinburgh 4	
1905		† Turner, Arthur Logan, M.D., LL.D., F.R.C.S.E., 27 Walker Street, Edinburgh 3	1926–29. V-P. 1930–33.
1925		* Turner, Harry Moreton Stanley, M.B.E., M.D., M.R.C.S., L.R.C.P., D.T.M. and H., Chevalier de l'Ordre Royale du Sauveur de Grèce. Wing Commander, R.A.F. (retired). Meerbrook House, Overdale, Ashtead, Surrey	
1924		* Turner, Richard, O.B.E., M.B., C.M., Hotel Hydropathic, Peebles	
1918	C. N.	* Tyrrell, George Walter, A.R.C.S., D.Sc., F.G.S., Lecturer in Petrology, Geological Department, University of Glasgow	1926–29. 1937–
1930	C.	* Voge, Cecil Innes Bothwell, Ph.D. (Edin.), 46 Roxborough Park, Harrow-on-the-Hill, London	
1932		* Wade, Henry, C.M.G., D.S.O., M.D., Senior Lecturer in Clinical Surgery, University of Edinburgh (Royal Infirmary). 6 Manor Place, Edinburgh 3	

Date of Election			Service on Council, etc.
1926		* Wakeley, Cecil Pembrey Grey, F.R.C.S., Surgeon to King's College Hospital, London, Lecturer in Anatomy, King's College, London. 14 Devonshire Street, Portland Place, London, W. 1	
1925	C.	* Walker, Frederick, M.A., Ph.D., D.Sc., Department of Geology, Columbia University, New York City, U.S.A.	
1931		* Walker, William James Stirling, Ph.D. (Edin.), A.H.-W.C., F.I.C., Scientific Officer, H.M. Fuel Research Station, East Greenwich, London, S.E. 10. C/o Harrison, 64 Sandtoft Road, Charlton, London, S.E. 7	
1902		Wallace, Alexander G., M.A., 56 Fonthill Road, Aberdeen	
1886	C.	Wallace, Robert, M.A., LL.D., F.L.S., Emeritus Professor of Agriculture and Rural Economy, University of Edinburgh. Mid Park House, Kincardine-on-Forth, Fife	
1898		Wallace, William, M.A., Campsie, Alta, Canada	
1920		* Walmsley, Thomas, M.D. (Glas.), Professor of Anatomy, Queen's University, Belfast	
1931	C.	* Walton, John, M.A. (Cantab.), D.Sc. (Manchester) (VICE-PRESIDENT), Regius Professor of Botany, University of Glasgow. 23 Lilybank Gardens, Glasgow, W. 2	1934–37. V-P 1937–
1927	C.	* Wardlaw, Claude Wilson, D.Sc. (Glas.), Imperial College of Tropical Agriculture, Trinidad, B.W.I.	
1936		* Warr, The Very Rev. Charles Laing, C.V.O., M.A., D.D., LL.D., Hon. R.S.A., Dean of the Thistle and of the Chapel Royal in Scotland, Chaplain to H.M. The King, Minister of St Giles' Cathedral, Edinburgh. 63 Northumberland Street, Edinburgh 3	
1923		* Warren, John Alexander, M.Inst.C.E. 74 Balshagray Avenue, Partick, Glasgow	
1901	C.	Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, University of St Andrews	1916–19, 1925–28.
1927		* Watson, Charles Brodie Boog, F.S.A.Scot. 24 Garscube Terrace, Edinburgh 12	
1923		* Watson, H. Ferguson, M.D., F.R.F.P.S., Ph.D., D.P.H. (Glas.), formerly H.M. Senior Deputy Commissioner, General Board of Control for Scotland. 109 Montgomery Street, Edinburgh 7	
1923	C.	* Watson, William, M.A., B.Sc. (Edin.), Lecturer in Physics, Heriot-Watt College, Edinburgh. 17 Braidburn Crescent, Edinburgh 10	
1911		† Watt, James, W.S., F.F.A., LL.D. (VICE-PRESIDENT), 7 Blackford Road, Edinburgh 9	1924–26. Treasurer 1926–37. V-P 1937–
1933		* Watt, John Mitchell, M.B., Ch.B., M.R.C.P. (Edin.), F.R.S.S.A., Professor of Pharmacology, University of the Witwatersrand, Johannesburg, South Africa	
1911		Watt, Very Rev. Lauchlan MacLean, M.A., D.D., LL.D., Kinloch, Lochcarron, Ross-shire	
1928		* Watters, Alexander Marshall, M.A., B.Sc. (Glas.), Rector of Hawick High School. High School House, Hawick	
1896		† Webster, John Clarence, C.M.G., M.D., D.Sc., LL.D., F.R.C.P.E., formerly Professor of Obstetrics and Gynæcology, Rush Medical College, Shediac, N.B., Canada	
1907	M-B. C.	† Wedderburn, Ernest Maclagan, O.B.E., D.K.S., M.A., D.Sc., LL.B. (TREASURER). 6 Succoth Gardens, Edinburgh 12	1913–16, 1921–24, 1932–35, 1936–37, Treasurer 1937–
1903	M-B. C.	† Wedderburn, J. H. Maclagan, M.A., D.Sc., F.R.S., Professor of Mathematics, Princeton University. Fine Hall, Princeton, N.J., U.S.A.	
1904		Wedderspoon, William Gibson, M.A., LL.D.	
1934	C.	* Weir, John, M.A., Ph.D., D.Sc. (Glas.), Lecturer in Palæontology, University of Glasgow. 18 Botanic Crescent, Glasgow, N.W.	
1930		* White, Adam Cairns, M.B., Ch.B., Ph.D., Assistant Pharmacologist, Wellcome Physiological Research Laboratory, Beckenham, Kent	
1933		* Whitley, William Frederic James, M.D. (Edin.), D.P.H. (Oxon.), Medical Officer of Health, Northumberland County Council. Westfield, Cramlington, Northumberland	

Date of Election			Service on Council, etc.
1931		* Whitson, Sir Thomas Barnby, D.L., LL.D., C.A., Lord Provost of the City of Edinburgh (1929–32). 27 Eglinton Crescent, Edinburgh 12	
1911		Whittaker, Charles Richard, F.R.C.S.E., F.S.A.Scot., Lynwood, Hatton Place, Edinburgh 9	
1912	C. V. J. B-P.	Whittaker, Edmund Taylor, M.A., Hon. Sc.D. (Dubl.), Hon. LL.D. (St Andrews and California), F.R.S., Foreign Member of the R. Accademia dei Lincei, Rome, Member of the Pontifical Academy of Sciences (VICE-PRESIDENT), Professor of Mathematics, University of Edinburgh (16 Ghambers Street). 48 George Square, Edinburgh 8	1912–15, 1922–25. Sec. 1916–22. V-P 1925–28, 1937–
1928	C.	* Whittaker, John Macnaghten, M.A. (Edin.), M.A. (Cantab.), D.Sc., Professor of Pure Mathematics, University of Liverpool	
1936		* Whyte, Andrew, A.C.A., F.R.G.S., Chartered Accountant. The Knoll, 1 Linden Avenue, Darlington	
1918		* Whyte, Rev. Charles, M.A., LL.D., F.R.A.S., U.F. Church Manse, Kingswells, Aberdeen	
1935		* Whyte, Sir William Edward, Kt., O.B.E., J.P., Solicitor. Balgay, Uddingston	
1934		* Whyte, William, Cashier and General Manager of the Royal Bank of Scotland. Baberton House, Juniper Green, Edinburgh	
1929	C.	* Wiesner, Bertold Paul, D.Sc., formerly Macaulay Lecturer, Institute of Animal Genetics, University of Edinburgh, 14A Manchester Square, London, W. 1	
1918		* Wight, John Thomas, M.I.Mech.E., M.I.Mar.E., Joint Managing Director, Messrs MacTaggart, Scott & Co., Ltd., Loanhead. Calderwood Villa, Lasswade	
1934		* Wightman, William Persehouse Delisle, Ph.D., M.Sc. (Lond.), Science Master, Edinburgh Academy. 36 Coltbridge Terrace, Edinburgh 12	
1925		* Wilkie, Sir David Percival Dalbreck, O.B.E., M.D., Ch.M., F.R.C.S., Professor of Surgery, University of Edinburgh (Royal Infirmary). 9 Ainslie Place, Edinburgh 3	
1926	C. N.	* Williams, Samuel, Ph.D., Lecturer in Plant Morphology, University of Glasgow. 27 Lindsay Place, Kelvindale, Glasgow	
1908		Williamson, Henry Charles, M.A., D.Sc., formerly Naturalist to the Fishery Board for Scotland. 13 Windsor Street, Dundee	
1928	C.	* Williamson, John, M.A. (Edin.), Ph.D. (Chicago), Associate Professor of Mathematics, Johns Hopkins University, Baltimore, U.S.A.	
1910	C.	Williamson, William, F.L.S., 47 St Alban's Road, Edinburgh 9	
1927	C.	* Williamson, William Turner Horace, B.Sc. (Aberd.), Ph.D. (Edin.), Chief Chemist, Egyptian Ministry of Agriculture, Cotton Research Board, Giza, Egypt	
1911		Wilson, Andrew, O.B.E., D.L., M.Inst.C.E., 66 Netherby Road, Edinburgh 5	
1902	V. J.	† Wilson, Charles Thomson Rees, C.H., M.A., LL.D., D.Sc., F.R.S., Nobel Prize, Physics, 1927, Emeritus Professor of Natural Philosophy, University of Cambridge. 196 Grange Loan, Edinburgh 9	1937–
1922		* Wilson, John, F.R.I.B.A., Chief Architect, Scottish Department of Health. 20 Lomond Road, Edinburgh 5	
1920	C.	* Wilson, Malcolm, D.Sc. (London), A.R.C.S., F.L.S., Reader in Mycology and Bacteriology, University of Edinburgh (Royal Botanic Garden). Brent Knoll, Kinnear Road, Edinburgh 4	1931–34.
1924		* Wilson, William, M.A., LL.B., Advocate, Regius Professor of Public Law, University of Edinburgh (South Bridge). 38 Moray Place, Edinburgh 3	
1895		Wilson-Barker, Sir David, Kt., R.D., R.N.R., F.R.G.S., formerly Captain-Superintendent, Thames Nautical Training College, H.M.S. "Worcester." 12 Bolan Street, London, S.W. 11	
1934		* Winstanley, Arthur, M.B.E., D.Sc. (Eng.) (Lond.), M.I.Min.E., Mining Engineer to Safety in Mines Research Board. 18 St John's Road, Edinburgh 12	
1931	C.	* Wishart, John, M.A., B.Sc. (Edin.), M.A. (Cantab.), D.Sc. (Lond.), Reader in Statistics, University of Cambridge. Croft Lodge, Barton Road, Cambridge	
1922	C. B.	* Wordie, James Mann, M.A. (Cantab.), B.Sc. (Glas.), St John's College, Cambridge	
1937		* Wright, Edward Maitland, B.A. (Lond.), M.A., D.Phil. (Oxon.), Professor of Mathematics, University of Aberdeen. 52 College Bounds, Aberdeen.	

Date of Election			Se Cou
1933		* Wright, James, F.G.S., Balado, 212 Colinton Road, Edinburgh 11	
1896		† Wright, Sir Robert Patrick, LL.D., formerly Chairman of the Board of Agriculture for Scotland. The Heugh, North Berwick, East Lothian	
1911	C.	Wrigley, Ruric Whitehead, M.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh	
1937	C.	* Young, Andrew White, M.A., B.Sc., LL.B. (Edin.), W.S., 30 Queen Street, Edinburgh 2	
1882		Young, Frank W., C.B.E., F.C.S., H.M. Inspector of Schools (Emeritus). Panera, Shortheath, Farnham, Surrey	
1904		Young, Robert B., M.A., D.Sc., F.G.S., Professor of Geology, University of the Witwatersrand (South African School of Mines and Technology), Johannesburg, Transvaal	

Number of Fellows, 738.

HONORARY FELLOWS OF THE SOCIETY.

(*At 25th October* 1937.)

1920 HIS ROYAL HIGHNESS THE DUKE OF WINDSOR, K.G.

FOREIGNERS (LIMITED TO FORTY-FOUR BY LAW I).

Elected
1933 John Jacob Abel, Emeritus Professor of Pharmacology, Johns Hopkins University, Baltimore.
1936 Leo Hendrik Baekeland, Professor (Honorary) of Chemical Engineering, Columbia University, New York. Bakelite Corporation, 247 Park Avenue, New York City.
1916 Charles Eugène Barrois, formerly Professor of Geology and Mineralogy, Université, Lille, France. 41, Rue Pascal, Lille.
1923 Henri Bergson, Honorary Professor, College of France, Paris.
1930 Vilhelm Frimann Koren Bjerknes, Professor of Physics, University of Oslo.
1937 Marston Taylor Bogert, Professor of Organic Chemistry, Columbia University, N.Y.
1927 Niels Bohr, Nobel Laureate, Physics, 1922, Professor of Physics, University of Copenhagen.
1927 Jules Bordet, Nobel Laureate, Medicine, 1919, Professor of Bacteriology, University of Brussels.
1933 Filippo Bottazzi, Professor of Experimental Physiology, Royal Institute of Physiology, S. Andrea delle Dame, 21, Naples.
1923 Marcellin Boule, Director of the Institute of Human Palæontology, 1, Rue René-Panhard, Paris, XIII^e.
1905 Waldemar Christofer Brögger, Professor of Mineralogy and Geology, K. Frederiks Universitet, Oslo.
1916 Douglas Houghton Campbell, Emeritus Professor of Botany, Stanford University, California.
1920 William Wallace Campbell, President-Emeritus of the University of California, Berkeley, Director-Emeritus of the Lick Observatory, Mt. Hamilton, California, and formerly President of the National Academy of Sciences.
1930 Walter Bradford Cannon, Professor of Physiology, Harvard University, Cambridge, Mass.
1930 Maurice Caullery, Professor of Zoology in the University of Paris. Laboratoire d'Évolution des Êtres organisés, 105 Bould. Raspail, Paris, VI^e.
1933 Edwin Grant Conklin, formerly Professor of Biology, Princeton University, N.J.
1921 Reginald Aldworth Daly, Professor of Geology, Harvard University, Cambridge, Mass.
1927 Albert Einstein, Nobel Laureate, Physics, 1921, Princeton University, N.J.
1913 George Ellery Hale, Honorary Director of Mount Wilson Observatory (Carnegie Institution of Washington), Pasadena, California.
1934 Björn Helland-Hansen, Geophysical Institute, Bergen.
1921 Johan Hjort, Professor of Marine Biology, University, Oslo.
1923 Arnold Frederik Holleman, Emeritus Professor of Organic Chemistry, University, Amsterdam. Boekensteijn Parkweg 7, Bloemendaal.
1934 Bernardo Houssay, Professor of Physiology, National University of Buenos Aires.
1937 C. U. Ariëns Kappers, Director of the Central Institute of Brain Research, Amsterdam, and Professor of Comparative Neurology, University, Amsterdam.
1933 Nikolaj Konstantinovic Koltzoff, formerly Professor of Zoology, State University, Moscow; Director of the Research Institute of Experimental Biology. Moscow 64, Voronzovo Pole 6.
1923 Tullio Levi-Civita, Professor of Mathematics, Regia Università, Rome.
1934 Frank Rattray Lillie, Emeritus Professor of Zoology and Embryology, University of Chicago, and President, National Academy of Sciences, Washington, D.C.
1936 Maurice Lugeon, Professor of Geology, University, Lausanne.
1927 Hans Horst Meyer, Emeritus Professor of Pharmacology, University of Vienna.
1934 Thomas Hunt Morgan, Nobel Laureate, Medicine, 1933, Professor of Biology, California Institute of Technology, Pasadena.
1933 Albrecht Penck, Geheimrat, Emeritus Professor of Geography, Friedrich-Wilhelms-Universität, Berlin. Geographisches Institut, Universitäts-strasse 3*b*, Berlin, N.W. 7
1920 Charles Émile Picard, Perpetual Secretary, Academy of Sciences, Paris.
1937 Max Planck, Nobel Laureate, Physics, 1918, Professor Ordinarius Emeritus of Theoretical Physics, Director of the Institute for Theoretical Physics, University of Berlin.
1934 Paul Sabatier, Nobel Laureate, Chemistry, 1912, Professor of Chemistry, University of Toulouse.

Elected

1936 George Sarton, Editor of *Isis* and *Osiris*, Harvard Library, 185, Cambridge, Mass.

1930 Erik Helge Oswald Stensiö, Professor of Palæontology and Historical Geology, Royal University of Upsala.

1936 George Linius Streeter, Director, Department of Embryology, Carnegie Institution of Washington, Corner Wolfe and Madison Streets, Baltimore, Md. 3707 St Paul Street, Baltimore, Md.

1936 Nikolai Ivanovic Vavilov, Director of the Institute of Plant Industry, Academy of Sciences, Leningrad, U.S.S.R.

1913 Vito Volterra, formerly Professor of Mathematical Physics, Regia Università. Villa Volterra, Ariccia, near Rome.

1927 Richard Willstätter, Nobel Laureate, Chemistry, 1915, Professor of Chemistry, University of Munich. Moehlstrasse 29, Munich 27.

1923 Edmund Beecher Wilson, formerly Professor of Zoology, Columbia University, New York.

1933 Pieter Zeeman, Nobel Laureate, Physics, 1902, Emeritus Professor of Physics, University, Amsterdam. Stadhonderskade 158, Amsterdam.

Total, 42.

British Subjects (limited to twenty-two by Law I).

1937 John Logie Baird, Inventor of the Televisor. 3 Crescent Wood Road, Sydenham, London, S.E.

1936 Sir Charles Vernon Boys, Kt., A.R.S.M., LL.D., F.R.S., St Marybourne, Andover.

1927 Sir William Henry Bragg, O.M., K.B.E., M.A., D.Sc., LL.D., President R.S., Nobel Laureate, Physics, 1915, Fullerian Professor of Chemistry, Royal Institution, London.

1937 William Thomas Calman, C.B., D.Sc., LL.D. (St And.), F.R.S., formerly Keeper of Zoology, British Museum. 24 Lexham Gardens, London, W. 8.

1936 Sir Henry Hallett Dale, Kt., C.B.E., M.D., Hon. D.Sc., F.R.S., Joint Nobel Laureate, Medicine, 1936, Director of the National Institute for Medical Research, Mount Vernon, London, N.W. 3.

1936 Frederick George Donnan, C.B.E., M.A., Ph.D., D.Sc., LL.D., F.R.S., formerly Professor of Chemistry, University of London, and Director of Chemical Laboratories, University College. 23 Woburn Square, London, W.C.

1930 Sir Arthur Stanley Eddington, Kt., M.A., Hon. D.Sc., LL.D., F.R.S., Plumian Professor of Astronomy and Experimental Philosophy, University of Cambridge.

1927 Sir John Bretland Farmer, Kt., M.A., D.Sc., LL.D., F.R.S., Emeritus Professor of Botany, Imperial College of Science and Technology, London. St Leonard's, Weston Road, Bath.

1900 Andrew Russell Forsyth, M.A., Sc.D., LL.D., Hon. Math.D., F.R.S., Emeritus Professor of Mathematics, Imperial College of Science and Technology, London; formerly Sadleirian Professor of Pure Mathematics, University of Cambridge.

1910 Sir James George Frazer, Kt., O.M., D.C.L., LL.D., Litt.D., F.B.A., F.R.S., Commandeur de la Légion d'Honneur. Trinity College, Cambridge.

1927 Sir Frederick Gowland Hopkins, Kt., O.M., M.A., M.B., Hon. D.Sc., LL.D., Past President R.S., Joint Nobel Laureate, Medicine, 1929, Sir William Dunn Professor of Biochemistry, University of Cambridge

1930 Sir Arthur Keith, Kt., M.D., LL.D., F.R.S., Master, Buckston Browne Research Farm, Downe, Farnborough, Kent.

1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L., F.R.S., formerly Lucasian Professor of Mathematics, University of Cambridge. St John's College, Cambridge.

1933 Sir George Macdonald, K.C.B., D.Litt., Litt.D., LL.D., F.B.A., formerly Secretary, Scottish Education Department. 17 Learmonth Gardens, Edinburgh 4.

1934 Sir Edward Bagnall Poulton, Kt., M.A., D.Sc., LL.D., F.R.S., formerly Hope Professor of Zoology, University of Oxford. Wykeham House, Banbury Road, Oxford.

1930 Robert Robinson, M.A., D.Sc., LL.D., F.R.S., Waynflete Professor of Chemistry, University of Oxford.

1933 Sir William Napier Shaw, Kt., Sc.D.(Cantab.), LL.D., F.R.S., formerly Director Meteorological Office. 10 Moreton Gardens, London, S.W. 5.

1908 Sir Charles Scott Sherrington, O.M., G.B.E., M.A., D.Sc., M.D., LL.D., Past President R.S., Joint Laureate, Nobel Prize, Medicine, 1932, formerly Waynflete Professor of Physiology, University of Oxford.

1905 Sir Joseph John Thomson, Kt., O.M., D.Sc., LL.D., Past President R.S., Nobel Laureate, Physics, 1906, formerly Cavendish Professor of Experimental Physics, now Professor of Physics, University of Cambridge, Master of Trinity College, Cambridge.

1934 William Whitehead Watts, Sc.D., M.Sc., LL.D., F.R.S., Emeritus Professor of Geology, Imperial College of Science and Technology, London. Hillside, Langley Park, Sutton, Surrey.

Total, 20.

CHANGES IN FELLOWSHIP DURING SESSION 1936–1937.

FELLOWS OF THE SOCIETY ELECTED.

ADAM JOHN GUILBERT BARNETT.
KALIPADA BISWAS.
JOSEPH JOHN BLACKIE.
MAX BORN.
GEORGE BERNARD BROOK.
ARCHIBALD GRAY ROBERTSON BROWN.
CHARLES JOHNSTONE COUSLAND.
WILLIAM STUART McRAE CRAIG.
SASINDRA CHANDRA DHAR.
HUGH PATERSON DONALD.
DERRICK MELVILLE DUNLOP.
ROBERT CAMPBELL GARRY.
ANDREW RAE GILCHRIST.
GEORGE GREEN.
WILLIAM ALLAN FORSYTH HEPBURN.
PERCIVAL ROBSON KIRBY.
PEO CHARLES KOLLER.
WILLIAM ORR LEITCH.
HARRY WORK MELVILLE.
JAMES MILLER.
JOHN MORISON.
CHARLES ROY NASMITH.
THOMAS T. PATERSON.
JOHN DONALD POLLOCK.
BADRI NARAYAN PRASAD.
FRANCIS IAN GREGORY RAWLINS.
MOWBRAY RITCHIE.
JOHN WATSON ROBERTSON.
THE HON. LORD ROBERTSON.
WILLIAM RITCHIE RUSSELL.
JAMES KIRKWOOD SLATER.
ALEXANDER MARTIN SMITH.
HORACE GEORGE SMITH.
HENRY MARSHALL STEVEN.
THE RT. HON. THE EARL OF SUFFOLK AND BERKSHIRE.
JOHN GUTHRIE TAIT.
HENRY TOD.
MATHEW McKERROW TURNBULL.
EDWARD MAITLAND WRIGHT.
ANDREW WHITE YOUNG.

HONORARY FELLOWS ELECTED.

FOREIGN.

MARSTON TAYLOR BOGERT.
C. U. ARIËNS KAPPERS.
MAX PLANCK.

BRITISH.

JOHN LOGIE BAIRD.
WILLIAM THOMAS CALMAN.

FELLOWS DECEASED.

DE BURGH BIRCH.
ALBERT WILLIAM BORTHWICK.
W. BRODIE BRODIE.
ROBERT CRAIG COWAN.
ALFRED DANIELL.
JOHN EDWARDS.
DAVID ELLIS.
MUNGO McCALLUM FAIRGRIEVE.
D. F. FRASER-HARRIS.
ALFRED WILLIAM GIBB.
JOHN ANDERSON GILRUTH.
SIR PATRICK HEHIR.
WILLIAM HUNTER.
G. A. FRANK KNIGHT.
JAMES KNIGHT.
ALBERT ERNEST LAURIE.
JAMES ALEXANDER MACDONALD.
MAGNUS MACLEAN.
SIR DAVID ORME MASSON.
HENRY MOIR.
WILLIAM JOHN OWEN.
ARTHUR GEORGE PERKIN.
JOHN SMITH PURDY (1935–36).
WILLIAM RAMSAY SMITH.

FOREIGN HONORARY FELLOWS DECEASED.

RICHARD ANSCHÜTZ.
SALVATORE PINCHERLE.
WILLIAM MORTON WHEELER.

BRITISH HONORARY FELLOWS DECEASED.

HENRY EDWARD ARMSTRONG.
THE RT. HON. LORD RUTHERFORD OF NELSON.
SIR GRAFTON ELLIOT SMITH.

FELLOWS RESIGNED.

THOMAS PURVES BLACK.
ALEXANDER BOWMAN.
SUDHAMOY GHOSH.
THOMAS JOHNSON.
JOHN TAIT.

FELLOW REMOVED FROM ROLL.

ERIC PONDER

LAWS OF THE SOCIETY.

Adopted July 3, 1916; *amended December* 18, 1916.

LAW I, amended February 5, 1934.		LAW IX, amended May 3, 1920.
„ VI, „ „ 7, 1921.		„ XIII, „ May 3, 1920.
„ „ „ July 2, 1928.		„ XIX, „ June 16, 1924.
„ VIII, „ May 3, 1920.		

I.

THE ROYAL SOCIETY OF EDINBURGH, which was instituted by Royal Charter in 1783 for the promotion of Science and Literature, shall consist of Ordinary Fellows (hereinafter to be termed Fellows) and Honorary Fellows. The number of Honorary Fellows shall not exceed sixty-six, of whom not more than twenty-two may be British subjects, and not more than forty-four subjects of Foreign States.

Fellows only shall be eligible to hold office or to vote at any Meeting of the Society.

ELECTION OF FELLOWS.

II.

Each Candidate for admission as a Fellow shall be proposed by at least four Fellows, two of whom must certify from personal knowledge. The Official Certificate shall specify the name, rank, profession, place of residence, and the qualifications of the Candidate. The Certificate shall be delivered to the General Secretary before the 30th of November, and, subject to the approval of the Council, shall be exhibited in the Society's House during the month of January following. All Certificates so exhibited shall be considered by the Council at its first meeting in February, and a list of the Candidates approved by the Council for election shall be issued to the Fellows not later than the 21st of February.

III.

The election of Fellows shall be by Ballot, and shall take place at the first Ordinary Meeting in March. Only Candidates approved by the Council shall be eligible for election. A Candidate shall be held not elected, unless he is supported by a majority of two-thirds of the Fellows present and voting.

IV.

On the day of election of Fellows two scrutineers, nominated by the President, shall examine the votes and hand their report to the President, who shall declare the result.

V.

Each Fellow, after his election, is expected to attend an Ordinary Meeting, and sign the Roll of Fellows, he having first made the payments required by Law VI. He shall be introduced to the President, who shall address him in these words:—

In the name and by the authority of THE ROYAL SOCIETY OF EDINBURGH, I admit you a Fellow thereof.

PAYMENTS BY FELLOWS.

VI.

Each Fellow shall, before he is admitted to the privileges of Fellowship, pay an admission fee of three guineas, and a subscription of three guineas for the year of election. He shall continue to pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected subsequent to December 1916 and previous to December 1920 shall also pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected previous to December 1916, and who has not completed his twenty-five annual payments, shall, at the beginning of each session, pay three guineas until his twenty-five annual payments are made. Each Fellow who has completed or shall complete his payments shall be invited to contribute one guinea per annum or to pay a single sum of ten guineas.

A Fellow may compound for the annual subscriptions by a single payment of fifty guineas, or on such other terms as the Council may from time to time fix.

VII.

A Fellow who, after application made by the Treasurer, fails to pay any contribution due by him, shall be reported to the Council, and, if the Council see fit, shall be declared no longer a Fellow. Notwithstanding such declaration all arrears of contributions shall remain exigible.

ELECTION OF HONORARY FELLOWS.

VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the First Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

EXPULSION OF FELLOWS.

X.

If, in the opinion of the Council, the conduct of any Fellow is injurious to the character or interests of the Society, the Council may, by registered letter, request him to resign. If he fail to do so within one month of such request, the Council shall call a Special Meeting of the Society to consider the matter. If a majority consisting of not less than two-thirds of the Fellows present and voting decide for expulsion, he shall be expelled by declaration from the Chair, his name shall be erased from the Roll, and he shall forfeit all right or claim in or to the property of the Society.

XI.

It shall be competent for the Council to remove any person from the Roll of Honorary Fellows if, in their opinion, his remaining on the Roll would be injurious to the character or interests of the Society. Reasonable notice of such proposal shall be given to each member of the Council, and, if possible, to the Honorary Fellow himself. Thereafter the decision on the question shall not be taken until the matter has been discussed at two Meetings of Council, separated by an interval of not less than fourteen days. A majority of two-thirds of the members present and voting shall be required for such removal.

MEETINGS OF THE SOCIETY.

XII.

A Statutory Meeting for the election of Council and Office-Bearers, for the presentation of the Annual Reports, and for such other business as may be arranged by the Council, shall be held on the fourth Monday of October. Each Session of the Society shall begin at the date of the Statutory Meeting.

XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

XIV.

A Special Meeting of the Society may be called at any time by direction of the Council, or on a requisition to the Council signed by not fewer than six Fellows. The date and hour of such Meeting shall be determined by the Council, who shall give not less than seven days' notice of such Meeting. The notice shall state the purpose for which the Special Meeting is summoned; no other business shall be transacted.

PUBLICATION OF PAPERS.

XV.

The Society shall publish Transactions and Proceedings. The consideration of the acceptance, reading, and publication of papers is vested in the Council, whose decision shall be final. Acceptance for reading shall not necessarily imply acceptance for publication.

DISTRIBUTION OF PUBLICATIONS.

XVI.

Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the Transactions and the Proceedings published subsequently to their admission.

Copies of the Parts of the Proceedings shall be distributed by post or otherwise, as soon as may be convenient after publication; copies of the Transactions or Parts thereof shall be obtainable upon application, either personally or by an authorised agent, to the Librarian, provided the application is made within five years after the date of publication.

CONSTITUTION OF COUNCIL.

XVII.

The Council shall consist of a President, six Vice-Presidents, a Treasurer, a General Secretary, two Secretaries to the Ordinary Meetings (the one representing the Biological group and the other the Physical group of Sciences),* a Curator of the Library and Museum, and twelve ordinary members of Council.

ELECTION OF COUNCIL.

XVIII.

The election of the Council and Office-Bearers for the ensuing Session shall be held at the Statutory Meeting on the fourth Monday of October. The list of the names recommended by the Council shall be issued to the Fellows not less than one week before the Meeting. The election shall be by Ballot, and shall be determined by a majority of the Fellows present and voting. Scrutineers shall be nominated as in Law IV.

XIX.

The President may hold office for a period not exceeding five consecutive years; the Vice-Presidents, not exceeding three; the Secretaries to the Ordinary Meetings, not exceeding five; the General Secretary, the Treasurer, and the Curator of the Library and Museum, not exceeding ten; and ordinary members of Council, not exceeding three consecutive years; provided that the Treasurer may be re-elected for more than ten successive years in cases where the Council declares to the Society that an emergency exists.

XX.

In the event of a vacancy arising in the Council or in any of the Offices enumerated in Law XVII, the Council shall proceed, as soon as con-

* The Biological group includes Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology; the Physical group includes Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, Physics.

venient, to elect a Fellow to fill such vacancy for the period up to the next Statutory Meeting.

POWERS OF THE COUNCIL.

XXI.

The Council shall have the following powers:—(1) To manage all business concerning the affairs of the Society. (2) To decide what papers shall be accepted for communication to the Society, and what papers shall be printed in whole or in part in the Transactions and Proceedings. (3) To appoint Committees. (4) To appoint employees and determine their remuneration. (5) To award the various prizes vested in the Society, in accordance with the terms of the respective deeds of gift, provided that no member of the existing Council shall be eligible for any such award. (6) To make from time to time Standing Orders for the regulation of the affairs of the Society. (7) To control the investment or expenditure of the Funds of the Society.

At Meetings of the Council the President or Chairman shall have a casting as well as a deliberative vote.

DUTIES OF PRESIDENT AND VICE-PRESIDENTS.

XXII.

The President shall take the Chair at Meetings of Council and of the Fellows. It shall be his duty to see that the business is conducted in accordance with the Charter and Laws of the Society. When unable to be present at any Meetings or attend to current business, he shall give notice to the General Secretary, in order that his place may be supplied. In the absence of the President his duties shall be discharged by one of the Vice-Presidents.

DUTIES OF THE TREASURER.

XXIII.

The Treasurer shall receive the monies due to the Society and shall make payments authorised by the Council. He shall lay before the Council a list of arrears in accordance with Rule VII. He shall keep accounts of all receipts and payments, and at the Statutory Meeting shall present the accounts for the preceding Session, balanced to the 30th of September, and audited by a professional accountant appointed annually by the Society.

DUTIES OF THE GENERAL SECRETARY.

XXIV.

The General Secretary shall be responsible to the Council for the conduct of the Society's correspondence, publications, and all other business except that which relates to finance. He shall keep Minutes of the Statutory and Special Meetings of the Society and Minutes of the Meetings of Council. He shall superintend, with the aid of the Assistant Secretary, the publication of the Transactions and Proceedings. He shall supervise the employees in the discharge of their duties.

DUTIES OF SECRETARIES TO ORDINARY MEETINGS.

XXV.

The Secretaries to Ordinary Meetings shall keep Minutes of the Ordinary Meetings. They shall assist the General Secretary, when necessary, in superintending the publication of the Transactions and Proceedings. In his absence, one of them shall perform his duties.

DUTIES OF CURATOR OF LIBRARY AND MUSEUM.

XXVI.

The Curator of the Library and Museum shall have charge of the Books, Manuscripts, Maps, and other articles belonging to the Society. He shall keep the Card Catalogue up to date. He shall purchase Books sanctioned by the Council.

ASSISTANT SECRETARY AND LIBRARIAN.

XXVII.

The Council shall appoint an Assistant Secretary and Librarian, who shall hold office during the pleasure of the Council. He shall give all his time, during prescribed hours, to the work of the Society, and shall be paid according to the determination of the Council. When necessary he shall act under the Treasurer in receiving subscriptions, giving out receipts, and paying employees.

ALTERATION OF LAWS.

XXVIII.

Any proposed alteration in the Laws shall be considered by the Council, due notice having been given to each member of Council. Such alteration, if approved by the Council, shall be proposed from the Chair at the next Ordinary Meeting of the Society, and, in accordance with the Charter, shall be considered and voted upon at a Meeting held at least one month after that at which the motion for alteration shall have been proposed.

Additions to the Library—Presentations, etc.—1936–1937.

Account of the Life and Writings of William Robertson, D.D., F.R.S.E. By Dugald Stewart. 4to. London, 1801. (*Purchased.*)

Anniversary Volume Dedicated to Professor Kôtarô Honda on the Completion of Twenty-Five Years of his Professorship by his Friends and Pupils. (Science Reports of the Tôhoku Imperial University: First Series.) La. 8vo. Sendai, 1936. (*Presented.*)

Anthropological Study: Department of Conservation, State of Louisiana. No. 1→. 8vo. New Orleans, 1935. (*Exchange:* Louisiana State University.)

Archivos da Fundação Gaffrée e Guinle, 1936–1937. 8vo. Rio de Janeiro, 1937. (*Presented.*)

Aristophron, P. Z. Plato's Academia. (Printed in Greek characters.) 4to. Oxford, 1937. (*Presented by the Author.*)

Army Medical Library, Washington:—

One Hundredth Anniversary of the Founding of the Army Medical Library, Washington: 1836–November 16–1936. Program. 8vo.

Routine Operation of the Army Medical Library. Compiled by the Librarian. 8vo.

The Celebration of the Centenary of the Army Medical Library: 1836–November 16–1936. By Edgar Erskine Hume. 8vo. 1936.

The Centennial of the World's Largest Medical Library: The Army Medical Library of Washington. Founded 1836. By Edgar Erskine Hume. 8vo. 1936.

(*Presented by Col. Edgar Erskine Hume.*)

Association d'Océanographie physique: Union Géodésique et Géophysique Internationale. Procès-Verbeaux No. 2: General Assembly at Edinburgh, September 1936. 8vo. Liverpool, 1937. (*Presented.*)

Balk, Robert. Structural Behavior of Igneous Rocks (with special reference to interpolations by H. Cloos and collaborators). (Memoir 5: Geological Society of America.) 8vo. Washington, 1937. (*Exchange.*)

Bell, T. R. D., and Scott, F. B. Moths: Vol. V. Sphingidæ. (Fauna of British India, including Ceylon and Burma.) 8vo. London, 1937. (*Presented.*)

Bibliographie Géodésique Internationale. Tome I. Introduction et années 1928–1929–1930, par Georges Perrier et Pierre Tardi. (Published by l'Association de Géodésie de l'Union Géodésique et Géophysique Internationale.) 4to. Paris, 1935. (*Presented.*)

Boys, Sir Charles V. Weeds, Weeds, Weeds. 8vo. London, 1937. (*Presented.*)

British Association Report for 1936. 8vo. London, 1936. (*Presented by Dr James Watt.*)

British Museum (Natural History):—

An Index to the Authors (other than Linnæus) mentioned in the Catalogue of the Works of Linnæus preserved in the Libraries of the British Museum. 2nd Ed., 1933. 4to. London, 1936.

Flora of Jamaica containing Descriptions of the Flowering Plants known from the Island. By William Fawcett and Alfred Barton Rendle. Vol. VII. By Spencer Le Marchant Moore and A. B. Rendle: Dicotyledons; Families Rubiaceæ to Compositæ. 8vo. London, 1936. (*Presented.*)

British Polar Year Expedition, Fort Rae, N.W. Canada, 1932-1933. Vols. I, II. 4to. London, 1937. (*Presented.*)

Bulletin de la Société Scientifique de Skoplјé: Section des Sciences Humaines. Tome 14→. 4to. Skoplјé, Jugoslavia, 1935. (*Exchange.*)

Bulletin International de l'Académie Yougoslave des Sciences et des Beaux-Arts. Classe des Sciences matématiques et naturelles. Livres XXIX→. 8vo. Zagreb, 1936. (*Exchange.*)

Canadian Pacific Fauna. 1: Protozoa. By G. H. Wailes. (Published by the Biological Board of Canada.) 8vo. Toronto, 1937. (*Exchange.*)

Canal Rays. (Special Number, Current Science, September 1937.) 8vo. Bangalore, 1937. (*Purchased.*)

Carnegie Institution of Washington:—

No. 436. Contributions to Pre-Cambrian Geology of Western North America. By Norman E. A. Hinds. 8vo. 1936. Carnegie Institution of Washington.

No. 474. Physiology of the Elephant. By Francis G. Benedict. 8vo. 1936.

No. 478. The Vegetation of Petén (with an Appendix: Studies of Mexican and Central American Plants)—I. By Cyrus Longworth Lundell. 4to. 1937.

No. 480. Delta, Estuary, and Lower Portion of the Channel of the Colorado River 1933 to 1935. By Godfrey Sykes. 8vo. 1937.

No. 483. Contributions to American Archæology. Vol. 4, Nos. 20–23, 4to. 1937.

Supplementary Publications, No. 23. The Composition of Glass. By George W. Morey. 8vo. 1936.

No. 24. Environment and Life in the Great Plains. By Frederic E. Clements and Ralph W. Chaney. 8vo. 1936.

No. 25. Nutritional Improvement in Health and Longevity. By Henry C. Sherman. 8vo. 1936.

No. 27. The Earth's Interior, its Nature and Composition. By Leason H. Adams. 8vo. 1937.

(*Exchange.*)

Central Geophysical Observatory:—

Mémoires de l'Observatoire Géophysique Central: Institut d'Actinométrie et d'Optique Atmosphérique. Vol. I (No. 1), Liv. 1→. 8vo. Leningrad, 1934.

Transactions of the Central Geophysical Observatory. 8vo. Leningrad, 1935–1936. Fasc. 4: Theoretical Meteorology. Fasc. 5: Terrestrial Magnetism. Fasc. 6: Klimatology.

(*Exchange.*)

Conspectus Floræ Angolensis. Elaborado pelo Instituto Botanico de Coimbra com a colaboração do Museu Britanico (British Museum). Editado por L. Wittnich Carrisso. Vol. I, Fasc. 1→. 8vo. Coimbra, 1937. (*Exchange.*)

Dansk-Engelsk Ordbog af Johs. Magnussen, Otto Madsen og Hermann Vinterberg. 8vo. København, 1936. (*Purchased.*)

A Description of the Admirable Table of Logarithmes: with a declaration of the Most Plentifull, Easie and Speedy use thereof in both kinds of Trigonometry, as also in all Mathematicall Calculations. Invented and published in *Latine* by that Honourable Lord Iohn Nepair. Sm. 8vo. London, 1618. (*Presented by Mrs Stanley Robertson.*)

Early Science in Oxford. By R. T. Gunther. Vol. IX, De Corde, by Richard Lower, London, 1669; with Introduction and Translation by K. J. Franklin. 8vo. Oxford, 1932.

—— —— Vol. X, The Life of Robert Hooke (Part IV). Tract on Capillary Attraction, 1661; Diary, 1688 to 1693. 8vo. Oxford, 1935.

—— —— Vol. XI, Oxford Colleges and their Men of Science. 8vo. Oxford, 1937.

(*Purchased.*)

Econometrica: Journal of the Econometric Society. Vol. 1, No. 1→. 8vo. Menasha, 1932. (*Exchange.*)

Fleming, Walter L. Louisiana State University, 1860–1896. 8vo. Baton Rouge, 1936. (*Presented.*)

Flett, Sir John Smith. The First Hundred Years of the Geological Survey of Great Britain. 8vo. London, 1937. (*Presented.*)

Fraser, F. C. Odonata. Vol. III. (Fauna of British India, including Ceylon and Burma.) 8vo. London, 1936. (*Presented.*)

Geological Survey of Canada: Index to Memoirs, Bulletins, Summary Reports, Sessional Papers (Administrative), 1910–1926. Compiled by Frank Nicolas. 8vo. Ottawa, 1932. (*Presented.*)

The Geology of the Sanquhar Coalfield and Adjacent Basin of Thornhill. By J. B. Simpson and J. E. Richey, with Contributions by A. G. MacGregor. Palæontology by J. Pringle. (Memoirs of the Geological Survey of Scotland.) 8vo. Edinburgh, 1936. (*Exchange.*)

A Handbook of Home-Grown Timbers. (Forest Products Research, published by the Department of Scientific and Industrial Research.) 8vo. London, 1936. (*Presented.*)

Helgoländer Wissenschaftliche Meeresuntersuchungen. Band 1, Heft 1→. La. 4to. Helgoland, 1937. (*Exchange.*)

Hogben, Lancelot T. Inaugural Lecture: The Theoretical Leadership of Scottish Science in the English Industrial Revolution. 8vo. Aberdeen, 1937. (*Presented by the Author.*)

Index to Supplements to the Geographical Journal: Recent Geographical Literature Supplements. 1918–1932. 8vo. London, 1936. (*Purchased.*)

Indian Science Abstracts (being an Annotated Bibliography of Science in India). 1935, Part 1→. La. 8vo. Calcutta, 1936. (Published by the National Institute of Sciences of India.) (*Exchange.*)

Iowa Coal Studies: Technical Paper No. 3. (Published by the Iowa Geological Survey.) 8vo. Des Moines, 1936. (*Exchange.*)

Izquierdo, J. Joaquin. Harvey, Iniciador del Metodo Experimental. La. 8vo. Mexico, 1936. (*Presented.*)

Jubilé Louis Lumière: 6 Novembre 1935. 4to. Paris, 1936. (*Presented.*)

Jubilé Scientifique de M. Jacques Hadamard: Allocutions prononcées a la cérémonie du 7 janvier, 1936. 8vo. Paris, 1937. (*Purchased.*)

Kendall, James. At Home Among the Atoms: a First Book of Congenial Chemistry. 8vo. London, 1934.

Kieler Meeresforschungen. Bd. I, Heft 1→. 8vo. Kiel, 1936. (*Exchange.*)

Killington, Frederick James. A Monograph of the British Neuroptera. Vol. II. (Ray Society Publication.) 8vo. London, 1937. (*Purchased.*)

Knight, William. Lord Monboddo and some of his Contemporaries. 8vo. London, 1900. (*Presented by Dr Leonard Dobbin.*)

Laue Diagrams: 25 Years of Research on X-Ray Diffraction following Prof. Max von Laue's Discovery. (Current Science, Special Number, January 1937.) 8vo. Bangalore, 1937. (*Purchased.*)

List of Additions to the Library of the University of Aberdeen: August 1935 to July 1936. 8vo. Aberdeen, 1937. (*Exchange.*)

Lorentz, H. A. Collected Papers. Vol. IV. 4to. The Hague, 1937. (*Purchased.*)

McClean, W. N. River Flow Records of the Ness Basin (Inverness-shire). Series A: River Garry, September 1929 to December 1935. Series B: River Moriston, September 1929 to December 1935. Series C: River Ness, September 1929 to December 1935. Fol. London, 1937. (*Presented.*)

A MS. Miscellany: once the property of Alexander Brounlie and his Partners. Sm. 8vo. (*Presented by Mrs Stanley Robertson.*)

The March of Science: A First Quinquennial Review, 1931–1935. By Various Authors. (British Association.) 8vo. London, 1937. (*Purchased.*)

Memórias da Academia das Ciências de Lisboa: Classe de Letras. Tomo I→. 4to. Lisbon, 1936. (*Exchange.*)

Occasional Papers: San Diego Society of Natural History. No. 1→. 8vo. San Diego, California, 1936. (*Exchange.*)

Osborn, Henry Fairfield. Proboscidea. A Monograph of the Discovery, Evolution, Migration and Extinction of the Mastodonts and Elephants of the World. By Henry Fairfield Osborn. Edited by Mabel Rice Percy. Vol. I: Mœritherioidea, Deinotherioidea, Mastodontoidea. La. 4to. New York, 1936. (*Presented by the American Museum of Natural History, New York.*)

Phemister, J. Scotland: The Northern Highlands. (British Regional Geology.) 8vo. London, 1936. (*Geological Survey of Great Britain.*)

Pontificia Academia Scientiarum:—

Acta. Anno I, Vol. I, No. I→. 8vo. Vatican City, 1937.

Commentationes. Anno I, Vol. I, No. I→. 8vo. Vatican City, 1937. (*Exchange.*)

Ramon y Cajal, Santiago. Recollections of My Life. Translated by E. Horne Craigie. (Memoirs of the American Philosophical Society, Vol. VIII, Parts I, II.) 8vo. Philadelphia, 1937. (*Exchange.*)

Report of the Institute of Scientific Research of Manchoukuo. Vol. I, Nos. 1, 2. 8vo. Tatsung Tachie, Hshinching, 1936. (*Presented.*)

Report on the Progress of the *Discovery* Committee's Investigations. 8vo. London, 1937. (*Presented.*)

Road Research: Bulletin No. 1→. 8vo. London, 1936.

Road Research: Technical Paper No. 1→. 8vo. London, 1936. (*Presented by the Department of Scientific and Industrial Research.*)

Rogers, A. W. The Pioneers in South African Geology and their Work. (Annexure to vol. xxxix, Transactions of the Geological Society of South Africa.) 8vo. Johannesburg, 1937. (*Exchange.*)

St Kilda Papers 1931. 4to. Oxford, 1937. (*Presented.*)

The Science Reports of the Tôhoku Imperial University: 2nd Ser. (Geology). Special Volume No. I. Recent Reef-building Corals from Japan and the South Sea Islands under the Japanese Mandate. I. By Hisakatsu Yabe, Toshio Sugiyama and Motoki Eguchi. Fol. Sendai, Japan, 1936. (*Exchange.*)

Scott, C. M. Some Quantitative Aspects of the Biological Action of X- and γ-Rays. (Special Report Series, No. 223, The Medical Research Council.) 8vo. London, 1937. (*Presented by the Medical Research Council.*)

Scott, Sir Walter. The Letters of Sir Walter Scott. Edited by Sir Herbert J. C. Grierson. Centenary Edition. Vol. XI, 1828–1831. Vol. XII, 1831–1832. 8vo. Constable, London, 1936, 1937. (*Presented.*)

Steuart, Daniel Rankin. Bygone Days: Some Recollections by Daniel Rankin Steuart and Some Other Family Stories. With Notes by W. O. Steuart. 8vo. Edinburgh, 1936. (*Presented by Mr W. O. Steuart.*)

Tables de Précession pour des changements d'équinoxe de 25 et de 50 ans et pour tout autre changement d'équinoxe. 4to. Orléans, 1935. (*Exchange: Observatoire de Paris.*)

Transactions of the All-Union Scientific Research Institute of Economic Mineralogy. Fasc. 78→. 8vo. Leningrad and Moscow, 1935. (*Exchange.*)

Transactions of the Physiological Institute at the Leningrad State University of the Name of A. S. Boubnoff. Nos. 16, 17. 1936. 8vo. Leningrad, 1936. (*Presented.*)

Transactions of the Western Surgical Association. 45th Annual Meeting, 1935. 8vo. Chicago, 1936. (*Presented.*)

Troisième centenaire de l'Académie Française. 4to. Paris, 1935. (*Exchange.*)

Union Catalogue of the Periodical Publications in the University Libraries of the British Isles. Compiled on behalf of the Standing Committee of Library Co-operation by Marion C. Roupell. 4to. London, 1937. (*Purchased.*)

Union Géodésique et Géophysique Internationale. Sixième Assemblée Générale réunie à Edimbourg, 14–25 Septembre 1936. 8vo. London, 1937. (*Presented.*)

Watson, William J. The History of the Celtic Place-names of Scotland: Being the Rhind Lectures on Archæology (expanded), delivered in 1916. 8vo. Edinburgh and London, 1926. (*Presented by the late Mr R. C. Cowan.*)

INDEX.

Accounts of the Society, 1936–37, 463.
Additions to Library, 512.
Address to H.M. The King, 443, 447.
Aitken (A. C.). Studies in Practical Mathematics. I. The Evaluation, with Applications, of a Certain Triple Product Matrix, 172–181.
——— Studies in Practical Mathematics. II. The Evaluation of the Latent Roots and Latent Vectors of a Matrix, 269–304.
Algebra, Invariant System of a Quadratic Complex, by H. W. Turnbull, 155–162.
Amphipoda, Benthic, of North-Western North Sea and Adjacent Waters, by D. S. Raitt, 241–254.
Anderson (E. M.). Awarded Makdougall-Brisbane Prize, 444, 446, 449.
Anderson-Berry Prize. *See* Prizes.
Anschütz (R.), Obituary Notice of, 400.
Arc Equation: Studies in Clocks and Time-keeping: No. 6, by R. A. Sampson, 55–63.
Auditor, Reappointment of, 449.
Auerbach (Charlotte). *See* Crew (F. A. E.) and Auerbach (Charlotte).
Awards of Prizes, 455–462.
Axially Symmetric Gravitational Field, by W. J. van Stockum, 135–154.

Bailey (E. B.) and McCallien (W. J.). Perthshire Tectonics: Schiehallion to Glen Lyon. (*Title only*: published in *Trans.*, vol. lix.) 442.
Bessel Functions, Some Formulæ for, by T. M. MacRobert, 19–25.
Birch (de Burgh), Obituary Notice of, 402.
Born (M). Some Philosophical Aspects of Modern Physics, 1–18.
Borthwick (A. W.), Obituary Notice of, 404.
Boys (Sir C. V.). On Rotating Mirrors at High Speed, 377–378.
Brodie (W. Brodie), Obituary Notice of, 435.
Brown (R. S.). On the Anatomy of *Ophelia cluthensis* McGuire, 1935. (*Title only*: published in *Proc.*, vol. lviii.) 446.
Bruce Prize. *See* Prizes.
Bruce-Preller Lecture Fund. *See* Prizes.

Carnegie Trust, Universities of Scotland, Thanks for Grants, 449.
Clocks and Time-keeping, Studies in: No. 6. The Arc Equation, by R. A. Sampson, 55–63.
Cochrane (Flora). An Histological Analysis of Eye Pigment Development in *Drosophila pseudo-obscura*, 385–399.
Compositæ Dp-ages in Relation to Time, by J. Small, 215–220.
——— Quantitative Evolution in, by J. Small and I. K. Johnston, 26–54.
Contributions, Voluntary, 471.
Coronation of H.M. King George VI, 447.
Council, 1936–37, 441; 1937–38, 449, 472.
Cowan (R. C.), Obituary Notice of, 435.
Crew (F. A. E.) and Auerbach (Charlotte). "Spheroidal": A Mutant in *Drosophila funebris* affecting Egg Size and Shape, and Fecundity, 255–268.

Daniell (A.), Obituary Notice of, 405.
Darling (F. F.). Observations on Animal Sociality. Address. (*Title only*.) 443.
Darwin (C. G.). Awarded Gunning Victoria Jubilee Prize, 444, 446, 449.
——— Opens Discussion on the Origin of the Laws of Nature. (*Title only*.) 446.
David Anderson-Berry Prize. *See* Prizes.
Dennell (R.). On the Feeding Mechanism of *Apseudes talpa*, and the Evolution of the Peracaridan Feeding Mechanisms. (*Title only*: published in *Trans.*, vol. lix.) 442.
Dirac's Equations, Geometry and Tensorization of, by H. S. Ruse, 97–127.
Donald (H. P.) and Lamy (Rowena). Ovarian Rhythm in *Drosophila*, 78–96.
Drosophila, Ovarian Rhythm in, by H. P. Donald and Rowena Lamy, 78–96.
Drosophila funebris, Mutant affecting Egg Size and Shape, and Fecundity in, by F. A. E. Crew and Charlotte Auerbach, 255–268.
Drosophila pseudo-obscura, Eye Pigment Development in, by Flora Cochrane, 385–399.

Edwards (J.), Obituary Notice of, 407.
Egg, Size and Shape of, Mutant in *Drosophila funebris* affecting, by F. A. E. Crew and Charlotte Auerbach, 255–268.
Election of Fellows, 443, 448, 503.
——— Honorary Fellows, 444, 445, 448, 503.
Ellis (D.), Obituary Notice of, 408.
Equations, Simultaneous, Solution of, by A. C. Aitken, 172–181.
Evolution, Quantitative, in Compositæ, by J. Small and I. K. Johnston, 26–54.
——— II. Compositæ Dp-ages in Relation to Time, by J. Small, 215–220.
——— III. Dp-ages of Gramineæ, by J. Small, 221–227.

Eye-defects (Microphthalmia, etc.) in Rats, Inheritance of, by A. M. Hain, 64–77.

Fairgrieve (M. McC.), Obituary Notice of, 410.
Fecundity, Mutant in *Drosophila funebris* affecting, by F. A. E. Crew and Charlotte Auerbach, 255–268.
Fellows, Deceased and Resigned, 503.
——— Elected, 443, 448, 503.
——— Honorary, 444, 445, 448, 503.
——— List of, 473–500.
——— List of Honorary, 501–502.
——— Obituary Notices, 400–436.
——— Removed from Roll, 503.
Forster-Cooper (C.). The Middle Devonian Fish Fauna of Achanarras. (*Title only*: published in *Trans.*, vol. lix.) 446.
Fraser (Mabel S.). A Study of the Vascular Supply to the Carpels in the Follicle-bearing Ranunculaceæ. (*Title only*: published in *Trans.*, vol. lix.) 442.
Fraser-Harris (D. F.), Obituary Notice of, 411.
Fretter (Vera). The Structure and Function of the Alimentary Canal of some Species of Polyplacophora (Mollusca). (*Title only*: published in *Trans.*, vol. lix.) 444.

Geonemertes dendyi Dakin, a Land Nemertean, in Wales, by A. R. Waterston and H. E. Quick, 379–384.
Gibb (A. W.), Obituary Notice of, 413.
Gilruth (J. A.), Obituary Notice of, 415.
Graham (A.). On the Ciliary Currents on the Gills of Some *Tellinacea* (Lamellibranchiata), 128–134.
Graham-Smith (W.) and Westoll (T. S.). On a New Long-headed Dipnoan Fish from the Upper Devonian of Scaumenac Bay, P.Q., Canada. (*Title only*: published in *Trans.*, vol. lix.) 445.
Gramineæ, Dp-ages of, by J. Small, 221–227.
Grants towards Publications, 449.
Greenshields (F.). Studies in the Cytology of Parthenogenetic Reproduction of *Hymenoptera Symphyta*. I. Chromosome Number and Individuality in Three Arrhenotokous Species. (*Title only*.) 445.
Gregory (James). Tercentenary, 449.
Gunning Victoria Jubilee Prize. *See* Prizes.

Hain (A. M.). Microphthalmia and other Eye-defects throughout Fourteen Generations of Albino Rats, 64–77.
Hamilton (W. J.). The Early Stages in the Development of the Ferret: The Formation of the Mesoblast and Notochord. (*Title only*: published in *Trans.*, vol. lix.) 445.
Harris (D. F. Fraser-), Obituary Notice of, 411.
Hehir (Sir P.), Obituary Notice of, 416.
Honorary Fellows, Deceased, 503.
——— Elected, 444, 445, 448, 503.
——— List of, 501–502.
Houstoun (R. A.). The Time Lag of the Vacuum Photo-cell, 163–171.
Hunter (W.), Obituary Notice of, 417.

Inheritance of Microphthalmia and other Eye-defects in Rats, by A. M. Hain, 64–77.
Ions and Isotopes, by J. Kendall, 182–193.
Isotopes, Ions and, by J. Kendall, 182–193.

Johnston (I. K.). *See* Small (J.) and Johnston (I. K).

Keith Prize. *See* Prizes.
Kendall (J.). Ions and Isotopes, 182–193.
Kermack (W. O.) and McKendrick (A. G.). Tests for Randomness in a Series of Numerical Observations, 228–240.
——— ——— Some Distributions associated with a Randomly Arranged Set of Numbers, 332–376.
King George VI, Address to, 443, 447; Proclamation, 447; Coronation, 447; Patron of the Society, 447; signs Roll of Fellows as Patron and as Honorary Fellow, 447.
Knight (G. A. F.), Obituary Notice of, 418.
Knight (J.), Obituary Notice of, 435.
Koller (P. C.). The Genetical and Mechanical Properties of Sex Chromosomes. III. Man, 194–214.

Lal (K. B.). On the Immature Stages of some Scottish and other Psyllidæ, 305–331.
Lamy (Rowena). *See* Donald (H. P.) and Lamy (Rowena).
Laurie (A. E.), Obituary Notice of, 419.
Laws, 504–511.
Legendre Functions, Some Formulæ for, by T. M. MacRobert, 19–25.
Levee at Palace of Holyroodhouse, 448.
Library, Additions, 512.

McCallien (W. J.). *See* Bailey (E. B.) and McCallien (W. J.).
Macdonald (J. A.), Obituary Notice of, 436.
McKendrick (A. G.). *See* Kermack (W. O.) and McKendrick (A. G.).
Maclean (M.), Obituary Notice of, 420.
MacRobert (T. M.). Some Formulæ for the Associated Legendre Functions of the Second Kind; with corresponding Formulæ for the Bessel Functions, 19–25.
Makdougall-Brisbane Prize. *See* Prizes.
Man, Sex Chromosome in, by P. C. Koller, 194–214.
Marr (J. W. S.). Presented with Bruce Prize, 450.
——— Antarctic Surveys: The Work of the "Discovery" Investigations. Address. (*Title only*.)
Matrix, Evaluation of Latent Roots and Latent Vectors of, by A. C. Aitken, 269–304.
——— Reciprocal, Calculation of, by A. C. Aitken, 172–181.
Meetings, Proceedings of Ordinary, 442–446.
——— Proceedings of Statutory, 1936, 439; 1937, 447.
Mirrors, Rotating, at High Speed, by Sir C. V. Boys, 377–378.
Moir (H.), Obituary Notice of, 422.

National Committees. R.S.E. Representatives, 448.
Neill Prize. *See* Prizes.
Nemertean Land, *Geonemertes dendyi* Dakin, in Wales, by A. R. Waterston and H. E. Quick, 379–384.
North Sea, Benthic Amphipoda of North-Western Reaches and Adjacent Waters, by D. S. Raitt, 241–254.

Obituary Notices:—Richard Anschütz, de Burgh Birch, Albert William Borthwick, W. Brodie Brodie, Robert Craig Cowan, Alfred Daniell, John Edwards, David Ellis, Mungo McCallum Fairgrieve, D. F. Fraser-Harris, Alfred William Gibb, John Anderson Gilruth, Sir Patrick Hehir, William Hunter, G. A. Frank Knight, James Knight, Albert Ernest Laurie, James A. Macdonald, Magnus Maclean, Henry Moir, William John Owen, Arthur George Perkin, Salvatore Pincherle, John Smith Purdy, Rt. Hon. Lord Rutherford of Nelson, Sir Grafton Elliot Smith, William Ramsay Smith, William Morton Wheeler, 400–436.
Origin of the Laws of Nature. Discussion. (*Title only*.) 446.
Ovarian Rhythm in *Drosophila*, by H. P. Donald and Rowena Lamy, 78–96.
Owen (W. J.), Obituary Notice of, 423.

Papers read, 442–450.
Patronage of H.M. King George VI, 447; signs Roll as Patron and Honorary Fellow, 447.
Perkin (A. G.), Obituary Notice of, 424.
Philosophy of Physics, by M. Born, 1–18.
Photo-cell, Vacuum, Time Lag of, by R. A. Houstoun, 163–171.
Pigment Development, Histological Study of, in *Drosophila pseudo-obscura*, by Flora Cochrane, 385–399.
Pincherle (S.), Obituary Notice of, 426.
Positivism, Critical considerations on, by M. Born, 1–18.
Prizes, etc., Bruce, 450, 453, 462.
——— Bruce-Preller Lecture Fund, 444, 445, 447, 453, 462.
——— David Anderson-Berry Fund, 453, 462.
——— Gunning Victoria Jubilee, 444, 446, 449, 452, 461.
——— Keith, 451, 455.
——— Makdougall-Brisbane, 444, 446, 449, 451, 457.
——— Neill, 452, 459.
——— Scott, 452, 462.
——— Rules and Awards, 451–462.
——— *See* also in Accounts.
Proceedings, Ordinary Meetings, 442–446.
——— Statutory Meetings, 1936, 439; 1937, 447.
Proclamation of H.M. King George VI, 447.
Psyllidæ, On the Immature Stages of some Scottish and other, by K. B. Lal, 305–331.
Purdy (J. S.), Obituary Notice of, 436.

Quadratic Complex, Complete System of, by H. W. Turnbull, 155–162.
Quantum Mechanics, Geometry and Tensorization of Dirac's Equations in, by H. S. Ruse, 97–127.
Quick (H. E.). *See* Waterston (A. R.) and Quick (H. E.).

Raitt (D. S.). The Benthic Amphipoda of the North-Western North Sea and Adjacent Waters, 241–254.
Randomly Arranged Set of Numbers, Some Distributions associated with, by W. O. Kermack and A. G. McKendrick, 332–376.
Randomness, Tests for, in a Series of Numerical Observations, by W. O. Kermack and A. G. McKendrick, 228–240.
Read (H. H.). Metamorphic Correlation in the Polymetamorphic Rocks of the Valla Field Block, Unst, Shetland Islands. (*Title only*: published in *Trans.*, vol. lix.) 445.
Reports, General Secretary's, 447; Treasurer's, 449.
Representatives, R.S.E., Committees, Congresses, etc., 447–449.
Roll of Fellows signed by H.M. King George VI, 447.
Roots, Evaluation of Latent, by A. C. Aitken, 269–304.
Rotating Mirrors at High Speed, by Sir C. V. Boys, 377–378.
Rotating System of Particles, Gravitational Field of, by W. J. van Stockum, 135–154.
Royal Society, Government Publication Grant, 449.
Ruse (H. S). On the Geometry of Dirac's Equations and their Expression in Tensor Form, 97–127.
Rutherford (Rt. Hon. Lord), Obituary Notice of, 427.
——— Tribute to, 450.

Sampson (R. A.). Studies in Clocks and Time-keeping: No. 6. The Arc Equation, 55–63.
Scott Prize. *See* Prizes.
Sex Chromosome in Man, by P. C. Koller, 194–214.
Small (J.). Quantitative Evolution. II. Compositæ Dp-ages in Relation to Time, 215–220.
——— Quantitative Evolution. III. Dp-ages of Gramineæ, 221–227.
Small (J.) and Johnston (I. K.). Quantitative Evolution in Compositæ, 26–54.
Smith (Sir G. E.), Obituary Notice of, 430.
Smith (W. R)., Obituary Notice of, 436.
Statutory Meetings, 1936, 439; 1937, 447.
Stephen (A. C.). Production of Large Broods in certain Lamellibranchs in relation to Weather Conditions. (*Title only*.) 444.
Stockum (W. J. van). The Gravitational Field of a Distribution of Particles Rotating about an Axis of Symmetry, 135–154.

Taylor (H. S.). Heavy Hydrogen in Scientific Research. Bruce-Preller Lecture. (*Title only.*) 444, 445.
Tellinacea (Lamellibranchiata), Ciliary Currents on the Gills of, by A. Graham, 128–134.
Transactions R.S.E., Index of Papers, 1936–37, 520.
Turnbull (H. W.). The Revised Complete System of a Quadratic Complex, 155–162.

Vacuum Photo-cell, Time Lag of, by R. A. Houstoun, 163–171.
Voluntary Contributors, 471.

Waterston (A. R.) and Quick (H. E.). *Geonemertes dendyi* Dakin, a Land Nemertean, in Wales, 379–384.
Watt (J.). Services as Treasurer, 450.
Westoll (T. S.). *See* Graham-Smith (W.) and Westoll (T. S.).
Wheeler (W. M.), Obituary Notice of, 432.

Index of Papers published in the "Transactions" during Session 1936–37.

Bailey (E. B.) and McCallien (W. J.). Perthshire Tectonics: Schiehallion to Glen Lyon, vol. lix, no. 3, p. 79.

Dennell (R.). On the Feeding Mechanism of *Apseudes talpa*, and the Evolution of the Peracaridan Feeding Mechanisms, vol. lix, no. 2, p. 57.

Forster-Cooper (C.). The Middle Devonian Fish Fauna of Achanarras, vol. lix, no. 7, p. 223.

Fraser (Mabel S.). A Study of the Vascular Supply to the Carpels in the Follicle-bearing Ranunculaceæ, vol. lix, no. 1, p. 1.

Fretter (Vera). The Structure and Function of the Alimentary Canal of some Species of Polyplacophora (Mollusca), vol. lix, no. 4, p. 119.

Graham-Smith (W.) and Westoll (T. S.). On a New Long-headed Dipnoan Fish from the Upper Devonian of Scaumenac Bay, P.Q., Canada, vol. lix, no. 8, p. 241.

Hamilton (W. J.). The Early Stages in the Development of the Ferret: The Formation of the Mesoblast and Notochord, vol. lix, no. 5, p. 165.

McCallien (W. J.). *See* Bailey (E. B.) and McCallien (W. J.).

Read (H. H.). Metamorphic Correlation in the Polymetamorphic Rocks of the Valla Field Block, Unst, Shetland Islands, vol. lix, no. 6, p. 195.

Westoll (T. S.). *See* Graham-Smith (W.) and Westoll (T. S.).

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